THE SYSTEM OF PERMUTATIONS OF A FINE MIXED SUBDIVISION OF A SIMPLEX

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CESAR AUGUSTO CEBALLOS LOPEZ

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Advisor
Professor Federico Ardila
San Francisco State University

Departamento de Matemáticas
Universidad de los Andes
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CERTIFICATION OF APPROVAL

I certify that I have read *The system of permutations of a fine mixed subdivision of a simplex* by Cesar Augusto Ceballos Lopez and that in my opinion this work meets the criteria for approving a thesis submitted in partial fulllment of the requirements for the degree: Magister in Mathematics at Universidad de los Andes.

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Federico Ardila
Professor of Mathematics
San Francisco State University

____

Matthias Beck
Professor of Mathematics
San Francisco State University

____

Jean Carlos Cortissoz
Professor of Mathematics
Universidad de los Andes

THE SYSTEM OF PERMUTATIONS OF A FINE MIXED SUBDIVISION OF A SIMPLEX

Cesar Augusto Ceballos Lopez
Universidad de los Andes
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Abstract

A coloring of a fine mixed subdivision of a simplex gives rise to an acyclic system of permutations on the edges of the simplex. In particular, we prove that a system on the edges of an equilateral triangle is achievable through a coloring if and only if it is acyclic, and provide evidence to conjecture that the same result is true for simplices in any dimension. Our work is related to the results on triangulations of products of simplices, Schubert calculus, tropical hyperplane arrangements and tropical oriented matroids, obtained by Santos, Ardila-Billey, Develin-Sturmfels and Ardila-Develin, among others. It also settles a special case of a conjecture of Ardila-Billey about the positions of the simplices in a fine mixed subdivision and provides precise explanation about the behavior of a generic pseudo tropical hyperplane at infinity.

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Contents

1 Introduction

The study of colorings of fine mixed subdivisions of the delated simplex $n\Delta_{d-1}$ gives us geometrical and combinatorial tools that can be used in many topics in mathematics. They are pseudo tropical hyperplane arrangements in tropical geometry, and the study of such arrangements at infinity can be regarded as the system of colors on the edges of the simplex. In this paper we explore the rich combinatorial structure of these systems and find that they have similar properties to the ones that appear in matroid theory [5]. A well developed theory of colorings could simplify the understanding and developing of the theory of tropical oriented matroids; for instance, it could prove conjectures about the representability of an oriented matroid as a tropical pseudo hyperplane arrangement.

The paper is organized as follows. In Section 2, we introduce the ideas of lozenge tilings of a triangle, fine mixed subdivisions of a simplex, and present Conjecture 2.2 of Ardila-Billey in [1]. In Section 3, we define the system of colors associated to a lozenge tiling of a triangle, and prove one of the main results of the thesis: Theorem 3.1. It basically says that a system of permutations on the edges of a triangle is achievable through a coloring if and only if it is acyclic. Then, we define the notion of colorings of fine mixed subdivisions of a simplex, prove that they give rise to acyclic systems of colors on the edges of the simplex, Theorem 3.13, and conjecture that every acyclic system of permutations on the edges of a simplex is achievable through a coloring, Conjecture 3.14. In section 4, we define dual systems and introduce dual subdivisions using the Cayley trick in [2]. In Section 5, we define operations of deletion and contraction for fine mixed subdivisions and acyclic systems, and use them to give a proof of Proposition 5.10, which says that the dual system of colors of a given subdivision is equal to the system of the dual subdivision. Finally, in Section 6 we use the theory of acyclic systems in order to give an other proof of Theorem 6.3 first proved by Ardila and Billey in [1]. We also prove a special case of Conjecture 3.14 that says that every acyclic system on the edges of a simplex of size 3 in every dimension is achievable through a coloring, and Theorem 6.5 that is the special case of Conjecture 2.2 of Ardila-Billey for simplices of size 3 in every dimension.

2 Lozenge tilings and fine mixed subdivisions of a simplex

The combinatorial and geometrical properties of subdivisions of a simplex are interesting by themselves and have been studied from different points of view by Santos [2], Ardila-Billey [1], Develin-Sturmfels [4], Ardila-Develin [3], among others. A recent source of interest in these objects comes from tropical geometry: results by Develin-Sturmfels [4] and Santos [2] show that arrangements of n tropical hyperplanes in the tropical space of dimension $d-1$ are in bijective correspondence with regular fine mixed subdivisions of the simplex $n\Delta_{d-1}$. Via this relation, we study the behavior of a *(pseudo)* tropical hyperplane arrangement at infinity by studying the combinatorics of the corresponding subdivision. Before thinking about fine mixed subdivisions of a simplex, let us start by studying the slight easier problem of understanding lozenge tilings of an equilateral triangle.

2.1 Lozenge tilings of an equilateral triangle

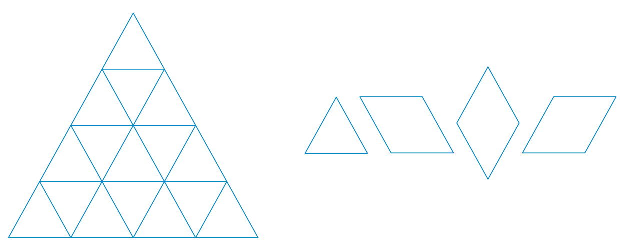


Figure 1: $T(4)$ and the four tiles allowed

Let $T(n)$ be an equilateral triangle with length n. Suppose we wanted to tile $T(n)$ using unit rhombi with angles equal to 60° and 120°. It is easy to see that this task is impossible, for the following reason. Cut $T(n)$ into n^2 unit equilateral triangles, as illustrated in Figure 1; the number of upward triangles is equal to n plus the number of downward triangles, since a rhombus always covers one upward triangle and one downward triangle, we cannot use them to tile $T(n)$. Suppose that we can also use n upward triangles. Now, it may or may not be possible to tile the remaining shape with rhombi. Figure 2 shows a Lozenge tiling of $T(4)$, which is defined as a tiling of $T(n)$ using n upward triangles and rhombi.

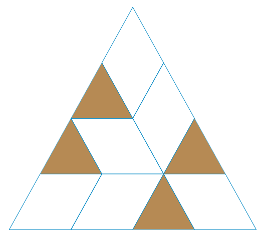


Figure 2: A Lozenge tiling of $T(4)$

The main question we address in this section is the following: Given the n positions of the little upward triangles in $T(n)$, is there a simple criterion to determine whether there exists a lozenge tiling containing the triangles at those positions?. Theorem 6.2. in [1] by Ardila-Billey, gives us an answer to this question.

Theorem 2.1 (Ardila-Billey, Theorem 6.2. in [1])**.** *Let S be a set of n little upward triangles in an equilateral triangle $T(n)$. The triangle $T(n)$ with holes at S can be tiled with unit rhombi if and only if every sub-triangle $T(k)$ in $T(n)$ contains at most k holes of S for all $k \leq n$.*

2.2 Fine mixed subdivisions of a simplex

Now we want to generalize lozenge tilings to high dimensional subdivisions of a simplex. A good high-dimensional analogue of lozenge tilings of the triangle $n\Delta_2$ are *fine mixed subdivisions* of the simplex $n\Delta_{d-1}$; we briefly recall their definition. The *Minkowski sum* of polytopes $P_1, ..., P_k$ in $\mathbb{R}^m$ is:

$$P = P_1 + ... + P_k := \{p_1 + ... + p_k | p_1 \in P_1, ..., p_k \in P_k\}.$$

We are interested in a $(d-1)$-dimensional simplex of size n which is the Minkowski sum $n\Delta_{d-1}$ of n simplices

$$\Delta_{d-1} = \{x = (x_1, \ldots, x_d) \in \mathbb{R}^d : x_i \geq 0 \text{ and } x_1 + \ldots + x_d = 1\}.$$

A fine mixed subdivision of $n\Delta_{d-1}$ is a subdivision[1] of $n\Delta_{d-1}$ into fine mixed cells, where a fine mixed cell is a Minkowski sum $B_1 +$

[1]A subdivision of a polytope P is a tiling of P with polyhedral cells whose vertices are vertices of P, such that the intersection of any two cells is a face of both of them.

