A COMBINATORIAL MODEL FOR A TORUS FIBRATION OF A $K3$ SURFACE IN THE LARGE COMPLEX STRUCTURE

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Contents

Abstract

This paper is a little survey of the phenomenon of mirror symmetry towards its connections with tropical geometry. In particular, using ideas from Haase and Zharkov, we describe a tropical Calabi-Yau structure on the complement of 24 points on a sphere. More precisely, we construct in purely combinatorial terms dual pairs of integral affine structures on a sphere, and construct a topological torus fibration of a $K3$ surface that coincides with the combinatorial model in the large complex structure limit.

1 Introduction

The phenomenon of mirror symmetry became of great interest for mathematicians when theoretical physicists made predictions about the number of rational curves on a Calabi-Yau manifold by invoking the "mirror" description. A full mathematical understanding of this phenomenon is still being developed, even though, it has inspired many mathematical contributions. Batyrev [2] gave a powerful mirror symmetry construction for Calabi-Yau hypersurfaces in toric varieties, later generalized by Batyrev and Borisov [4] to complete intersections. Strominger, Yau and Zaslow [16] conjectured a geometric interpretation for mirror symmetry in their paper "Mirror Symmetry is T -duality." Following this direction, names as Gross, Siebert and Wilson, among others have made great progress and have found some connections between tropical geometry and mirror symmetry. Specific and elegant examples of tropical Calabi-Yau's can be found in [9] and [10] by Haase and Zharkov. The present paper provides a little survey of mirror symmetry following the ideas above.

The first part of the paper is concerned to Mirror symmetry. In Section 2 we show a brief introduction to toric varietes constructed from polyhedral fans in $\mathbb{R}^d$. Section 3 and 4 are devoted to present the Batyrev construction of mirror symmetry for Calabi-Yau hypersurfaces in toric varieties, and the interpretation of mirror symmetry as duality of special Lagrangian torus brations by Strominger, Yau and Zaslow.

In the second part of the paper we apply the constructions of Haase and Zharkov to a specific example, namely a $K3$ hypersurface of a toric variety. In Section 5 we construct an integral affine structure on the complement of 24 points on a sphere Σ, which gives rise to a natural torus fibration by taking fiber wise quotients. In Section 6 we link the model to the topology of toric $K3$ hypersurfaces H_s in a toric variety. The main result Theorem 6.3 asserts that for any neighborhood N of the 24 points, and a hypersurface with large enough complex structure, there is a torus bration of H_s^{sm}, a portion of the hypersurface, over $\Sigma\backslash N$, which is diffeomorphic to the restriction of our model fibration.

2 A brief introduction to toric varieties

There are two standard ways of defining toric varieties, one of them is via integral polyhedra, and the second and more general way is via rational polyhedral fans. For our applications it is more convenient to work with the construction associated to fans in $\mathbb{R}^d$. We will describe toric varieties from this point of view, but first we recall some definitions.

2.1 Toric varieties via polyhedral fans

A subset $C \subset \mathbb{R}^d$ is called a *convex polyhedral cone* if there is a finite set $\{u_1, \ldots, u_r\}$ of non-zero vectors in $\mathbb{R}^d$ such that

$$C = \{t_1 u_1 + \ldots + t_r u_r \in \mathbb{R}^d : \forall t_j \geq 0\}$$

C is said to be a *rational convex polyhedral cone* if it is generated by a set of vectors $\{u_1 \ldots u_r\}$ in $\mathbb{Z}^d$. We say that C is *strongly convex* if C contains no line through the origin.

The *dual cone* of $C \subset \mathbb{R}^d$ is the set

$$C^\vee = \{b \in (\mathbb{R}^d)^\vee : \langle b, u\rangle \geq 0 \text{ for all } u \in C\}$$

A *fan* in $\mathbb{R}^d$ is a finite set $\mathfrak{F}$ of rational strongly convex polyhedral cones in $\mathbb{R}^d$ such that

1. If $C \in \mathfrak{F}$ and σ is a face of C, then $\sigma \in \mathfrak{F}$.
2. If $C, C' \in \mathfrak{F}$, the $C \cap C'$ is a face of both C and C'.

Let $\mathfrak{F}$ be a fan in $\mathbb{R}^d$ and F be a field. For every cone $C \in \mathfrak{F}$ we define the semi-latice

$$S_{C^\vee} := \mathbb{Z}^d \cap C^\vee = \{b \in \mathbb{Z}^d : \langle b, u\rangle \geq 0, \text{ for all } u \in C\}$$

and the associated toric chart

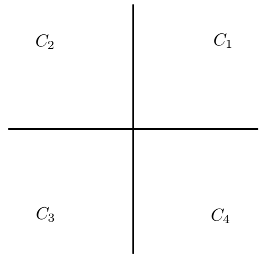
$$U_{\mathbb{Z}[S_{C^\vee}]}(F) := \mathrm{Hom}_{sg}(S_{C^\vee}, F^\times).$$

These collection of charts glue together to form the toric variety $\mathbb{X}_{\mathfrak{F}}$ associated to the fan $\mathfrak{F}$. In the following example we illustrate the glueing conditions between charts.

2.2 Examples

All projective spaces are special cases of toric varieties. In examples below we describe a toric construction for the product $\mathbb{P}^1 \times \ldots \times \mathbb{P}^1$ of projective lines.

$$C_1 = \mathbb{R}_{\geq 0}\begin{pmatrix} 1 \\ 0 \end{pmatrix} + \mathbb{R}_{\geq 0}\begin{pmatrix} 0 \\ 1 \end{pmatrix}$$
$$C_2 = \mathbb{R}_{\geq 0}\begin{pmatrix} 0 \\ 1 \end{pmatrix} + \mathbb{R}_{\geq 0}\begin{pmatrix} -1 \\ 0 \end{pmatrix}$$
$$C_3 = \mathbb{R}_{\geq 0}\begin{pmatrix} -1 \\ 0 \end{pmatrix} + \mathbb{R}_{\geq 0}\begin{pmatrix} 0 \\ -1 \end{pmatrix}$$
$$C_4 = \mathbb{R}_{\geq 0}\begin{pmatrix} 0 \\ -1 \end{pmatrix} + \mathbb{R}_{\geq 0}\begin{pmatrix} 1 \\ 0 \end{pmatrix}$$



$$C_{12} = \mathbb{R}_{\geq 0}\begin{pmatrix} 0 \\ 1 \end{pmatrix}$$
$$C_{23} = \mathbb{R}_{\geq 0}\begin{pmatrix} -1 \\ 0 \end{pmatrix}$$
$$C_{34} = \mathbb{R}_{\geq 0}\begin{pmatrix} 0 \\ -1 \end{pmatrix}$$
$$C_{41} = \mathbb{R}_{\geq 0}\begin{pmatrix} 1 \\ 0 \end{pmatrix}$$
$$C_0 = \{0\}$$

Figure 1: The fan $\mathfrak{F}_2$.

