Hopf algebras and diagonal harmonics

Cesar Ceballos
joint work with Nantel Bergeron and Vincent Pilaud

Graz University of Technology

Der Wissenschaftsfonds.

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Plan

To give an insight into two apparently unrelated areas:

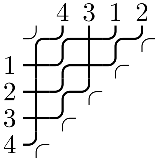
- the theory of Hopf algebras
  motivated by work of Hopf on algebraic topology and of Dieudonné on algebraic groups, 1940' and 1950'.
- the theory of diagonal harmonics
  initiated by Garsia and Haiman in the early 1990's in order to understand properties of Macdonald polynomials.

Main objective: present connections among them.
We will need certain combinatorial objects: pipe dreams

Pipe dreams

Pipe dreams

Fill a triangular shape with crosses $+$ and elbows :



A pipe dream $P \in \Pi_4$ where $\omega_P = [4,3,1,2]$.

Conditions:

- pipes entering on the left exit on the top.
- two pipes cross at most once.
- the top left corner is an elbow .

Pipe dreams

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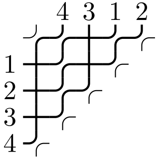
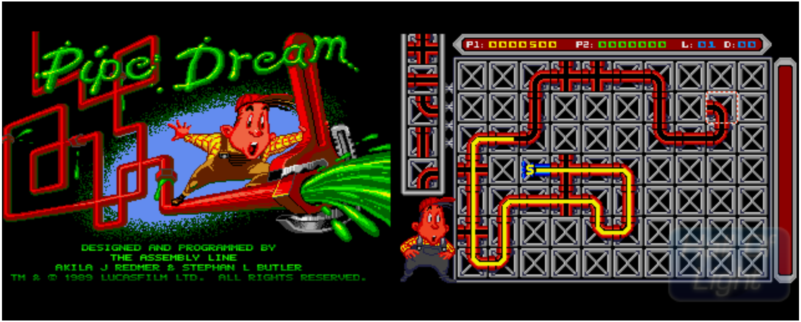


A pipe dream $P \in \Pi_4$ where $\omega_P = [4, 3, 1, 2]$.

Introduced and studied by:

- S. Fomin and A. N. Kirillov. The Yang-Baxter equation, symmetric functions, and Schubert polynomials. (FPSAC 1993)
- N. Bergeron and S. Billey. RC-graphs and Schubert polynomials. (Experiment. Math. 1993)
- A. Knutson and E. Miller. Gröbner geometry of Schubert polynomials. (Ann. of Math. 2005)
- ...

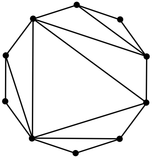
Pipe dreams



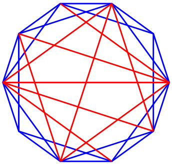
Pipe Dream
DESIGNED AND PROGRAMMED BY
THE ASSEMBLY LINE
AKILA J REDMER & STEPHAN L BUTLER
TM & © 1989 LUCASFILM LTD. ALL RIGHTS RESERVED.
P1: 0000500
P2: 0000000
L: 01
D: 00

Pipe dreams: why are they interesting?

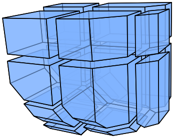
1. They give a combinatorial understanding of Schubert polynomials in the study of Schubert varieties.
2. Pipe dreams of certain families of permutations encode interesting combinatorial-geometric objects:



triangulations



multitriangulations



ν-Tamari lattices

Goal

- Introduce a Hopf algebra structure on pipe dreams.
- Present some applications.

Hopf algebras

Hopf algebras

Hopf algebra: Vector space whose generators can be multiplied and comultiplied in a compatible way. Also there is an antipode.

Example

$\mathbf{k}G$: $\Delta(g) = g \otimes g \quad m(g \otimes h) = gh$.

- Polynomial rings
- Permutations
- Cohomology of Lie groups
- Universal enveloping algebra of Lie algebras
- Quantum groups
- Many more ...