$...+B_n$ such that the $B_i s$ are faces of $n\Delta_{d-1}$ which lie in independent affine subspaces, and whose dimensions add up to $d-1$. Figure 3 shows all the possible cells in dimension 3, and Figure 4 shows an example of a fine mixed subdivision of $3\Delta_{4-1}$.

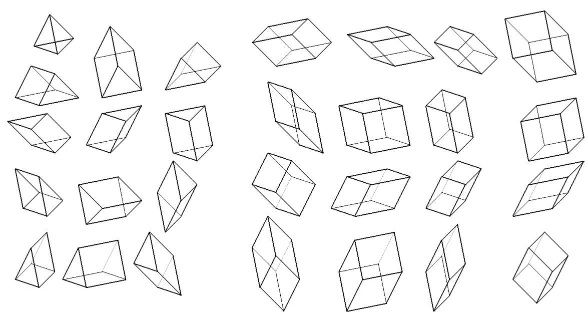
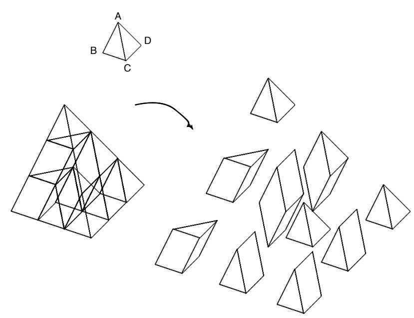


Figure 3: The cells in dimension 3



$ABCD$	+	A	+	A
BCD	+	A	+	AB
CD	+	AC	+	AB
D	+	ACD	+	AB
BCD	+	AB	+	B
CD	+	ABC	+	B
CD	+	C	+	ABC
D	+	CD	+	ABC
D	+	D	+	$ABCD$
D	+	$ABCD$	+	B

Figure 4: A fine mixed subdivision of $3\Delta_{4-1}$ and its corresponding Minkowski sums

Surprisingly, for high dimensional simplices, the number of little simplices in every fine mixed subdivision is equal to the size of the initial simplex. Ardila and Billey in [1] conjecture that

Conjecture 2.2 (Ardila-Billey, Conjecture 7.1. in [1])**.** *The possible positions of the simplices in a fine mixed subdivision of $n\Delta_{d-1}$ are precisely those for which every sub-simplex of size k contains at most k of them.*

3 Colorings of fine mixed subdivisions of simplices

In this section we define the coloring of a fine mixed subdivision. Such colorings were introduced by Ardila-Develin in [3], represent arrangements of tropical pseudo hyperplanes and play an important role in the theory of tropical oriented matroids. We focus our attention just in one part of the coloring, namely the system of permutations of colors on the edges of the simplex. These systems satisfy interesting properties that will be used in section 6 to make progress towards Conjecture 2.2.

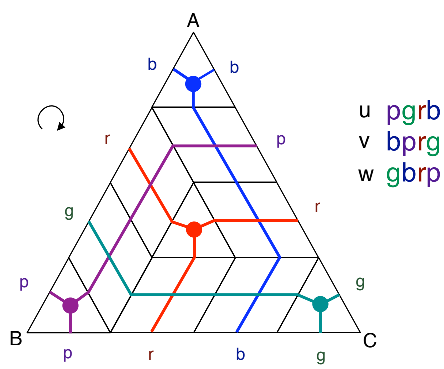


Figure 5: Coloring and system of permutations of a lozenge tiling of a triangle.

3.1 Coloring of a lozenge tiling

Given a lozenge tiling of a triangle, we assign different colors to each one of the triangles and spread them via the rhombi as shown in Figure 5; it appears a natural system of colors on the edges of the triangle defined as follows: let $u(j)$ be the color which is in position j on the directed edge BA, $v(j)$ be the color which is in position j on the edge AC, and $w(j)$ be the color which is in position j on the edge CB. The main goal of this section is to characterize all the possible systems u, v, w which come from a coloring, and prove that the positions of the triangles in a tiling which gives rise to a given system, are completely determined.

Theorem 3.1. *Let u, v, w be a system of permutations on the edges of a triangle, then*

1. *The system u, v, w comes from a coloring of a tiling if and only if there does not appear something of the form $a...b, a...b, a...b$ clockwise around the triangle. In other words, u, v, w is achievable if and only if there is not a, b such that* $u^{-1}(a) < u^{-1}(b)$, $v^{-1}(a) < v^{-1}(b)$ *and* $w^{-1}(a) < w^{-1}(b)$.

2. *Given an achievable system, the positions of the triangles of any tiling achieving it are completely determined.*

First we need some technical results in order to prove this theorem. Let G_n be the directed graph whose set of vertices is the triangular array $T_{n,3}$, where each dot not on the bottom row is connected to the two dots directly below it (see Figure 6.). We assign coordinates (M, m) to the vertices of G_n as follows: The coordinate M of a point $p = (M, m)$ increases as p gets close to the edge AC, being $M = 1$ for the farthest vertex and $M = n$ for the closest vertices. On the other hand, coordinate m tell us how close is p to the edge AB, being $m = 1$ for the closest vertices and $m = n$ for the farthest vertex. Figure 6 shows a routing and coordinates of the graph G_4.

For every tiling of an equilateral triangle T_n of size n, there is a corresponding routing of the graph G_n. Given such a routing, one can easily recover the tiling that gave rise to it: simply place one rhombus over each edge in the routing, one vertical rhombus over each isolated vertex, and one triangle over the upper vertex of each rout. It is easy to check that this is a bijection between the rhombus tilings of triangles of size n, and the routings in the graph G_n which start anywhere and end at vertices on the edge BC.

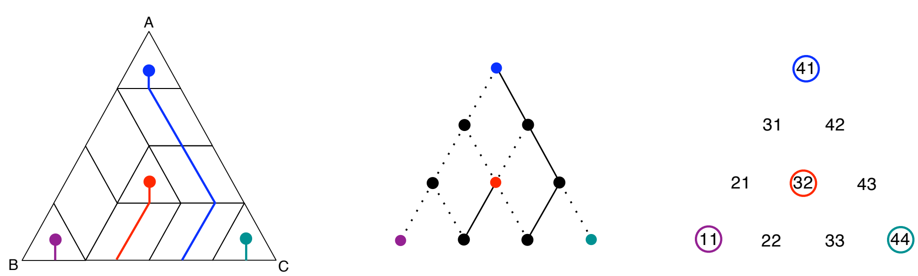


Figure 6: Tiling of T_4, Routing of G_4
and Coordinates (M, m)

If we call $1, ..., n$ the colors from left to right on the edge BC, and (M_i, m_i) the position of the triangle which contains color i, then the following two algorithms give us all the information we need to calculate the system u, v, w.