Example 2.1. Take in $\mathbb{R}^2$ the fan $\mathfrak{F}_2 = \{C_1, C_2, C_3, C_4, C_{12}, C_{23}, C_{34}, C_{41}, C_0\}$ corresponding to the fan in Figure 1.

Here $C_j^\vee = C_j$ and the homomorphisms $\varphi \in U_{\mathbb{Z}[S_{C_j^\vee}]}(F)$ can be identified with pairs $(x_j, y_j) \in F^2$ where

$$\begin{array}{ll} x_1 = \varphi(1,0) & y_1 = \varphi(0,1) \\ x_2 = \varphi(-1,0) & y_2 = \varphi(0,1) \\ x_3 = \varphi(-1,0) & y_3 = \varphi(0,-1) \\ x_4 = \varphi(1,0) & y_4 = \varphi(0,-1) \end{array}$$

The dual cone $C_{12}^\vee$ is equal to the upper half plane and the homomorphisms $\varphi \in U_{\mathbb{Z}[S_{C_{12}^\vee}]}(F)$ are given by pairs $(x, y) \in (F \backslash \{0\}) \times F$ with

$$x = \varphi(1,0) \qquad y = \varphi(0,1)$$

The charts $U_{\mathbb{Z}[S_{C_1^\vee}]}(F)$ and $U_{\mathbb{Z}[S_{C_2^\vee}]}(F)$ intersect in $U_{\mathbb{Z}[S_{C_{12}^\vee}]}(F)$; the change of coordinates is given by

$$\begin{array}{l} x_2 = x_1^{-1} \\ y_2 = y_1 \end{array}$$

The toric variety $\mathbb{X}_{\mathfrak{F}_2}$ is then covered by four affine charts corresponding to the cones C_1, C_2, C_3 and C_4. They intersect in the charts of the other cones, and the change of coordinates between charts are similar to the one above. In this case, $\mathbb{X}_{\mathfrak{F}_2}$ turns to be equal to the product $\mathbb{P}^1 \times \mathbb{P}^1$ of two projective lines. More explicitly, for $([x, x'], [y, y']) \in \mathbb{P}^1 \times \mathbb{P}^1$ the bijection $\mathbb{P}^1 \times \mathbb{P}^1 \to \mathbb{X}_{\mathfrak{F}_2}$ is characterized by the equations

$$\begin{array}{ll} x_1 = \frac{x}{x'} & y_1 = \frac{y}{y'} \\ x_2 = \frac{x'}{x} & y_2 = \frac{y}{y'} \\ x_3 = \frac{x'}{x} & y_3 = \frac{y'}{y} \\ x_4 = \frac{x}{x'} & y_4 = \frac{y'}{y} \end{array}$$

Example 2.2. Take in $\mathbb{R}^d$ the fan $\mathfrak{F}_d$ of which the maximal cones correspond to the 2^d cones of the form

$$C = \mathbb{R}_{\geq 0}(\pm e_1) + \ldots + \mathbb{R}_{\geq 0}(\pm e_d)$$

where $e_1, \ldots, e_d$ is the standard basis of $\mathbb{R}^d$. The fan $\mathfrak{F}_d$ is the set of all possible intersections of its maximal cones. In particular, for $d = 2$ this fan coincides with the fan presented in Example 2.1 above. The toric variety $\mathbb{X}_{\mathfrak{F}_d}$ is equal the product $\mathbb{P}^1 \times \ldots \times \mathbb{P}^1$ of d projective lines.

3 Batyrev interpretation of mirror symmetry vs dual reflexive polytopes

Mirror symmetry is a phenomenon that was first discovered by physicists, it conjectures that for any 3-dimensional Calabi-Yau manifold V there exists a Calabi-Yau manifold V^*, called the mirror manifold, for which two $\mathcal{N} = (2,2)$ supersymmetric quantum field theories associated to them are equivalent as quantum field theories. The full understanding of mirror symmetry from the mathematical point of view is still open and has inspired many mathematical contributions in algebraic geometry, toric geometry, hodge theory among others. The first explicit examples of mirror symmetry in physics were given by Greene and Plesser in [?]. And later, Batyrev found and interesting toric generalization of Greene-Plesser construction, that strongly uses the notion of duality between reflexive polytopes.

Definition 3.1. A reflexive polytope $\Delta^\vee \in \mathbb{R}^d$ is a convex polytope with vertices in $\mathbb{Z}^d$ that contains the origin in its interior, and such that the vertices of the dual polytope $\Delta = \{m \in (\mathbb{R}^d)^* : \langle m, n\rangle \leq 1 \text{ for all } n \in \Delta^\vee\}$ belong to the dual lattice $(\mathbb{Z}^d)^*$.

Consider two central triangulations of the polytopes Δ and $\Delta^\vee$. Let S and T be induced triangulations of the boundaries $\partial\Delta$ and $\partial\Delta^\vee$, and denote by $\mathfrak{F}$ (respectively $\mathfrak{F}^\vee$) the fan composed by the cones spanned by the faces of S (respectively T). Denote by H_f^{aff} the affine hypersurface

$$H_f^{\text{aff}} = \{x \in (\mathbb{C}\backslash\{0\})^d : f(x) = \sum_{m \in \Delta \cap (\mathbb{Z}^d)^*} a_m x^m = 0\}$$

where the set $\{a_m\}_{m \in \Delta \cap (\mathbb{Z}^d)^*}$ consist of generically chosen complex numbers. The fan $\mathfrak{F}^\vee$ defines a simplicial subdivision of the normal fan to Δ, the projective toric variety $\mathbb{X}_{\mathfrak{F}^\vee}$ associated to $\mathfrak{F}^\vee$ contains $(\mathbb{C}\backslash\{0\})^d = U_{\mathbb{Z}[S_{\{0\}^\vee}]}(\mathbb{C})$ (corresponding chart of the cone $\{0\}$ in $\mathfrak{F}^\vee$) as a dense open subset. Let H_f be the closure of H_f^{aff} in $\mathbb{X}_{\mathfrak{F}^\vee}$. If we repeat the same procedure with the dual polytope $\Delta^\vee$ we obtain the affine hypersurface

$$H_g^{\text{aff}} = \{x \in (\mathbb{C}\backslash\{0\})^d : g(x) = \sum_{m \in \Delta^\vee \cap \mathbb{Z}^d} a_m x^m = 0\}$$

and denote by H_g the closure of H_g^{aff} in $\mathbb{X}_{\mathfrak{F}}$.