Examples: Hopf algebra on permutations

$\mathfrak{S}_n$: collection of permutations of $[n]$
$\mathbf{k}\mathfrak{S}$: vector space spanned by all permutations

Theorem (Malvenuto, 1994, Malvenuto–Reutenauer, 1995)

$\mathbf{k}\mathfrak{S}$ *may be equipped with a structure of graded Hopf algebra.*

Comultiplication: sum of pairs obtained by cutting a permutation in two

$$\boldsymbol{\Delta}(\mathbf{312}) = \mathbf{312} \otimes \emptyset + \mathbf{31} \otimes \mathbf{2} + \mathbf{3} \otimes \mathbf{12} + \emptyset \otimes \mathbf{312}$$

C. Malvenuto and C. Reutenauer. Duality between quasi-symmetric functions and the Solomon descent algebra. (J. Algebra 1995)

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Multiplication: sum of all possible shuffles between two permutations

$$\mathbf{12} \cdot \mathbf{21} = \mathbf{1221} + \mathbf{1221} + \mathbf{1212} + \mathbf{2121} + \mathbf{2112} + \mathbf{2112}$$

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Multiplication: sum of all possible shuffles between two permutations

$$\mathbf{12} \cdot \mathbf{21} = \mathbf{1243} + \mathbf{1423} + \mathbf{1432} + \mathbf{4123} + \mathbf{4132} + \mathbf{4312}$$

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$$\Delta(312) = 312 \otimes \emptyset + 21 \otimes 1 + 1 \otimes 12 + \emptyset \otimes 312$$

Multiplication: sum of all possible shuffles between two permutations

$$12 \cdot 21 = 1243 + 1423 + 1432 + 4123 + 4132 + 4312$$

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Examples: Hopf algebra on binary trees

Y_n: collection of planar binary trees with n leaves
$\mathbf{k}Y$: vector space spanned by all planar binary trees

Theorem (Loday–Ronco, 1998)

$\mathbf{k}Y$ *may be equipped with a structure of graded Hopf algebra.*

Comultiplication

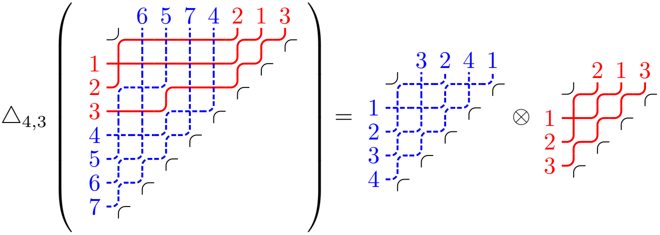
Multiplication

J.-L. Loday and M. O. Ronco. Hopf algebra of the planar binary trees. (Adv. Math. 1998)

A Hopf algebra on pipe dreams

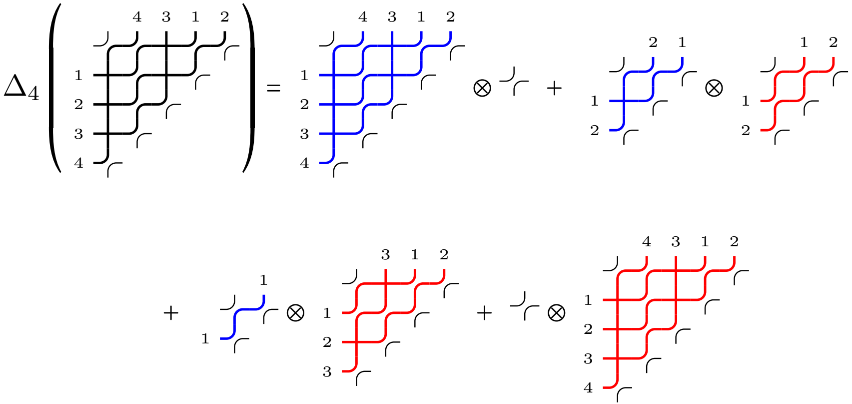
Comultiplication

$$\begin{array}{rcl} \triangle_n : \quad \Pi_n & \longrightarrow & \displaystyle\bigoplus_{\gamma=0}^{n} \Pi_\gamma \otimes \Pi_{n-\gamma} \\ P & \longmapsto & \displaystyle\sum_{\gamma \in GD(\omega_P)} \triangle_{\gamma, n-\gamma}(P). \end{array}$$