Algorithm 1:

1. Let $M_i(i) = M_i$.
2. For $k = i-1, i-2, ..., 1$, define $M_i(k)$ as follows:

$$M_i(k) = \begin{cases} M_i(k+1) - 1 & if,\ M_i(k+1) \leq M_k \\ M_i(k+1) & if,\ M_i(k+1) > M_k \end{cases}$$

For instance, in Figure 6 we have

k	1234
$M(k)$	1344
$M_4(k)$	2234
$M_3(k)$	444
$M_2(k)$	33
$M_1(k)$	1

Algorithm 2:

1. Let $m_i(i) = m_i$.
2. For $k = i+1, i+2, ..., n$, define $m_i(k)$ as follows:

$$m_i(k) = \begin{cases} m_i(k-1) + 1 & if,\ m_i(k-1) \geq m_k \\ m_i(k-1) & if,\ m_i(k-1) < m_k \end{cases}$$

For instance, in Figure 6 we have

k	1234
$m(k)$	1214
$m_1(k)$	1122
$m_2(k)$	233
$m_3(k)$	11
$m_4(k)$	4

Proposition 3.2. *$u^{-1}(i) = M_i(1)$, $v^{-1}(i) = m_i(n)$ and $w = n...21$.*

For instance, in the example above (Figure 6), 1,2,3 and 4 correspond to colors purple, red, blue and green respectively, $u = 1423$, $v = 3124$, $w = 4321$ and $M_i(1) = 1342$, $m_i(4) = 2314$.

Proof. Note that $u^{-1}(i)$ is the position of the edge BA on which the color i hits. Also note that the path of the color i going from the point (M_i, m_i) to the edge BA decrease its coordinate M in 1 every time that this path cross a horizontal rhombus, or equivalently every time that this path cross lines of the routing of G_n, which is precisely what algorithm 1 does. Then, the final position of this path, or equivalently, the final coordinate M of this path is equal to $M_i(1)$. Therefore $u^{-1}(i) = M_i(1)$. A similar argument apply for v, and $w = n...21$ just because we call $12...n$ the colors from left to right on the edge BC. □

Given a tiling of a triangle we have constructed two permutations u and w in a purely geometric way (we can always assume $w = n...1$). Now, algorithms 1 and 2 give us a way to calculate them that only depends on the coordinates (M_i, m_i).

Theorem 3.3. *Algorithms 1 and 2 determine a bijection $u^{-1}(i) = M_i(1)$, $v^{-1}(i) = m_i(n)$, between the n-tuples of coordinates (M_i, m_i) with $1 \le m_i \le i \le M_i \le n$, and all the $(n!)^2$ possible pairs of permutations u, v. More over*

$$\begin{aligned} u &= (n, ..., M_n) \circ ... \circ (2, ..., M_2) \circ (1, ..., M_1) \\ v &= (1, ..., m_1) \circ ... \circ (n-1, ..., m_{n-1}) \circ (n, ..., m_n) \end{aligned}$$

Corollary 3.4. *Given an achievable system of permutations u, v, w on the edges of a triangle, there exits an unique choice of the position of the triangles that are generating such system. However, some positions could generate several systems.*

Proof. Given a system, the positions (M_i, m_i) of the triangles are completely determined by the theorem above. □

We are ready to prove Theorem 3.1.

Proof theorem 3.1. Consider a coloring on a tiling of a triangle and its corresponding system u, v, w of permutations of colors. Since each color is composed by three lines which go from certain point inside of the triangle to each one of the sides as shown in Figure 7, then every pair of colors a and b intersect at least twice. Furthermore, every intersection occurs in a rhombus and the number of rhombi is equal to $\binom{n}{2}$ which is equal to the number of pairs of colors. As a consequence, every pair of colors a and b intersect exactly once. Now, we call A, B and C the three different regions that the triangle is subdivided in, for color a (see Figure 7). Without lost of generality suppose that triangle b is in region C, then the only possibility for v and w is $...a...b...$ and $...b...a...$. Therefore, if a system u, v, w is achievable through a coloring of a tiling, then it does not appear any $a...b, a...b, a...b$.

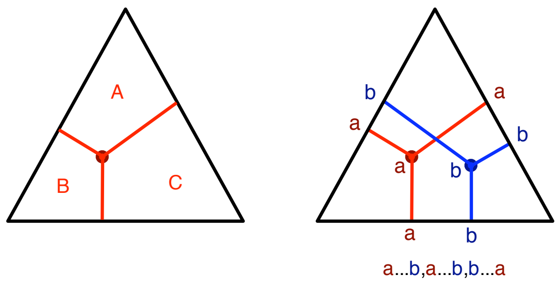


Figure 7: Behavior of the colors in a triangle

Now suppose we have a system u, v, w that is not achievable through a coloring of a tiling. After some relabel of the colors we can assume that $w = n...1$. Let (M_i, m_i), $i = 1, ..., n$, be the n-tuple in Theorem 3.3 for which algorithms 1 and 2 give us $u^{-1}(i) = M_i(1)$ and $v^{-1}(i) = m_i(n)$. Since the system is not achievable, it is impossible to make a routing of the graph G_n connecting the points (M_i, m_i) to (i, i) for $i = 1, ..., n$. Then, if we start drawing the routing from $i = 1$ to forward, there is a first element j for which the path going from (M_j, m_j) to (M_j, j) and then to (j, j), cross inevitably some preceding rout. This only could happen if one of the following three cases holds: Here we assume that the first $j-1$ routs are the most left possible.

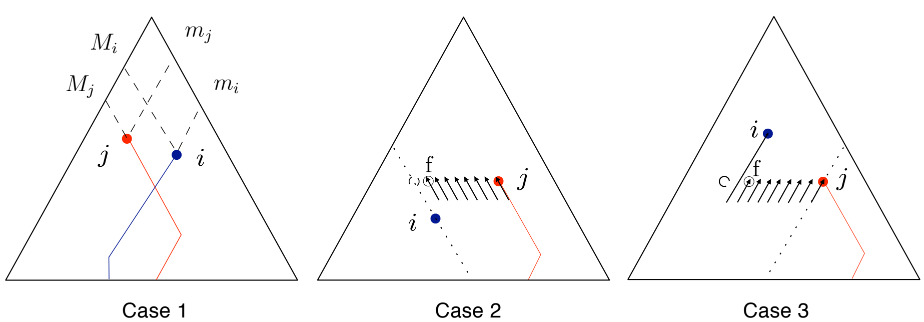


Figure 8: The three possible cases.

Case 1: There is an i such that $i < j$, $M_i \geq M_j$ and $m_i \geq m_j$. Then $M_j(j) \leq M_i(i)$, and so, $M_j(i) < M_i(i)$. Thus, $M_j(k) < M_i(k)$ for all $1 \leq k \leq i$, in particular $M_j(1) < M_i(1)$, which implies $u^{-1}(j) < u^{-1}(i)$. On the other hand, $m_j(j) \leq m_i(i)$, and so, $m_j(j) < m_i(j)$. Thus, $m_j(k) < m_i(k)$ for all $j \leq k \leq n$, in particular $m_j(n) < m_i(n)$, which implies $v^{-1}(j) < v^{-1}(i)$.

For the other two cases, consider the first vertex f of the graph G_n going from j to the left, such that the vertex which is one position on the left of f, is not in any of the first $j-1$ routs.

Case 2: The first edge of the rout going from f to BC is parallel to AC. Then, it is easy to see that $M_j(i) < M_i = M_i(i)$, and so $M_j(1) < M_i(1)$, which implies $u^{-1}(j) < u^{-1}(i)$. On the other hand, $m_j = m_j(j) < m_i(j)$, then $m_j(n) < m_i(n)$, i.e $v^{-1}(j) < v^{-1}(i)$.