The pair (H_f, H_g) equipped with some additional information about kahler structures, is conjectured to induce isomorphic superconformal field theories whose $N = 2$ superconformal representations are the same up to a sign change [3], [5]. Strictly speaking, the conjecture only applies when H_f and H_g are 3-folds, although the Batyrev mirror construction works in general. In particular, if $d = 4$, then H_f (respectively H_g) is birational to a smooth Calabi-Yau 3-fold $\widehat{H_f}$ (respectively $\widehat{H_g}$) and one has that

$$h^{1,1}(\widehat{H_f}) = h^{2,1}(\widehat{H_g}),\ h^{1,1}(\widehat{H_g}) = h^{2,1}(\widehat{H_f})$$

In general, as proved by Batyrev in [3], the Hodge numbers of H_f and H_g are related as follows

Theorem 3.2. *If H_g is the Batyrev mirror of H_f, then*

$$h^{1,1}(H_f) = h^{d-2,1}(H_g),\ h^{d-2,1}(H_f) = h^{1,1}(H_g).$$

which is a particular case of what is well known as the *topological mirror symmetry test* [12]:

$$h^{p,q}(H_f) = h^{d-1-p,q}(H_g), \quad 0 \le p, q \le d-1.$$

3.1 Examples

(Example 2.1 continued) Coming back to example 2.1. The fan $\mathfrak{F}_2$ is the fan $\mathfrak{F}^\vee$ associated to the triangulation of $\Delta_2^\vee$ in Figure 2. We saw before that the projective toric variety $\mathbb{X}_{\mathfrak{F}^\vee}$ is equal to the product $\mathbb{P}^1 \times \mathbb{P}^1$ of two projective lines.

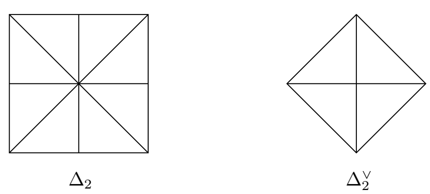


Figure 2: The traingulations $\{0\} * S$ of Δ_2 and $\{0\} * T$ of $\Delta_2^\vee$.

The family of affine hypersurfaces H_f^{aff} is defined by possible linear combinations of monomials corresponding to the lattice points in Δ_2:

$$axy + bx + cxy^{-1} + dy + e + fy^{-1} + gx^{-1}y + hx^{-1} + ix^{-1}y^{-1} = 0.$$

It determines a family of hypersurfaces H_f in $\mathbb{P}^1 \times \mathbb{P}^1$ whose defining equations are homogeneous polynomials of degree (2,2). if $([X_1, X_2], [Y_1, Y_2]) \in \mathbb{P}^1 \times \mathbb{P}^1$, the family is given by:

$$aX_1^2Y_1^2 + bX_1^2Y_1Y_2 + cX_1^2Y_2^2 + dX_1X_2Y_1^2 + eX_1X_2Y_1Y_2 + fX_1X_2Y_2^2 + gX_2^2Y_1^2 + hX_2^2Y_1Y_2 + iX_2^2Y_2^2 = 0$$

4 Mirror symmetry as duality of special Lagrangian torus fibrations

In 1996 Strominger, Yau and Zaslow [16] proposed a geometric construction of mirror manifold via special Lagrangian torus fibration. They conjecture that a Calabi-Yau 3-fold should admit a special Lagrangian torus fibration, and that the mirror manifold can be obtained by dualizing the fibers.

In the following Sections we mix ideas from both Batyrev and SYZ interpretations of mirror symmetry. More precisely, we describe a dual pair of torus fibrations of mirror $K3$ hypersurfaces in toric varieties.

5 The combinatorial model of Haase and Zharkov

In this Section we introduce a purely combinatorial model for an integral affine structure on the complement of 24 points on a sphere, this induces a topological torus fibration of a $K3$ surface that will be described in Section 6. All the ideas and constructions that we use in the rest of the paper are basically taken from the paper [9] by Haase and Zharkov.

We start with a dual pair of d-dimensional reflexive polytopes Δ and $\Delta^\vee$ as before. Let $\lambda \in \mathbb{Z}^{\Delta\cap(\mathbb{Z}^d)^*}$, and $\nu \in \mathbb{Z}^{\Delta^\vee\cap\mathbb{Z}^d}$ be two sufficiently generic vectors that induce central coherent triangulations of Δ and $\Delta^\vee$. These triangulations restrict to triangulations S and T on the boundaries $\partial\Delta$ and $\partial\Delta^\vee$, and induce fans $\mathfrak{F}$ and $\mathfrak{F}^\vee$ given by the cones spanned by the faces of S respectively T. We define polytopes

$$\Delta_\nu = \{m \in (\mathbb{R}^d)^* : \langle m, n\rangle \leq \nu(0) - \nu(n) \text{ for all } n \in \Delta^\vee \cap \mathbb{Z}^d\}$$

$$\Delta^\vee_\lambda = \{n \in \mathbb{R}^d : \langle m, n\rangle \leq \lambda(0) - \lambda(m) \text{ for all } m \in \Delta \cap (\mathbb{Z}^d)^*\}$$

whose normal fans are given by $\mathfrak{F}^\vee$ respectively $\mathfrak{F}$.

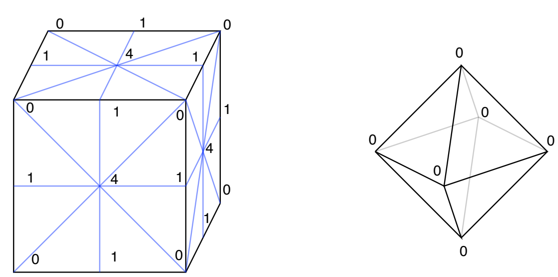


Figure 3: Δ =conv$(\pm e_1 \pm e_2 \pm e_3)$, $\Delta^\vee$ =conv$(\pm e_1, \pm e_2, \pm e_3)$
The values of λ and ν are marked on the vertices, $\lambda(0) = 16$, $\nu(0) = 1$