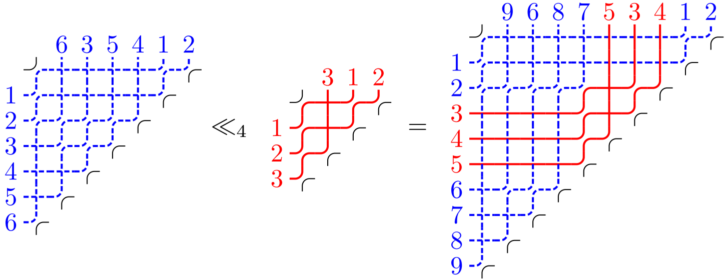
The sum ranges over allowable cuts of the permutation: global descents.

Comultiplication



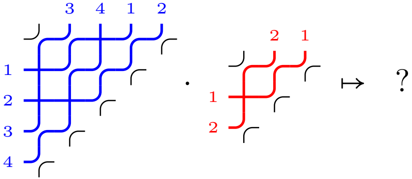
Multiplication

Inserting a pipe dream in another:

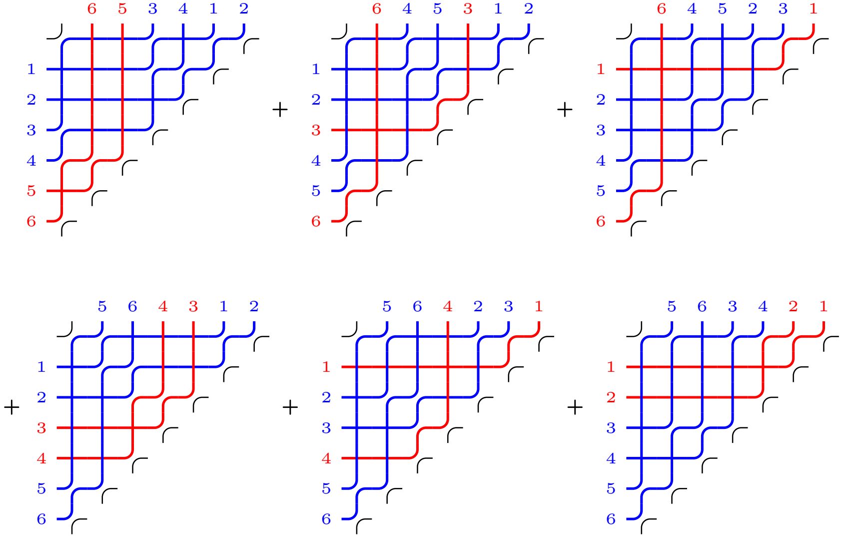


Multiplication

$$\begin{array}{rccc} \mu_{r,s}: & \Pi_r \otimes \Pi_s & \longrightarrow & \Pi_{r+s} \\ & P \cdot Q & \longmapsto & ? \end{array}$$



Multiplication



A Hopf algebra on pipe dreams

Π_n: collection of pipe dreams of permutations in $\mathfrak{S}_n$

$\mathbf{k}\Pi$: vector space spanned by pipe dreams

Theorem (N. Bergeron–C. C.–V. Pilaud)

These operations endow $\mathbf{k}\Pi$ with a graded Hopf algebra structure.
This Hopf algebra is free and cofree.

Hopf subalgebras

Hopf subalgebra of reversing pipe dreams

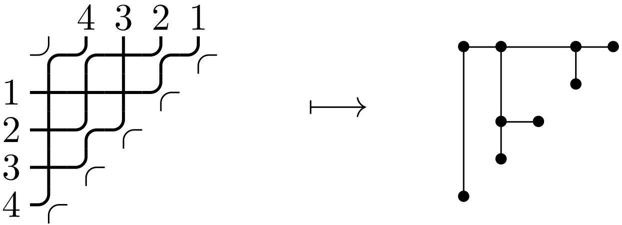
$\mathbf{k}\Pi_{\mathrm{rev}}$: vector space spanned by pipe dreams of permutations $n \dots 321$.

Theorem (N. Bergeron–C. C.–V. Pilaud)

$\mathbf{k}\Pi_{\mathrm{rev}}$ *is a Hopf subalgebra of* $\mathbf{k}\Pi$.
It is isomorphic to the Loday–Ronco Hopf algebra on planar binary trees.