Case 3: The first edge of the rout going from f to BC is parallel to AB. Then, it is not hard to see that there exist $i \leq k < j$ such that $m_k(j) > m_j = m_j(j)$, then $v^{-1}(j) < v^{-1}(k)$. On the other hand, $M_j(k) < M_k = M_k(k)$, then $u^{-1}(j) < u^{-1}(k)$

In summary, first we proved that if a system u, v, w is achievable then it does not appear something of the form $a...b, a...b, a...b$. Then we proved that if this system is not achievable, there exits $a > b$ ($a = j$ and $b = i$ or k) such that $u^{-1}(a) < u^{-1}(b)$, $v^{-1}(a) < v^{-1}(b)$ and $w^{-1}(a) = n+1-a < n+1-b = w^{-1}(b)$, i.e it appears something of the form $a...b, a...b, a...b$.

□

3.2 Acyclic systems of permutations on the edges of a simplex

Now we are interested in generalizing colorings of tilings to colorings of high dimensional fine mixed subdivisions. Intuitively, a coloring can be obtained by assigning different colors to each one of the simplices and spreading the colors via the mixed cells. Below, we formalize this notion and recover some of the geometrical properties of a subdivision by looking at the combinatorics of it.

Geometrical interpretation of Minkowski sums: Every color subdivides the simplex $n\Delta_{d-1}$ in d regions. Which one am I in?. If $A, B, ..., C$ are the regions containing vertices $A, B, ..., C$ respec-

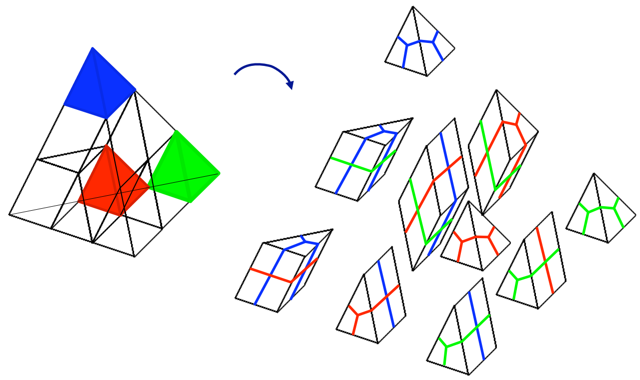


Figure 9: Coloration in dimension 3

tively, then a piece of the subdivision is represented by the Minkowski sum $S_1+...+S_n$, where i-component S_i is the set of all regions of the color i which the piece intersects (See Figure 10). Note that it not only works for the full dimensional pieces of the subdivision, but for any dimensional piece. Also note that in this paper, we often abuse notation by using the n-tuple $S_1, ..., S_n$ to talk about the Minkowski sum $S_1 + ... + S_n$ for simplicity.

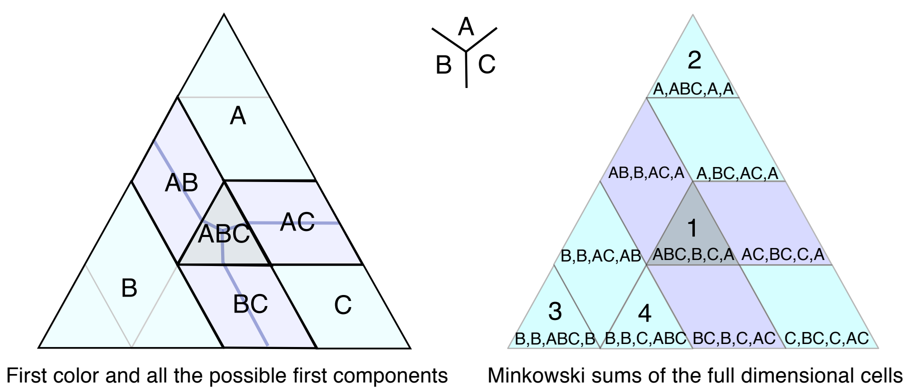


Figure 10: i-components or Minkowski sums on a subdivision of $5\Delta_2$.

A good high-dimensional analogue of a color on a fine mixed subdivision is a **Tropical Pseudo-hyperplane** in [3], that we define using the **Voronoi Subdivisions**; we briefly recall their definition. The **Voronoi Subdivision** of a k- simplex $A_1...A_k$ divides it in k regions, where region j consists of the points in the simplex for

which A_j is the closest vertex.
In this paper we will denote by S the set of the cells (of any dimension) on a fine mixed subdivision of $n\Delta_{d-1}$, and we will refer to S itself as a fine mixed subdivision.

Definition 3.5. The skeleton of the Voronoi subdivision of a simplex S_i is the set of points inside of S_i for which the minimum between the distances from the point to the vertices of S_i achieves at least twice.

Definition 3.6. The color i is the polyhedral complex whose cells are Minkowski sums $S_1 + ... + S_{i-1} + R_i + S_{i+1}, ..., S_n$ for which $(S_1 + ... + S_n) \in S$, $|S_i| \geq 2$ and R_i is the skeleton of the Voronoi subdivision of S_i. (See Figures 5, 9 and 10).

Color i corresponds to a tropical pseudo-hyperplane in tropical geometry. It has as central point the middle point of the simplex i and spread its leaves through the mixed cells of the subdivision. Each color hits the edges of the simplex exactly once, producing a natural system of colors defined as follows:

Definition 3.7 (The system of colors)**.** Let S be a fine mixed subdivision of the simplex $n\Delta_{d-1}$ with vertices $A_1, .., A_d$. The system of colors C_S is the set of all permutations $A_{d_1}A_{d_2} : [n] \to [n]$, $d_1 \neq d_2 \in [d]$,where $A_{d_1}A_{d_2}(j)$ is defined as the coordinate that is equal to $A_{d_1}A_{d_2}$ of the unique element of S that is written using only letters A_{d_1}, A_{d_2} and has exactly j letters equal to A_{d_2}.

The nature of the previous definition can be checked in Figure 11. and explained as follows: The one dimensional cells of the subdivision that are located at the edge $A_{d_1}A_{d_2}$ correspond to the elements of S which are written using only letters A_{d_1}, A_{d_2}. The one dimensional cell on this edge containing color i has i-component $A_{d_1}A_{d_2}$ and other components equal to either A_{d_1} or A_{d_2}. And the position j of the color i is equal to the number of letters A_{d_2}.

Note that now one can even calculate the system of colors geometrically by looking at the picture, or just by looking at the combinatorics of the list of elements of S. This definition has the advantage of a precise way of computing systems of colors for higher dimensional fine mixed subdivisions that can be used in order to prove theorems.

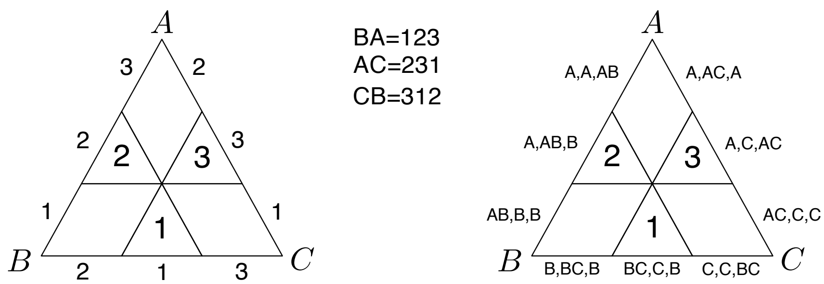


Figure 11: A system of colors and the Minkowski sums on the edges of $3\Delta_2$.

Definition 3.8. Let S be a fine mixed subdivision of the simplex $n\Delta_{d-1}$ with vertices $A_1, ..., A_d$, and $U \subset \{A_1, ..., A_d\}$. The restriction $S|_U$ is the set composed by the elements of S that are written using just letters from U.