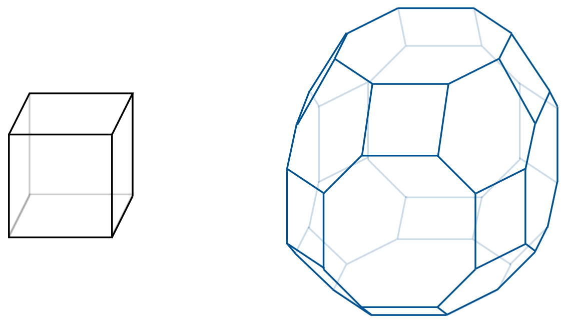


Figure 4: Δ_ν $\Delta_\lambda^\vee$

5.1 The base and the discriminant locus

The base of our torus fibration is going to be a subcomplex Σ of the $(d-1)$-dimensional complex:

$$|\Sigma| = \{(m,n) \in \Delta \times \Delta^\vee : \langle m,n \rangle = 1\}$$

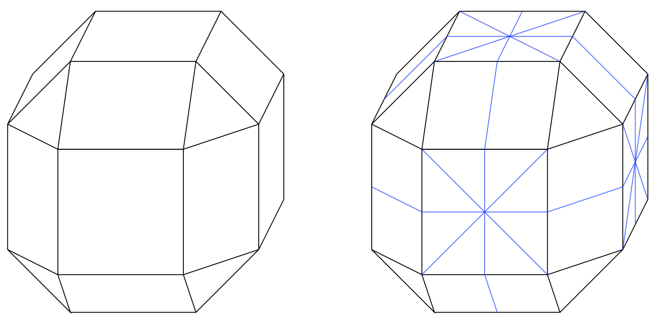


Figure 5: The complex $|\Sigma|$ and the subdivision $S \times T$ restricted to $|\Sigma|$

Figure 5 shows the complex $|\Sigma|$ for the polytopes Δ and $\Delta^\vee$ above. Notice that in this case, $|\Sigma|$ lives in the 6-dimensional euclidian space $\mathbb{R}^3 \times \mathbb{R}^3$, nevertheless one can draw a picture of it using just three dimensions. More over, it is proven in [9] that $|\Sigma|$ is topologically a $(d-1)$-sphere, we will come back to this in Section 5.4. For now, notice that the faces of $|\Sigma|$ are of the form $F \times F^\vee$, for F and $F^\vee$ dual faces of Δ and $\Delta^\vee$. For instance, each of the 8 vertices of the cube in our example is dual to one of the triangles on the boundary of the octahedron, each edge is dual to an edge and each square face is dual to a vertex. The 2-dimensional faces of $|\Sigma|$ are given by 8 faces of the form (vertex, triangle), 12 of the form (edge, edge) and 6 of the form (square, vertex). We define Σ and the singular locus D as follows:

For a poset $\mathcal{P}$, the poset/simplicial complex of chains in $\mathcal{P}$ is denoted by $\mathrm{bsd}(\mathcal{P})$.

Definition 5.1. Σ is the restriction to $|\Sigma|$ of the product subdivision $\mathrm{bsd}(S) \times \mathrm{bsd}(T)$ of $\Delta \times \Delta^\vee$

Geometrically speaking, the boundary subdivision $\mathrm{bds}(S)$ (respectively T) is the subdivision induced by the barycenters of simplices $\sigma \in S$ (respectively T), and the vertices of Σ are pairs $(\widehat{\sigma}, \widehat{\tau})$ of barycenters of simplices $\sigma \in S$ and $\tau \in T$ such that $\langle \sigma, \tau \rangle = 1$.

Definition 5.2. The singular locus D is the full subcomplex of Σ, induced by vertices $(\widehat{\sigma}, \widehat{\tau})$, such that neither σ nor τ is 0-dimensional.

Remark: The topology of $\Sigma \backslash D$ is very simple [9, Lemma 2.2]. It is homotopy equivalent to bipartite graph Γ with vertex set $\mathrm{vert}(S) \cup \mathrm{vert}(T)$ with an edge between $v \in \mathrm{vert}(S)$ and $w \in \mathrm{vert}(T)$ if and only if $\langle v, w \rangle = 1$. In our example, $\mathrm{vert}(T)$ corresponds to vertices of the octahedron, $\mathrm{vert}(S)$ are the lattice points in the boundary of the cube, and a vertex $w \in \mathrm{vert}(T)$ is connected to all lattice points in the square face of the cube which is dual to w.

We introduce below an open covering of $\Sigma \backslash D$. Consider the two natural projections

$$p_1 : \Sigma \to \mathrm{bsd}(S) \text{ and } p_2 : \Sigma \to \mathrm{bsd}(T).$$

For a vertex $v \in \mathrm{vert}(S)$ or $w \in \mathrm{vert}(T)$, define U_v respectively V_w to be the preimages

$$U_v = p_1^{-1}(\mathrm{star}_{\mathrm{bsd}(S)}(v)) \quad \text{and} \quad V_w = p_2^{-1}(\mathrm{star}_{\mathrm{bsd}(T)}(w))$$

of open stars in the barycentric subdivisions. Here, $\mathrm{star}_{\mathrm{bsd}}(S)(v)$ denotes the union of all faces in $\mathrm{bsd}(S)$ which contain v as a vertex.

The collection $\mathcal{U} \cup \mathcal{V}$ for $\mathcal{U} = (U_v)_{v \in \mathrm{vert}(S)}$ and $\mathcal{V} = (V_w)_{w \in \mathrm{vert}(T)}$, is an open covering of $\Sigma \backslash D$. The singular locus is given by $D = \partial \mathcal{U} \cap \partial \mathcal{V}$ where $\partial \mathcal{U} = \bigcup \partial U_v$ and $\partial \mathcal{V} = \bigcup \partial V_w$.

5.2 Integral affine structure and monodromy

We define an integral affine structure on $\Sigma \backslash D$, using the covering $\mathcal{U} \cup \mathcal{V}$. That is, a coordinate covering with transition maps in $SL(n, \mathbb{Z}) \ltimes \mathbb{R}^n$ on the non-empty overlaps, such that the usual cocycle condition is satisfied. Notice that two members U_v and V_w of our covering intersect if and only if $\langle v, w \rangle = 1$, the members of $\mathcal{U}$ are disjoint to each other as well as the members of $\mathcal{V}$.

Definition 5.3. For a point $q \in U_v$ we identify the tangent space $T_q(\Sigma \backslash D)$ and the lattice $T_q^{\mathbb{Z}}$ in it with the following codimension 1 subspace and sublattice of the pair $(\mathbb{R}^d, \mathbb{Z}^d)$:

$$T_q = \mathbb{R}^d_v = \{n \in \mathbb{R}^d : \langle v, n \rangle = 0\}, \qquad T_q^{\mathbb{Z}} = \mathbb{Z}^d_v = \{n \in \mathbb{Z}^d : \langle v, n \rangle = 0\}$$

For a point $q \in V_w$ we identify the tangent space $T_q(\Sigma \backslash D)$ and the lattice in it with the $(d-1)$-dimensional quotients

$$T_q = \mathbb{R}^d / w, \qquad T_q^{\mathbb{Z}} = \mathbb{Z}^d / w$$

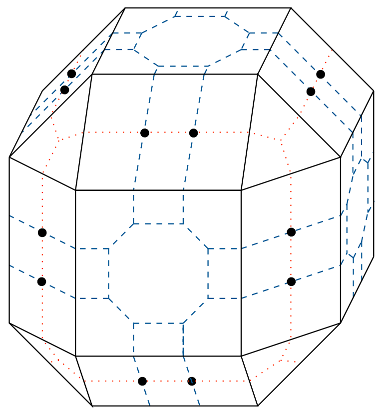


Figure 6: The doted lines are $\partial\mathcal{U}$, and the dashed lines are $\partial\mathcal{V}$. Their intersection D consists of 24 points.