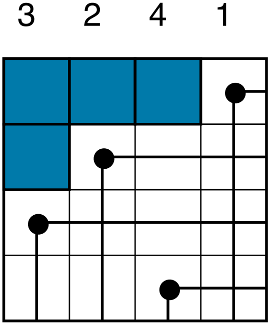
- *dim deg* $n = C_n$, *the nth Catalan number* $C_n = \frac{1}{n+1}\binom{2n}{n}$.

Bijection: replace elbows by nodes $\bullet$



Hopf subalgebra of dominant pipe dreams

A permutation ω is called *dominant* if its Rothe diagram is a Young diagram located at the top-left corner.



Schubert polynomials of dominant permutations are specially interesting.

Hopf subalgebra of dominant pipe dreams

$\mathbf{k}\Pi_{\mathrm{dom}}$: vector space spanned by pipe dreams of dominant permutations

Theorem (N. Bergeron–C. C.–V. Pilaud)

$\mathbf{k}\Pi_{\mathrm{dom}}$ is a Hopf subalgebra of $\mathbf{k}\Pi$.

- *dim deg n* $= \det \begin{vmatrix} C_n & C_{n+1} \\ C_{n+1} & C_{n+2} \end{vmatrix}$

Dominant pipe dreams are in bijection with pairs of nested Dyck paths.

L. Serrano and C. Stump. Maximal fillings of moon polyominoes, simplicial complexes, and Schubert polynomials. (Electron. J. Combin. 2012)

Application to multivariate diagonal harmonics

What is multivariate diagonal harmonics?

The story begins with the Macdonald positivity conjecture, regarding the coefficients of the Schur function expansion of Macdonald polynomials:

$$H_\mu(\mathbf{x}; q, t) = \sum_{\nu \vdash \mu} k_{\mu\nu}(q, t) s_\nu(\mathbf{x}).$$

Conjecture (Macdonald Positivity Conjecture, 1988)

$k_{\mu\nu}(q, t)$ are polynomials in q and t with non-negative coefficients.

Garsia–Haiman's combinatorial approach:
study a representation of the symmetric group on a space ∂D_μ

Garsia–Haiman's combinatorial approach

Theorem (The $n!$ conjecture, Haiman 2001)

For any $\mu \vdash n$, we have

$$\dim_{\mathbb{C}} \partial D_\mu = n!.$$

Theorem (Haiman 2001)

$$k_{\mu\nu}(q,t) = \sum_{i,j} t^i q^j \operatorname{mult}(\chi^\nu, \operatorname{ch}(D_\mu)_{i,j})$$

In particular, it is a polynomial with non-negative integer coefficients and the Macdonald positivity conjecture holds.

For $\mu = (1, 1, \ldots, 1)$, ∂D_μ is the *space of harmonics*.

M. Haiman. Hilbert schemes, polygraphs, and the Macdonald positivity conjecture. (J. Amer. Math. Soc. 2001)

The space of harmonics

$\mathbb{Q}[\mathbf{x}] := \mathbb{Q}[x_1, \ldots, x_n]$ is the polynomial ring in n variables,
$I :=$ ideal generated by $\mathfrak{S}_n$ invariant polynomials with no constant term,
$\partial\mathbf{x} = (\frac{\partial}{\partial x_1}, \ldots, \frac{\partial}{\partial x_n})$.

Definition

The *space of harmonics* is defined by

$$H_n = \{h \in \mathbb{Q}[\mathbf{x}] : f(\partial\mathbf{x})h = 0, \ \forall f \in I\}.$$

Fact

As $\mathfrak{S}_n$-modules,

$$H_n \cong \mathbb{Q}[\mathbf{x}]/I.$$

The space of diagonal harmonics

$\mathbb{Q}[\mathbf{x}, \mathbf{y}] := \mathbb{Q}[x_1, \ldots, x_n, y_1, \ldots, y_n]$
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$\partial\mathbf{x} = (\frac{\partial}{\partial x_1}, \ldots, \frac{\partial}{\partial x_n})$.