Geometrically, the restriction $S|_U$ corresponds to the set of Minkowski sums of the restriction of the fine mixed subdivision S over the face of the simplex with vertices at U. In fact

Proposition 3.9. *The restriction $S|_U$ is the set of Minkowski sums of a fine mixed subdivision of $n\Delta_{|U|-1}$ with vertices at U.*

Proof. An special case of the Cayley Trick in [2] that will discussed in Section 4.2 give us a bijection between fine mixed subdivisions of $n\Delta_{d-1}$ and triangulation of the polytope $\Delta_{n-1} \times \Delta_{d-1}$. Then, the restriction of the corresponding triangulation of S to the face $\Delta_{n-1} \times \Delta_{|U|-1}$ where the second component is the simplex with vertices at U, corresponds to the fine mixed subdivision $S|_U$. $\square$

Definition 3.10. A system of permutations C on the edges of $n\Delta_{d-1}$ is a set of permutations $A_{d_1}A_{d_2} : [n] \to [n]$, $1 \leq d_1, d_2 \leq d$, such that $A_{d_1}A_{d_2}(j) = A_{d_2}A_{d_1}(n-j+1)$.

Definition 3.11. Let C be a system of permutations on the edges of $n\Delta_{d-1}$. For every $1 \leq i, j \leq n$, the graph $G_{ij}(C)$ has vertices at $[d]$, and, we draw a directed edge $d_1 \to d_2$ if the permutation $A_{d_1}A_{d_2}$ is of the form $...i...j....$

Figure 14 shows two examples of the graphs G_{12}, G_{13}, G_{23} associated to systems of colors of fine mixed subdivisions of $3\Delta_2$ and $3\Delta_3$.

Definition 3.12. Let C be a system of permutations on the edges of $n\Delta_{d-1}$. We will say that C is acyclic if the graphs $G_{ij}(C)$ are acyclic for all $1 \leq i, j \leq n$. Some times we call $G_{ij} = G_{ij}(C)$ for simplicity.

In other words, we will say that a system of permutations on the edges of a simplex is acyclic if there is not a cycle of the form $a...b, a...b, ..., a...b$.

Theorem 3.13. *Let S be a fine mixed subdivision of $n\Delta_{d-1}$. Then, the system of colors C_S is acyclic.*

Proof. The case $d = 1, 2$ are trivial. The case $d = 3$ was already proved before in Theorem 3.1. Now, let's assume the result is true for all $d < k$; suppose there exist a fine mixed subdivision S of $n\Delta_{k-1}$ such that the corresponding system of colors C_S is not acyclic. Then, there exist $a, b \in [n]$ such that the graph G_{ab} has a cycle $d_1 \to ... \to d_m \to d_1$.
Case 1: Suppose $m < k$. If we take $U = \{A_{d_1}, ..., A_{d_m}\}$, then $C_{S|_U}$ is not acyclic where $S|_U$ is a fine mixed subdivision of $n\Delta_{|U|-1}$ with $|U| < k$, which contradicts our assumption.
Case 2: Suppose $m = k \geq 4$. If $A_{d_1}A_{d_{m-1}}$ is of the form $...a...b...$ then we have a cycle $d_1 \to d_{m-1} \to d_m \to d_1$. On the other hand, if $A_{d_1}A_{d_{m-1}}$ is of the form $...b...a...$ then we have a cycle $d_1 \to ... \to d_{m-1} \to d_1$. In both situations we use case 1 to get a contradiction. $\square$

We end this section by describing the behavior of a *generic* pseudo tropical hyperplane arrangement at infinity. Generic arrangements of n tropical hyperplanes in the tropical space of dimension $d-1$ are in bijective correspondence [4] with *regular* fine mixed subdivisions of the dilated simplex $n\Delta_{d-1}$. In general, we define generic arrangements of n tropical pseudo-hyperplanes in the tropical space of dimension $d-1$ as the coloring of a fine mixed subdivision (possible non regular) of $n\Delta_{d-1}$. Theorem 3.13 says that such arrangements behave in an acyclic way at infinity. The following conjecture predicts that any acyclic behavior can be represented as the system of colors of a pseudo tropical hyperplane arrangement.

Conjecture 3.14. *Every acyclic system C on the edges of $n\Delta_{d-1}$ is achievable as the system of colors of a fine mixed subdivision.*

We have already proved in Theorem 3.1 that Conjecture 3.14 is true in the case of acyclic systems on the edges of an equilateral triangle $n\Delta_2$. A proof of this conjecture would give as a simple way of constructing fine mixed subdivisions to make progress towards Conjecture 2.2. Indeed, in section 6 we prove Conjecture 3.14 for the special case of acyclic systems on the edges of $3\Delta_{d-1}$, a simplex of size three in any dimension, and use it in order to prove Conjecture 2.2 also for simplices of size three in any dimension.

4 Dual systems and dual subdivisions

4.1 Dual systems

Every pair of numbers $i, j \in [n]$ of an acyclic system C of $n\Delta_{d-1}$ determine an acyclic orientation G_{ij} on the edges of the complete graph K_d. On the other hand, for any acyclic orientation G of K_d one can define the permutation $P(G) = k_1...k_d$, where k_m is the vertex of G with $d-m$ outgoing arrows. It is easy to check that this determines a bijection between acyclic orientations of the complete graph K_d and permutations of the set $[d]$. This biection gives us a form of duality for acyclic systems of permutations.

Definition 4.1. Let C be an acyclic system on the edges of $n\Delta_{d-1}$. Define C^* be a system on the edges of $d\Delta_{n-1}$ such that the permutation on the edge ij is equal to $P(G_{ij})$.

Proposition 4.2. *The permutation AB is of the form $...i...j...$ in system C if and only if the permutation ij is of the form $...A...B...$ in system C^*.*

Proof. The permutation on the edge AB is of the form $...i...j...$ if and only if the graph G_{ij} contains an arrow from A to B, if and only if the permutation ij is of the form $...A...B....$ $\square$

Proposition 4.3. *The system C^* is acyclic and $(C^*)^* = C$*

Proof. Suppose C^* has a cycle $..A...B.., ..A...B.., ..., ..A...B..$ passing through the vertices $i_1, ..., i_r$. Then, the permutation AB is of the form $..i_1...i_2...i_r...i_1..$ which is a contradiction. Thus, C^* is an acyclic, and we can define the system $(C^*)^*$ on the edges of $n\Delta_{d-1}$; let AB' be the permutation of $(C^*)^*$ on the edge AB. Then, the permutation AB is of the form $...i...j...$ in system C if and only if the permutation

ij is of the form $...A...B...$ in system C^* if and only if the permutation AB' is of the form $...i...j...$ in system $(C^*)^*$. The result follows from this. $\square$

4.2 Dual subdivisions

We start by recalling the one-to-one correspondence between fine mixed subdivisions of $n\Delta_{d-1}$ and triangulations of the polytope $\Delta_{n-1} \times \Delta_{d-1}$. This equivalent point of view has the drawback of bringing us to a higher-dimensional picture. Its advantage is that it simplifies greatly the combinatorics of the tiles, which are now just simplices. Let $v_1, ..., v_n$ and $w_1, ..., w_d$ be the vertices of Δ_{n-1} and Δ_{d-1}, so that the vertices of $\Delta_{n-1} \times \Delta_{d-1}$ are of the form $v_i \times w_j$. A triangulation T of $\Delta_{n-1} \times \Delta_{d-1}$ is given by a collection of simplices. For each simplex t in T, consider the fine mixed cell whose i-th summand is $w_a w_b ... w_c$, where a, b,..., c are the indexes j such that $v_i \times w_j$ is a vertex of t. These fine mixed cells constitute the fine mixed subdivision of $n\Delta_{d-1}$ corresponding to T. This bijection is only a special case of the more general Cayley trick, which is discussed in detail by Santos in a very nice paper [2].
For instance, Figure 12 shows a triangulation of the triangular prism $\Delta_1 \times \Delta_2 = 12 \times ABC$, and the corresponding fine mixed subdivision of $2\Delta_2$, whose three tiles are $ABC + B$, $AC + AB$, and $C + ABC$.