On the overlap $U_v \cap V_w$, we define the transition map $f_{vw} : \mathbb{R}^d_v \to \mathbb{R}^d/w$ to be the restriction to the subspace $\mathbb{R}^d_v$ of the natural proection $\mathbb{R}^d \to \mathbb{R}^d/w$.

These transition maps respect the integral structure: $f_{vw} \in \mathrm{Hom}(\mathbb{Z}^d_v, \mathbb{Z}^d/w)$, and the condition $\langle v, w\rangle = 1$ ensures that f_{vw} is an isomorphism. The cocycle condition for the graph-type covering is trivial.

Monodromy: The monodromy around a singularity is completely determined by monodromy around *simple* loops in the graph Γ: they consist of 4 edges: $(v_0, w_0), (w_0, v_1), (v_1, w_1), (w_1, v_0)$ for some pair of edges $\{v_0, v_1\} \in S$, $\{w_0, w_1\} \in T$. In our example, w_0, w_1 are any vertices of the octahedron, and v_0, v_1 are the middle point and a vertex of the dual edge of $\{w_0, w_1\}$ in the cube. For instance, if $w_0 = (1, 0, 0), w_1 = (0, 1, 0)$ and $v_0 = (1, 1, 0), v_1 = (1, 1, 1)$, we can choose $\{e_1, e_2\} = \{(-1, 1, 0), (0, 0, 1)\}$ as a basis of T_{v_0}. The monodromy transformation $T(v_0w_0v_1w_0) : T_{v_0} \to T_{v_0}$ along the loop $(v_0w_0v_1w_0)$ is characterized by $T(v_0w_0v_1w_0)(e_1) = e_1$ and $T(v_0w_0v_1w_0)(e_2) = e_1 + e_2$. Hence, the monodromy along a simple loop around a singular point is given by

$$\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$$

5.3 The torus fibration

The torus fibration over $Y = \Sigma \backslash D$ is constructed as follows. We define the tori:

$$\mathbb{T} := \mathbb{R}^d/\mathbb{Z}^d, \qquad \mathbb{T}_v := (\mathbb{R}^d_v)/(\mathbb{Z}^d_v), \qquad \mathbb{T}/w := (\mathbb{R}^d/w)/(\mathbb{Z}^d/w).$$

For $\langle v, w\rangle$, the transition isomorphism $f_{vw} \in \mathrm{Hom}(\mathbb{Z}^d_v, \mathbb{Z}^d/w)$ induces an isomorphism of the tori, which we will denote by the same symbol

$$f_{vw} : \mathbb{T}_v \to \mathbb{T}/w$$

We form the relative quotient $W \to Y$ with fibers $W_q = T_qY/T_q^{\mathbb{Z}}Y$. Thus, the fibers are $W_q = \mathbb{T}_v$ when $q \in U_v$, and $W_q = \mathbb{T}/w$ when $q \in V_w$, with the canonical identifications $f_{vw} : \mathbb{T}_v \to \mathbb{T}/w$ for $q \in U_v \cap V_w$.

The duality between reflexive polytopes leaves invariant Σ and the discriminant locus D, if we interchange Δ and $\Delta^\vee$, and consider the integral affine structure to be dual to the original one, we obtain a dual torus fibration over the same base whose fibers are dual to the original ones.

Let $N(D) \subset \Sigma$ be aregular neighborhood of the discriminant locus. Let $W^\epsilon \to \Sigma \backslash N(D)$ denote the torus fibration associated to the original integral affine structure restricted to the complement of $N(D)$ in Σ. In Section 6 we will see that the torus fibration W^ϵ on $\Sigma \backslash N(D)$ embeds differentially into H_s for sufficiently large s. The dual torus fibration by symmetry embeds into the mirror hypersurface.

5.4 Isomorphism between $\partial\Delta_\lambda^\vee$ and $|\Sigma|$

Haase and Zharkov in [9] developed a nice generalization of boundary subdivisions, and used it to give a proof of the sphericity of $|\Sigma|$. More precisely, they describe a coherent subdivision of $\partial\Delta_\lambda^\vee$ (alternatively $\partial\Delta_\nu$), which is isomorphic to the restriction to $|\Sigma|$ of the product subdivision $\mathrm{bsd}(S) \times T$. This Section is devoted to explain such isomorphism in the particular case of the example in Figures 3 and 4. We will do so in two steps:

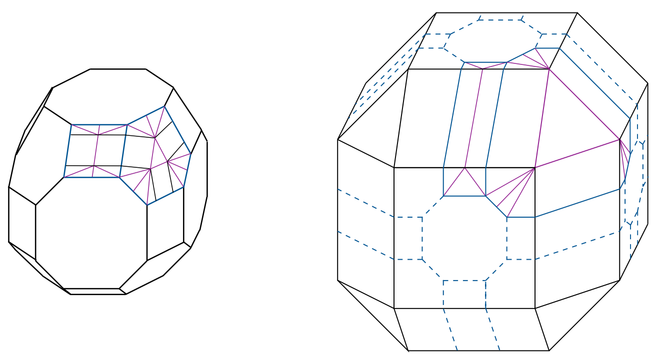


Figure 7: Subdivision of $\Delta_\lambda^\vee$ combinatorially isomorphic to the restriction to $|\Sigma|$ of the product subdivision $\mathrm{bsd}(S) \times T$.

Step 1: This step is concerned to a bijective correspondence between the 2-dimensional faces of the polytope $\Delta_\lambda^\vee$ and the open neighborhoods $\{U_v\}_{v \in vert(S)}$. There is natural duality between vertices of S and maximal-dimensional faces of $\partial\Delta_\lambda^\vee$. Every vertex $v \in \mathrm{vert}(S)$ determines a defining inequality of $\Delta_\lambda^\vee$, the dual face $\sigma_v \subset \partial\Delta_\lambda^\vee$ of v is the one that represents the inequality of v. This duality turns into a duality between maximal-dimensional faces of $\partial\Delta_\lambda^\vee$ and the open neighborhoods $\{U_v\}_{v \in \mathrm{vert}(S)}$ as we wanted. In our example, the six octagon faces of $\Delta_\lambda^\vee$ correspond to the six octagons $\{U_v\}$ for v the mid points of the square faces of Δ, the twelve rectangles of $\Delta_\lambda^\vee$ correspond to twelve open sets $\{U_v\}$ with v the mid points of the edges of the cube Δ, and the eight hexagons of $\Delta_\lambda^\vee$ correspond to the eight open sets $\{U_v\}$ with v a vertex of Δ.