Definition

The *space of diagonal harmonics* is defined by

$$DH_n = \{h \in \mathbb{Q}[\mathbf{x}, \mathbf{y}] : f(\partial\mathbf{x}, \partial\mathbf{y})h = 0, \ \forall f \in I\}.$$

Fact

as $\mathfrak{S}_n$-modules,

$$DH_n \cong \mathbb{Q}[\mathbf{x}, \mathbf{y}]/I.$$

The space of diagonal harmonics

The $(n+1)^{n-1}$ conjecture by Garsia and Haiman from 1993:

Theorem (Haiman 2002)

The dimension of DH_n is equal to $(n+1)^{n-1}$.

Theorem (Haiman 2002)

The dimension of the alternating component of DH_n is equal to $\frac{1}{n+1}\binom{2n}{n}$.

This led to the now famous q, t-Catalan polynomials!

M. Haiman. Vanishing theorems and character formulas for the Hilbert scheme of points in the plane. (Invent. Math. 2002)

Multivariate diagonal harmonics

The space DH_n can be generalized to three, or more sets of variables.

Conjecture (Haiman 1994)

In the trivariate case,

- *the dimension of* DH_n *is* $2^n(n+1)^{n-2}$.
- *the dimension of its alternating component is*

$$\frac{2}{n(n+1)}\binom{4n+1}{n-1}.$$

These two numbers can be combinatorially interpreted as the number of labeled and unlabeled intervals in the Tamari lattice, certain poset on Catalan objects.

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No conjectural formulas are known for more sets of variables.

In summary

The dimensions of the spaces of multivariate diagonal harmonics and their alternating components are

one set of variables	two sets of variables	three sets of variables	more sets of variables
$n!$	$(n+1)^{n-1}$	**Tamari lattice labelled intervals**	**Unknown**
1	$\frac{1}{n+1}\binom{2n}{n}$	**Tamari lattice intervals**	**Unknown**

Open problems

One may expect that dimensions for r sets of variables are counted by labeled and unlabeled chains $(\pi_1, \ldots, \pi_{r-1})$ in the Tamari lattice. But this is not true in general.

Back to pipe dreams

Hopf chains

Pipe dreams have a natural poset structure.
The number of intervals in the graded dimensions of $\mathbf{k}\Pi_{\mathrm{dom}}$ is:

$$1, 4, 29, 297, 3823, 57956, \ldots$$

They correspond to certain triples of Dyck paths.

Definition (Hopf chains)

A Hopf chain of length r and size n is a tuple $(\pi_1, \ldots, \pi_r)$ of Dyck paths of size n such that

- π_1 is the bottom diagonal path,
- every triple comes from an interval of dominant pipe dreams.

Counting Hopf chains

Example (n=4)

The number of Hopf chains $(\pi_1, \ldots, \pi_r)$ of Dyck paths of size $n = 4$ is

$$1, 14, 68, 217, 549, 1196, 2345, \ldots$$

Counting Hopf chains

Example (n=4)

The number of Hopf chains $(\pi_1, \ldots, \pi_r)$ of Dyck paths of size $n = 4$ is

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Example (n=4)

The dimension of the alternating component of the space of diagonal harmonics DH_n for fixed $n = 4$ and r variables is equal to

$$1, 14, 68, 217, 549, 1196, 2345, \ldots$$

Counting Hopf chains

Theorem (N. Bergeron–C. C.–V. Pilaud)

For $n \leq 4$ and any number r of sets of variables, the q,t-Frobenius characteristic of the multivariate diagonal harmonics space $DH_{n,r}$ is

$$\Phi_{n,r}(q,t) = \sum_{\substack{\text{Hopf chains} \\ \boldsymbol{\pi}=(\pi_1,\pi_2,\ldots,\pi_r)}} q^{\mathrm{col}(\boldsymbol{\pi})} \mathbb{L}_{\pi_r}(t),$$

where $\mathbb{L}_\pi(t)$ denotes the LLT polynomial of Lascoux, Leclerc and Thibon, and $\mathrm{col}(\boldsymbol{\pi})$ is certain statistic on Hopf chains.

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where $\mathbb{L}_{\pi}(t)$ denotes the LLT polynomial of Lascoux, Leclerc and Thibon, and $\mathrm{col}(\boldsymbol{\pi})$ is certain statistic on Hopf chains.

For $r = 2$, this recovers the former shuffle conjecture (for $n \leq 4$) recently proven by Carlsson and Mellit.