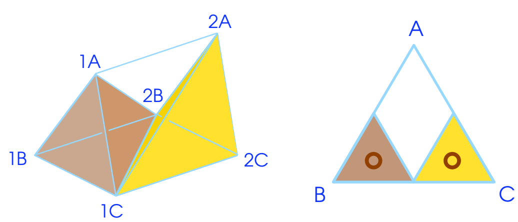


Figure 12: The Cayley Trick

This bijection gives us a form of duality for fine mixed subdivisions of simplices. This duality exist for triangulations of products of two simplices [2]. Since triangulations of $\Delta_{n-1} \times \Delta_{d-1} \cong \Delta_{d-1} \times \Delta_{n-1}$ are in canonical bijection with fine mixed subdivisions of $n\Delta_{d-1}$ and fine mixed subdivisions of $d\Delta_{n-1}$. One can then guess how the concept of duality should be defined for fine mixed subdivisions.

Definition 4.4. Let S be a fine mixed subdivision of $n\Delta_{d-1}$. Via the Cayley trick, it has a corresponding triangulation of $\Delta_{n-1} \times \Delta_{d-1} \cong \Delta_{d-1} \times \Delta_{n-1}$. S^* is the fine mixed subdivision of $d\Delta_{n-1}$ associated to the triangulation of $\Delta_{d-1} \times \Delta_{n-1}$ above. It is clear that $(S^*)^* = S$. Combinatorially, if we think of S as a collection of n-tuples $Z = (Z_1, ..., Z_n)$ and we define $Z^* = (Z_{A_1}, ..., Z_{A_d})$, where $Z_{A_k} = \{i | \ A_k \in Z_i\}$. Then $S^* = \{Z^* | \ Z \in S\}$. Figure 13 shows an example of a subdivision of $3\Delta_{4-1}$, its dual subdivision of $4\Delta_{3-1}$ and the Minkowski sums of the full dimensional cells.

5 Deletion and Contraction

In this section we introduce operations deletion and contraction first studied by Santos in [2] and Ardila-Develin in [3]. Then, we define operations deletion and contraction of a system of permutations on the edges of a simplex and study some properties that are similar to the ones that appear in Matroid Theory.

5.1 Deletion and Contraction of a fine mixed subdivision

Definition 5.1. Let S be a fine mixed subdivision of $n\Delta_{d-1}$ and $i \in [n]$. The deletion $S_{\setminus i}$ is the set with elements the n-tuples obtained by deleting the i-component of the elements of S.

Proposition 5.2. *$S_{\setminus i}$ is a fine mixed subdivision of $(n-1)\Delta_{d-1}$.*

Proof. Via the Cayley trick, consider the triangulation of $\Delta_{n-1} \times \Delta_{d-1}$ corresponding to S. Then $S_{\setminus i}$ corresponds to the restriction of such triangulation over $\Delta_{n-2} \times \Delta_{d-1}$, where the first component Δ_{n-2} is the simplex with vertices at $1, ..., \hat{i}, ..., n$. Then, $S_{\setminus i}$ is a fine mixed subdivision of $(n-1)\Delta_{d-1}$. $\square$

Definition 5.3. Let S be a fine mixed subdivision of the simplex $n\Delta_{d-1}$ with vertices $A_1, ..., A_d$. The contraction $S_{/A_i}$ is the set composed by the elements of S that are written without the letter A_i

Geometrically, the contraction $S_{/A_i}$ is the restriction of the fine mixed subdivision S over the facet of $n\Delta_{d-1}$ that does not contain the vertex A_i.

Proposition 5.4. *The contraction $S_{/A_i}$ is the set of Minkowski sums of a fine mixed subdivision of $n\Delta_{d-2}$ with vertices at $A_1, ..., \hat{A}_i, ..., A_d$.*

Proof. Via the Cayley trick, consider the triangulation of $\Delta_{n-1} \times \Delta_{d-1}$ corresponding to S. Then $S_{/A_i}$ corresponds to the restriction of such triangulation over $\Delta_{n-1} \times \Delta_{d-2}$, where the second component Δ_{d-2} is the simplex with vertices at $A_1, ..., \hat{A}_i, ..., A_d$. Then, $S_{/A_i}$ is the set of Minkowski sums a fine mixed subdivision of $n\Delta_{d-2}$ with vertices at $A_1, ..., \hat{A}_i, ..., A_d$. $\square$

5.2 Deletion and Contraction of an acyclic system

Definition 5.5. Let C be an acyclic system on the edges of $n\Delta_{d-1}$ and $i \in [n]$. The deletion $C_{\setminus i}$ is the acyclic system on the edges of $(n-1)\Delta_{d-1}$ which is obtained by deleting the number i from each permutation.

Proposition 5.6. *If $C = C_S$ is the system of colors of a fine mixed subdivision S, then $C_{\setminus i} = C_{S_{\setminus i}}$.*

Proof. Recall that the permutation on the edge $A_{d_1}A_{d_2}$ of the system $C_{S_{\setminus i}}$ is defined by $A_{d_1}A_{d_2}(j)$ equal to the coordinate that is equal to $A_{d_1}A_{d_2}$ of the unique element of S that is written using only letters A_{d_1}, A_{d_2} and has exactly j letters equal to A_{d_2}. Since the elements of $S_{\setminus i}$ have coordinates $1, .., \hat{i}, ..., n$, then the system $C_{S_{\setminus i}}$ can be obtained just by deleting the number i from all the permutations of C. Therefore, $C_{\setminus i} = C_{S_{\setminus i}}$. $\square$

Definition 5.7. Let C be an acyclic system on the edges of $n\Delta_{d-1}$ and $i \in [d]$. The contraction $C_{/A_i}$ is the acyclic system on the edges of $n\Delta_{d-2}$ equal to the restriction of the system C to the edges of the facet of $n\Delta_{d-1}$ which does not contain the vertex A_i.

Proposition 5.8. *If C is the system of colors of a fine mixed subdivision S, then $C_{/A_i}$ is the system of colors of the fine mixed subdivision $S_{/A_i}$.*

The following proposition tells us that Deletion and Contraction are dual to each other.

Proposition 5.9. *Let S and C be a subdivision and an acyclic system of $n\Delta_{d-1}$ respectively, then the following properties hold:*

1. $(C_{\setminus i})^* = C^*_{/i}$
2. $(C_{/A_i})^* = C^*_{\setminus A_i}$

3. $(S_{\backslash i})^* = S^*_{/i}$

4. $(S_{/A_i})^* = S^*_{/A_i}$

Proof. 1. Every pair of numbers $j_1, j_2 \in [n]\backslash i$ of the acyclic systems $C_{\backslash i}$ of $(n-1)\Delta_{d-1}$ and C of $n\Delta_{d-1}$ determine the same acyclic orientation $G_{j_1 j_2}$ of the complete graph K_d. Therefore, if $j_1, j_2 \neq i$ the permutation $j_1 j_2$ of $(C_{\backslash i})^*$ is equal to the permutation $j_1 j_2$ of C^*. Thus, it is clear that $(C_{\backslash i})^* = C^*_{/i}$.
Property 2. follows from 1. And properties 3 and 4 follow from the the interpretation of deletion and contraction via the Cayley trick. $\square$