Step 2: In this step, we construct a subdivision of σ_v combinatorially isomorphic to the subdivision bsd$(S) \times T$ restricted to $\overline{U_v}$.

There are three types of faces σ_v: Octagons, rectangles and hexagons. Figure 8 shows the subdivisions for each one of this types.

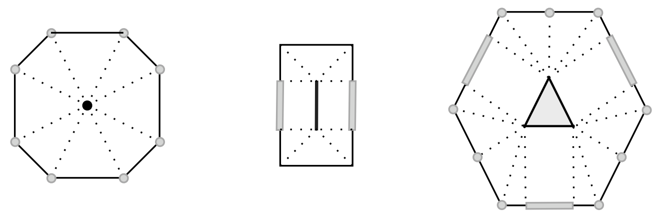


Figure 8: Subdivision of σ_v combinatorially isomorphic to the subdivision $\mathrm{bsd}(S) \times T$ restricted to $\overline{U_v}$, for v a vertex of the cube Δ.

6 Torus fibration of a $K3$ surface

We consider a family of affine hypersurfaces given by

$$H_s^{\mathrm{aff}} := \{x \in (\mathbb{C}\backslash\{0\})^d : \sum_{m \in \Delta \cap (\mathbb{Z}^d)^*} a_m s^{\lambda(m)} x^m = 0\}$$

The projective toric variety X_{Δ_ν} associated to the polytope Δ_ν is equivalent to the toric variety $\mathbb{X}_{\mathfrak{F}^\vee}$ for the normal fan $\mathfrak{F}^\vee$ of Δ_ν which is given by the cones spanned by the faces of T. It contains $U_{\mathbb{Z}[S_{\{0\}^\vee}]}(\mathbb{C}) = (\mathbb{C}\backslash\{0\})^d$ as a dense open subset, and we can think of H_s^{aff} as a hypersurface on this chart. Let H_s be the closure of H_s^{aff} in X_{Δ_ν}.

According to [6, Ch. 10], the hypersurfaces given by these particular equations are all diffeomorphic to each other (in the orbifold sense). For that reason, we can set the coefficients $a_m = 1$ without loss of generality.

Our $K3$ surface is the hypersurface H_s for our imput data in example of Figure 3. In this section we construct a torus fibration $H_s^{\mathrm{sm}} \to \Sigma\backslash N(D)$ on a "Smooth" part of H_s, for large enough s, and show that it is the same as our model fibration $W^\epsilon \to \Sigma\backslash N(D)$.

6.1 Amoebas of hypersurfaces

Let $\mathrm{Log}_s : (\mathbb{C}\backslash\{0\})^d \to \mathbb{R}^d$ be the logarithmic map with base $|s| \neq 1$:

$$\mathrm{Log}_s(x) := \frac{\log(|x|)}{\log|s|} = \left\{\frac{\log|x_1|}{\log|s|}, \ldots, \frac{\log|x_d|}{\log|s|}\right\}$$

The preimage of a point $n = (n_1, \dots, n_d) \in \mathbb{R}^d$ under the Log_s map is the torus:

$$\mathrm{Lot}_s^{-1}(n) = \{x \in (\mathbb{C}\backslash\{0\})^d : x_j = |s|^{n_j} e^{i\theta_j} \text{ and } 0 \leq \theta_j \leq 2\pi\}$$

Definition 6.1. ([6, Ch. 6]) The *Amoeba* associated to the family of affine hypersurfaces H_s^{aff} is the image of the log map:

$$\mathcal{A}_s^\lambda := \mathrm{Log}_s(H_s^{\mathrm{aff}})$$

The geometry of amoebas of affine hypersurfaces is a well developed subject that originated in the work of Gelfand, Kapranov and Zelevinsky [6]. The limiting behavior of amoebas as $s \to \infty$ can be described in terms of the Legendre transform $L_\lambda : \mathbb{R}^d \to \mathbb{R}$ of the vector λ:

$$L_\lambda(n) = \max_{m \in \Delta \cap (\mathbb{Z}^d)^*} \{\langle m, n\rangle + \lambda(m)\}$$

$L_\lambda(n)$ is a piecewise linear convex function. The *non-Archimedean amoeba* $\mathcal{A}_\infty^\lambda \subset \mathbb{R}^d$ is defined as the corner locus of $L_\lambda(n)$ (the set of points where $L_\lambda(n)$ is not smooth). $\mathcal{A}_\infty^\lambda$ induces a polyhedral complex subdivision of $\mathbb{R}^d$, whose face lattice is in a reverse order bijective correspondence with the face lattice of the triangulation $\{0\} * S$. The bounded maximal cell of this complex is precisely the polytope $\Delta_\lambda^\vee$.

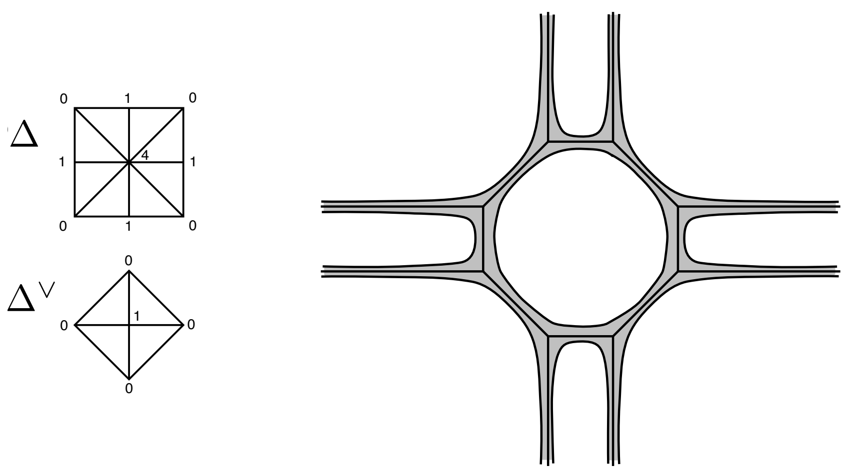


Figure 9: The affine amoeba $\mathcal{A}_s^\lambda$ with the corresponding spine $\mathcal{A}_\infty^\lambda$ for the family $H_s^{\mathrm{aff}} = [s^4 + sx + sy + sx^{-1} + sy^{-1} + xy + x^{-1}y + x^{-1}y^{-1} + xy^{-1} = 0]$

6.2 The foliation

In this section we will exhibit a vector field $\mathfrak{X}$ on $\mathbb{R}^3 \backslash \Delta_{\lambda^\epsilon}^\vee$ for our imput data in Figure 3. The desired foliation $\mathcal{F}$ is the one induced by $\mathfrak{X}$.