J. Haglund, M. Haiman, N. A. Loehr, J. B. Remmel, and A. Ulyanov. A combinatorial formula for the character of the diagonal coinvariants. (Duke Math. J. 2005)

E. Carlsson and A. Mellit. A proof of the shuffle conjecture. (J. Amer. Math. Soc. 2018)

Counting Hopf chains

Corollary

For $n \leq 4$ and any number r of sets of variables:

1. *The bigraded Hilbert series of* $\mathrm{Alt}(DH_{n,r})$ *is*

$$\widetilde{\Phi}_{n,r}(q,t) = \sum_{\substack{\text{Hopf chains} \\ \boldsymbol{\pi}=(\pi_1,\pi_2,\ldots,\pi_r)}} q^{\mathsf{col}(\boldsymbol{\pi})} t^{\mathsf{dinv}(\pi_r)}.$$

2. *The q-Frobenius characteristic of* $DH_{n,r}$ *is*

$$\Phi_{n,r}(q,1) = \sum_{\substack{\text{Hopf chains} \\ \boldsymbol{\pi}=(\pi_1,\pi_2,\ldots,\pi_r)}} q^{\mathsf{col}(\boldsymbol{\pi})} e_{\mathsf{type}(\pi_r)}.$$

Counting Hopf chains

Corollary

For $n \leq 4$ and any number r of sets of variables:

1. *The dimension of $\mathrm{Alt}(DH_{n,r})$ equals the number of Hopf chains of length r and size n.*
2. *The dimension of $DH_{n,r}$ equals to the number of labeled Hopf chains of length r and size n.*

Counting Hopf chains

The dimensions of the alternating and full component for fixed $n \leq 4$ and arbitrary r are given in the following table:

n	number of Hopf chains	number of laballed Hopf chains
$n = 1$	$\binom{r}{0}$	$\binom{r+1}{0}$
$n = 2$	$\binom{r}{1}$	$\binom{r+1}{1}$
$n = 3$	$\binom{r}{1} + 3\binom{r}{2} + \binom{r}{3}$	$\binom{r+1}{1} + 4\binom{r+1}{2} + \binom{r+1}{3}$
$n = 4$	$\binom{r}{1} + 12\binom{r}{2} + 29\binom{r}{3}$ $+25\binom{r}{4} + 9\binom{r}{5} + \binom{r}{6}$	$\binom{r+1}{1} + 22\binom{r+1}{2} + 56\binom{r+1}{3}$ $+40\binom{r+1}{4} + 11\binom{r+1}{5} + \binom{r+1}{6}$

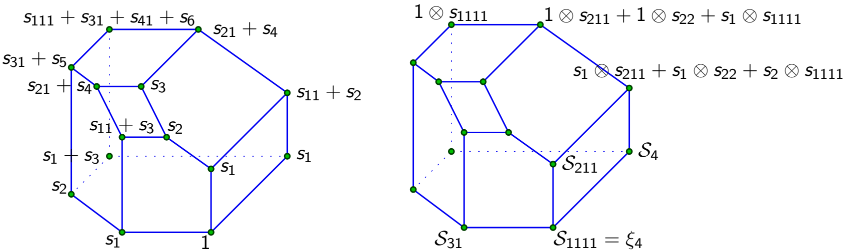
Counting Hopf chains

For $n = 5$ the result is not true. There is a small excess:

$$\mathrm{Excess}_{n=5} = \binom{k+4}{9} e_{[5]} + \binom{k+4}{8} e_{[4,1]}.$$

We have a few possible candidates that kill this excess but do not have a combinatorial rule to describe them at the moment.

The Multi-Shuffle Conjecture



Conjecture (F. Bergeron–N. Bergeron–C. C.–V. Pilaud)

The multi-graded Frobenius characteristic of the space of multivariate diagonal harmonics $DH_{n,r}$ is

$$\xi_n(\mathbf{q} + \mathbf{t}; \mathbf{z}) = \sum_{\mu \subseteq \delta_n} \sigma_\mu(\mathbf{q}) \otimes \mathbb{L}_\mu(\mathbf{t}; \mathbf{z}).$$

To be continued . . .

Thank you!