Proposition 5.10. ${C_S}^* = C_{S^*}$

Proof. Let S be a fine mixed subdivision of $n\Delta_{d-1}$. The result is equivalent to prove that the permutation associated to $G_{ij}(C_S)$ is equal to the permutation ij of C_{S^*} for all $1 \leq i, j \leq n$. Note that $G_{ij}(C_S) = G_{ij}(C_{S_{\backslash\{1\ldots\hat{i}\ldots\hat{j}\ldots n\}}})$. On the other hand, permutation ij of C_{S^*} is equal to $C_{S^*_{/\{1\ldots\hat{i}\ldots\hat{j}\ldots n\}}} = C_{(S_{\backslash\{1\ldots\hat{i}\ldots\hat{j}\ldots n\}})^*}$. Therefore the result is equivalent to prove that $G_{ij}(C_{\widetilde{S}})$ is equal to the permutation ij of $C_{\widetilde{S}^*}$, where $\widetilde{S} = S_{\backslash\{1\ldots\hat{i}\ldots\hat{j}\ldots n\}}$. Since $\widetilde{S}$ is a subdivision of $2\Delta_{d-1}$, it is enough to prove the proposition for fine mixed subdivisions S of $2\Delta_{d-1}$. It is not hard to see that they are of the form:

$$\begin{array}{rcl}
1 & & 2 \\
A_1A_2A_3 \ldots A_d & + & A_1 \\
A_2A_3 \ldots A_d & + & A_1A_2 \\
A_3 \ldots A_d & + & A_1A_2A_3 \\
\vdots & & \vdots \\
A_d & + & A_1A_2A_3 \ldots A_d
\end{array}$$

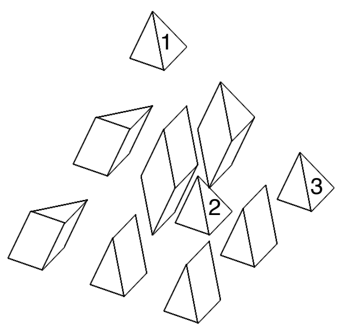
with corresponding dual subdivision S^* of $d\Delta_1$:

$$\begin{array}{ccccccccc}
A_1 & & A_2 & & A_3 & & \cdots & & A_d \\
12 & + & 1 & + & 1 & + & \cdots & + & 1 \\
2 & + & 12 & + & 1 & + & \cdots & + & 1 \\
2 & + & 2 & + & 12 & + & \cdots & + & 1 \\
\vdots & & \vdots & & \vdots & & & & \vdots \\
2 & + & 2 & + & 2 & + & \cdots & + & 12
\end{array}$$

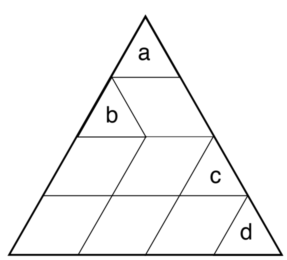
And then, $(C_S)^* = C_{S^*} = G_{12} = A_1 A_2 ... A_d$.

$\square$

The proof of this proposition also allow us to calculate in a very easy way the system of colors associated to a fine mixed subdivision in terms of the dual subdivision. We do not get into details but say that the use of dual subdivisions and dual systems of permutations give us an other point of view for understanding the properties of such subdivisions, for which in many cases arise difficult questions that are easier to answer by thinking on the dual problem instead of thinking on the problem by itself. Indeed, the following section is a clear example of it.



$ABCD$	+	A	+	A
BCD	+	A	+	AB
CD	+	AC	+	AB
D	+	ACD	+	AB
BCD	+	AB	+	B
CD	+	ABC	+	B
CD	+	C	+	ABC
D	+	CD	+	ABC
D	+	D	+	$ABCD$
D	+	$ABCD$	+	B



123	+	1	+	1	+	1
23	+	13	+	1	+	1
23	+	3	+	12	+	1
23	+	3	+	2	+	12
2	+	123	+	1	+	1
2	+	23	+	12	+	1
3	+	3	+	123	+	1
3	+	3	+	23	+	12
3	+	3	+	3	+	123
2	+	23	+	2	+	12

Figure 13: Subdivision of $3\Delta_{4-1}$, its dual subdivision of $4\Delta_{3-1}$ and Minkowski sums of the full dimensional cells.

6 Applications

6.1 Forward direction of Ardila-Billey's Conjecture

Now we are interested in the relationship between system of colors on the edges of a simplex and positions of the simplices of any subdivision achieving it. Such subdivisions are not necessarily unique, however the positions of the simplices are completely determined. Every pair of colors i, j define an acyclic orientation G_{ij} of the complete graph K_d; this acyclic orientation has an unique source that is precisely the relative position of the simplex i with respect to j. It can be formally expressed as follows:

Theorem 6.1. *Let S be a fine mixed subdivision of $n\Delta_{d-1}$. Then, the Minkowski sum corresponding to simplex i has i-component equal to $A_1...A_d$ and j-component equal to the source of $G_{ij}(C_S)$ for all $j \neq i$.*

Proof. Consider the dual subdivision S^* of S. The simplex i has i-component $A_1 \dots A_d$, and so, its dual cell contains the number i in all components A_r, for all $1 \leq r \leq d$. Therefore, the dual cell of the simplex i is the full dimensional cell of the subdivision S^* which is the closest one to the vertex i of $d\Delta_{n-1}$. This dual cell is then completely determined by the adjacent colors of the vertex i that are precisely the sources of $G_{ij}(C_S)$ with $j \neq i$. More explicitly, its Minkowski sum has A_r-component equal to $\{j \in [n] : j = i \text{ or } G_{ij} = A_r\}$. The result follows from dualizing again. $\square$

This is a powerful theorem that allows us to determine the positions of the simplices and more general, the corresponding Minkowski sums of the simplices just by using the system of colors. For instance, in the first example of Figure 14, 1 is in positions C and B with respect to 2 and 3, then the simplex 1 correspond to the Minkowski sum containing C and B in the second a third components respectively, thus $1 = ABC + C + B$ and so on.

Now, we define T_C be the table which has as rows the positions determined by an acyclic system of colors C. The table of the example in Figure 14 is

$$\begin{bmatrix} ABC & C & B \\ A & ABC & B \\ A & C & ABC \end{bmatrix}$$

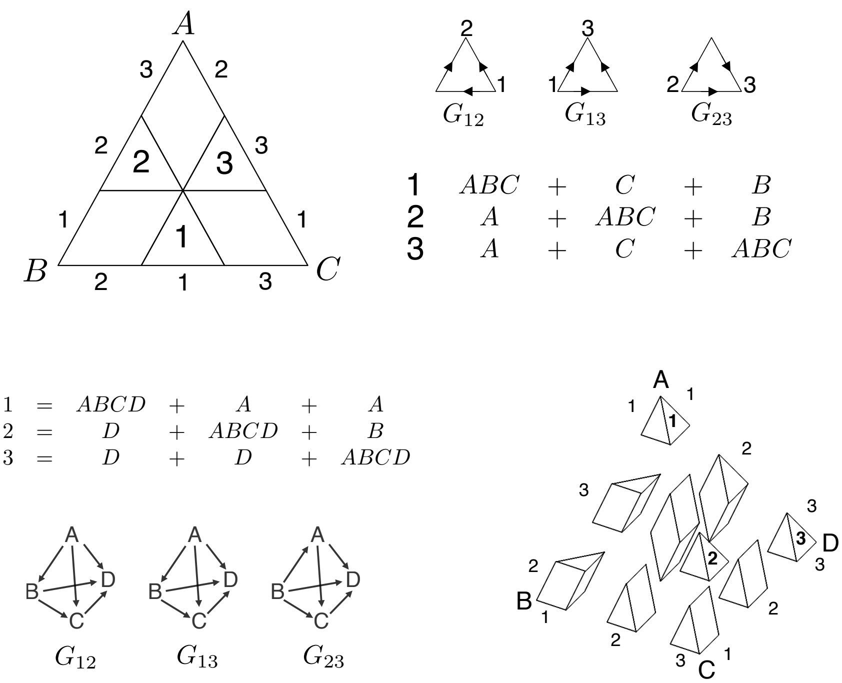


Figure 14: Acyclic orientations G_{12}, G_{13} and G_{23} of subdivisions of $3\Delta_2$ and $3\Delta_3$ together with the corresponding positions of the simplices.