Denote by $\lambda^\epsilon \in \mathbb{R}^{\Delta\cap(\mathbb{Z}^d)^*}$ the vector given by $\lambda^\epsilon(0) = \lambda(0)$, and $\lambda^\epsilon(v) = \lambda(v) + \epsilon$ for $v \in \mathrm{vert}(S)$. Suppose that $\epsilon > 0$ is small enough to ensure that λ and λ^ϵ induce the same triangulation. Then $\partial\Delta^\vee_{\lambda^\epsilon} \subset \partial\Delta^\vee_\lambda$ are combinatorially equivalent.

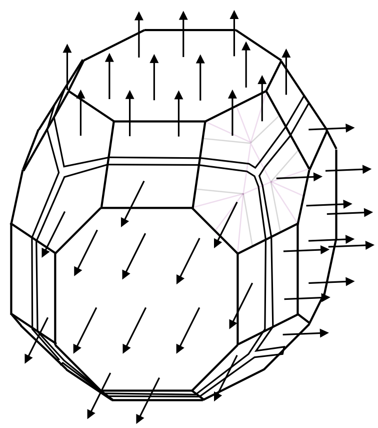


Figure 10: The vector field $\mathfrak{X}$ on $\mathbb{R}^3\backslash\Delta^\vee_{\lambda^\epsilon}$.

Haase and Zharkov [9, Section 3.3] introduced a vector field in a more general way. Given a neighborhood $N_2(\partial\mathcal{V})$ of $\partial\mathcal{V} \subset \partial\Delta^\vee_\lambda \cong \Sigma$, it satisfies that $\mathfrak{X}(q) = w$ for every $q \in V_w\backslash N_2(\partial\mathcal{V})$, and it smoothly changes from one open set V_w to other. Their construction applied to our example in Figure 3 is easy to describe and satisfies the following two main properties:

1. If $n \in \mathcal{F}_q$ with $q \in U^\epsilon_v$, then $\langle v, \mathfrak{X}(n)\rangle = 1$.
2. if $n \in V_w\backslash N_2(\partial\mathcal{V})$, the flow line $\mathcal{F}_n$ through n is a straight line parallel to w outside $\Delta^\vee_{\lambda^\epsilon}$

6.3 The torus fibration

Using the foliation $\mathcal{F}$ we define a decomposition of the hypersurface $H_s = H^{\mathrm{sm}}_s \sqcup H^{\mathrm{sing}}_s$, construct a torus fibration $H^{\mathrm{sm}}_s \to \Sigma\backslash N(D)$ and show that it is isomorphic to the fibration $W^\epsilon \to \Sigma\backslash N(D)$.

For any closed subset $J \subset \Sigma$ we will denote by $X_s(J) \subset X_{\Delta_\nu}$ the closure of $\mathrm{Log}^{-1}_s(\cup_{q\in J}\mathcal{F}_q)$ in X_{Δ_ν}.

Definition 6.2. Let $N(D)$ be a regular neighborhood of D in Σ. Then the *smooth* part of the hypersurface is $H^{sm}_s := H_s \cap X_s(\Sigma\backslash N(D))$, and the rest $H^{\mathrm{sing}}_s := H_s\backslash H^{\mathrm{sm}}_s$ is *singular.*

Since $D = \partial\mathcal{U} \cap \partial\mathcal{V}$, there exists regular neighborhoods $N_1(\partial\mathcal{U})$ of $\partial\mathcal{U}$ and $N_2(\partial\mathcal{V})$ of $\partial\mathcal{V}$ in Σ, such that $N(D) \supset N_1(\partial\mathcal{U}) \cap N_2(\partial\mathcal{V})$. Thus, $\Sigma\backslash N(D)$ can be covered by the union of the closed sets:

$$\mathcal{U}^\epsilon = \{U^\epsilon_v\} = \{U_v\backslash N_1(\partial\mathcal{U})\} \text{ and } \mathcal{V}^\delta = \{V^\delta_w\} = \{V_w\backslash N_2(\partial\mathcal{V})\}$$

The amoebas $\mathcal{A}_s^\lambda$, for a large enough s, all lie in $\mathbb{R}^d\backslash\Delta_{\lambda^\epsilon}^\vee$. This means that $\mathcal{F}$ defines a projection $\mathcal{A}_s^\lambda \to \Sigma$ and, by composition with Log_s, the projection $H_s^{\mathrm{aff}} \to \Sigma$.

The set U_v^ϵ lie in the interior of a two dimensional face of $\Delta_\lambda^\vee$. Since the unbounded ends of flow lines $\mathcal{F}_q$, for $q \in U_v^\epsilon$, do not intersect the amoeba for a large enough s, their closures do not contain any extra points of the hypersurface:

$$H_s^{\mathrm{aff}} \cap X_s(U_v^\epsilon) = H_s \cap X_s(U_v^\epsilon)$$

Thus the map $H_s \cap X_s(U_v^\epsilon) \to U_v^\epsilon$ is well defined. On the other hand, for two distinct points q_1, q_2 in V_w^δ the corresponding leaves are straight lines and the sets $X_s(q_1)$ and $X_s(q_2)$ are disjoint. Hence, the map $H_s \cap X_s(V_w^\delta) \to V_w^\delta$ is well defined. Combined together we have (for large enough s) the well defined projection

$$f_s : H_s^{\mathrm{sm}} \to \Sigma\backslash N(D), \qquad f_s(x) := q \Leftrightarrow x \in X_s(q).$$

Theorem 6.3. *There exits a real number s_0, such that for any s with $|s| \geq s_0$,*

$$f_s : H_s^{sm} \to \Sigma\backslash N(D)$$

is a torus fibration isomorphic to $W^\epsilon \to \Sigma\backslash N(D)$.

Proof. For $v \in \mathrm{vert}(S)$, we consider the $(\Delta \cap (\mathbb{Z}^d)^* - 2)$-parameter family of hypersurfaces $H_s^v(a)$ in $X_s(U_v^\epsilon)$:

$$s^{\lambda(0)} + s^{\lambda(v)} + \sum_{m \neq \{0\}, v} a_m s^{\lambda(m)} x^m = 0, \qquad 0 \leq a_m \leq 1.$$

Lemma 6.4. *[9, Lemma 3.8] There exist a real number s_0 such that whenever $|s| \geq s_0$, all $H_s^v(a)$ are smooth and transversal to $X_s(q)$ for every $q \in U_v^\epsilon$.*

As a consequence, we have that $H_s^v(a)$ restricted to $X_s(q)$ is diffeomorphic to $H_s^v(0)$ restricted to $X_s(q)$, which is characterized by the set of values x that satisfy:

$$s^{\lambda(0)} + s^{\lambda(v)} x^v = 0, \qquad \text{with } x_j = |s|^{n_j} e^{2\pi i \theta_j}$$

for some $n \in \mathcal{F}_q$ and arbitrary θ. In order to have a solution to this equation, we need the absolute values $|s^{\lambda(0)}| = |s|^{\lambda(0)}$ and $|s^{\lambda(v)} x^v| = |s|^{\lambda(v) + \langle v, n\rangle}$ to be equal to each other. Since $\langle v, \mathfrak{X}(n)\rangle = 1$ for all $n \in \mathcal{F}_q$, then there is only one point of $\mathcal{F}_q$ that satisfies this condition, this point is precisely $n = q$. On the other hand, the arguments of the two terms in the equation above should be opposite so that they cancel to each other. Therefore, if $2\pi\theta_s$ denotes the argument of the complex number s, then $x \in H_s^v(0) \cap X_s(q)$ if and only if $x_j = |s|^{q_j} e^{2\pi i \theta_j}$ with

$$\langle v, 2\pi\theta\rangle + (\lambda(v) - \lambda(0)) 2\pi\theta_s - \pi \equiv 0 \bmod 2\pi,$$

which is equivalent to

$$\langle v, \theta\rangle + (\lambda(v) - \lambda(0))\theta_s - 1/2 \equiv 0 \bmod \mathbb{Z}.$$

If θ_0 is a particular solution to this equation, then any other solution is of the form $\theta - \theta_0$ with $\theta \in \mathbb{R}^d$ satisfying the relation

$$\langle v, \theta \rangle \equiv 0 \bmod \mathbb{Z}.$$

Thus, for any $q \in U_v^\epsilon$ the fiber $F_q := H_s \cap X_s(q)$ is diffeomorphic to hypersurface $H_s^v(0)$ restricted to $X_s(q)$, which is a torus that can be naturally identified with the torus $\mathbb{T}_v = (\mathbb{R}_v^d)/(\mathbb{Z}_v^d)$: Recall from Section 5.2 that

$$\mathbb{R}_v^d = \{n \in \mathbb{R}^d : \langle v, n \rangle = 0\}, \qquad \mathbb{Z}_v^d = \{n \in \mathbb{Z}^d : \langle v, n \rangle = 0\}$$

For $n \in \mathbb{R}_v^d$, the identification $\theta = n$ gives rise to an identification of the two tori. In order to argue the last statement we need to check two things:

1. Every θ such that $\langle v, \theta \rangle \equiv 0 \bmod \mathbb{Z}$, has a representative $\widetilde{\theta}$ with $\langle v, \widetilde{\theta} \rangle = 0$. If $\langle v, \theta \rangle = k$, just take $\widetilde{\theta} = \theta - kw$ for some lattice point w with $\langle v, w \rangle = 1$.
2. Let $\theta_1, \theta_2 \in \mathbb{R}_v^d$. Then θ_1, θ_2 represent the same element of $H_s^v(0) \cap X_s(q)$ if and only if $\theta_1 - \theta_2$ belongs to the lattice $\mathbb{Z}_v^d$, which is trivial.

Similarly, for $w \in$vert(T) we consider the $(\Delta \cap (\mathbb{Z}^d)^* - w^\perp - 1)$-parameter family of hypersurfaces:

$$s^{\lambda(0)} + \sum_{m \in G_w} s^{\lambda(m)} x^m + \sum_{m \notin G_w \cup \{0\}} a_m s^{\lambda(m)} x^m = 0, \qquad 0 \leq a_m \leq 1,$$

where G_m is the set of lattice points of Δ whose inner product with w is equal to 1. We denote by $H_s^w(a)$ its closure in $X_s(V_w^\delta)$

Lemma 6.5. *[9, Lemma 3.9.] There exists a real number s_0 such that whenever $|s| \geq s_0$, all $H_s^w(a)$ are smooth and transversal to $X_s(q)$ for every $q \in V_w^\delta$.*

As before, this implies that for any $q \in V_w^\delta$ the fiber $F_q := H_s \cap X_s(q)$ is diffeomorphic to $F_q^w := H_s^w(0) \cap X_s(q)$.

But F_q^w can be identified with the torus $\mathbb{T}/w$ as follows. We choose a basis $\{e_i\}$ of $(\mathbb{Z}^d)^*$ with

$$\langle e_1, w \rangle = -1 \text{ and } \langle e_i, w \rangle = 0, \ i = 2, \ldots, d.$$

Then we can think of $y_i = x^{e_i} \neq 0$ as new coordinates in the toric variety $X_{\mathfrak{F}^\vee}$ that can be extended by allowing zero values for y_1 (when the flow line $\mathcal{F}_q$ goes to infinity in direction w). If $m \in G_w$ then $e_1 + m$ is orthogonal to w, and so $y_1 x^m = x^{e_1+m}$ does not depend on the variable y_1. Thus, multiplying the defining equation of the hypersurface $H_s^w(0)$ by y_1 we get:

$$s^{\lambda(0)} y_1 + P(y_2, \ldots, y_d) = 0$$

where $P(y_2, \ldots, y_d)$ is a Laurent polynomial independent of y_1. On the other hand, the flow line $\mathcal{F}_q$ through q is a line parallel to w, then, restricting the hypersurface to the fiber $X_s(q)$ means fixing absolute values of $y_i, \ i = 2, \ldots, d$

($|y_j| = |s|^{\langle q.e_j \rangle}$). A point on the torus $\mathbb{T}/w$ determines the phases of y_i, $i = 2, \ldots, d$. Onces y_i, $i = 2, \ldots, d$, are fixed, there is a unique solution to the equation of $H_s^w(0)$.

Thus, we have proven that $f_s : H_s^{\mathrm{sm}} \to \Sigma \backslash N(D)$ is a torus fibration over two kind of covering patches whose fibers are diffeomorphic to the ones obtained in the combinatorial model. The only thing left to check is that it has the correct monodromy.

Note that all diffeomorphisms $F_q \cong F_q^v$, $q \in U_v^\epsilon$, and $F_q \cong F_q^w$, $q \in V_w^\delta$, are deformation diffeomorphisms. Hence, the transitions mapas between $\mathbb{T}_v$ and $\mathbb{T}/w$, for $q \in U_v \cap V_w$, are homotopic to the map $f_{vw} : \mathbb{T}_v \to \mathbb{T}/w$. But monodromy is a homotopy invariant, hence, it has to be equal to the one given by the maps f_{vw}. This completes the proof.

□

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