We will make an arrow $i \to j$, from i to j $(i \neq j)$ of type $A_{d_1}A_{d_2}$ if there is an A_{d_1} in the row i and an A_{d_2} in the row j so that they are in the same column, here $d_1 \neq d_2$.

Lemma 6.2. *Let T be the table of positions given by an acyclic system. Then, there is not a cycle $i_1 \to i_2 \to ... \to i_r \to i_1$ of type $A_{d_1}A_{d_2}$ for any $d_1, d_2 \in [d]$.*

Proof. First we are going to prove that if there is an arrow $i \to j$ from i to j of type $A_{d_1}A_{d_2}$, then the permutation of colors on the edge $A_{d_1}A_{d_2}$ (reading from vertex A_{d_1} to A_{d_2}) is something of the form $...i...j....$ Suppose there is an arrow from i to j of type $A_{d_1}A_{d_2}$, i.e there is a column k such that $A_{d_1} \in T_{ik}$ and $A_{d_2} \in T_{jk}$.
Case 1: Suppose $k = j$. Then $T_{ij} = A_{d_1}$ which means that the graph G_{ij} has a source at A_{d_1}, and so the permutation on the edge $A_{d_1}A_{d_2}$ is of the form $...i...j....$
Case 2: If $k = i$, it is similar to case 1.
Case 3: Suppose $k \neq i, j$. Then $T_{ik} = A_{d_1}$ and $T_{jk} = A_{d_2}$, which

means that the graphs G_{ik} and G_{jk} have sources at A_{d_1} and A_{d_2} respectively. Thus, the permutation on the edge $A_{d_1}A_{d_2}$ is of the form $...i...k...j...$ as we wanted.
Finally, we can not have a cycle $i_1 \to i_2 \to ... \to i_r \to i_1$ of type $A_{d_1}A_{d_2}$, otherwise the permutation on the edge $A_{d_1}A_{d_2}$ is of the form $...i_1...i_2...i_r...i_1$ which is a contradiction. $\square$

The following result is another proof of the forward direction of Conjecture 2.2 of Ardila-Billey proved first by Ardila-Billey, Proposition 8.2.b in [1], and now proved using the language of system of colors of fine mixed subdivisions.

Theorem 6.3 (Forward direction of Ardila-Billey's conjecture)**.** *Let P be the set of positions of the simplices in a fine mixed subdivision of $n\Delta_{d-1}$, then every sub-simplex of size k contains at most k elements of P.*

Proof. Suppose we have a fine mixed subdivision of $n\Delta_{d-1}$ with vertices $A_1, ..., A_d$ and positions of simplices at P, such that there is a sub-simplex of size k containing more than k simplices of P. Let T_P be the table that has as rows the Minkowski sums of the simplices of P; we can assume that the first rows of T_P (rows in the small square in Figure 15) are those corresponding to the saturated (more than k) simplices contained in a sub-simplex of size k: $T_{a_1...a_d} = \{x = (x_1 \dots x_d) \in \mathbb{R}^d : x_i \geq a_i \text{ and } x_1 + \dots x_d = 1\}$ for some non negative integers $a_1 + ... + a_d = n - k$. Thus, each one of these more than k first rows satisfies that they contain (off of the diagonal) at least a_1 letters A_1,..., and at least a_d letters A_d.

Figure 15 represents the table of positions associated to the positions of the simplices and is composed by one white square, one dark rectangle and one white rectangle. We will prove that there are at least a_m letters of A_m on the first row that are in the dark rectangle. For simplicity, let's call $a = a_m$ and $A = A_m$; if $a = 0$ it is obvious. Suppose $a > 0$, if there is not any A on the intersection of first row with the white square, then the first row contains at least a letters A on the dark rectangle for free. Now suppose there is a letter A on the first row which is in the white square (off of the diagonal), then we can make arrows $1 = i_1 \to ... \to i_r$ of type AA_k for any $k = 1, ..., d$ with $A_k \neq A$, on the following way: start by making an arrow from $1 = i_1 \to i_2$ of type AA_k, where i_2 is the number of the column of the first letter A in consideration (this is possible because

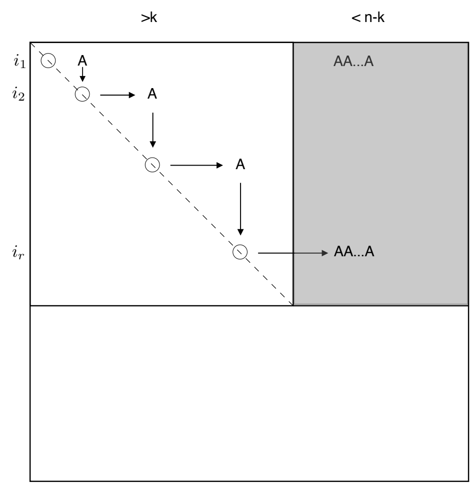


Figure 15: Table of positions

A is a letter in the row i_1, A_k is a letter in the row i_2, and they are in the same column i_2). Then, if there is a letter A in row i_2, column i_3 ($i_3 \neq i_2$) which is in the white square then we keep making an arrow $i_2 \to i_3$ of type AA_k on the same way as before. Since the table does not have any cycle of type AA_k then it is impossible to keep doing the process forever, and so we stop in some row i_r containing all the letters A (at least a) in the dark rectangle. Now look at the letters in the first row that are exactly above of these letters A, if some of those letters is equal to $B \neq A$, the table contains a cycle of type AB which is a contradiction. Therefore, the first row contains at least a letters A on the first row which are in the dark rectangle of the figure.
Now we finish the proof saying that the dark rectangle of the figure contains less than $n - k$ columns and in its first row it contains at least a_1 letters A_1,..., and at least a_d letters A_d which add up to at least $n - k$ letters, which is a contradiction. □

6.2 Ardila-Billey's conjecture for simplices of size three in every dimension

The following theorem is a special case of Conjecture 3.14.

Theorem 6.4. *Every acyclic system of* $3\Delta_{d-1}$ *is achievable as the system of colors of a fine mixed subdivision.*

Proof. Let C be an acyclic system of $3\Delta_{d-1}$. We proved in Theorem 3.1 that all acyclic systems of $d\Delta_{3-1}$ are achievable as the system of colors of a fine mixed subdivision. Let S^* be a subdivision of $d\Delta_{3-1}$ with corresponding system equal to C^*. The subdivision $S = (S^*)^*$ is a fine mixed subdivision of $3\Delta_{d-1}$ with corresponding system equal to C. $\square$

The following theorem is a special case of Conjecture 2.2 of Ardila-Billey.

Theorem 6.5. *Ardila-Billey's conjecture is true for simplices of size 3 in every dimension, i.e the possible positions of the three simplices in a fine mixed subdivision of* $3\Delta_{d-1}$ *are precisely those for which every sub-simplex of size* 2 *contains at most* 2 *of them.*

Proof. We already proved the forward direction in Theorem 6.3. For the backward direction, we will construct dual acyclic systems on the edges of the triangle $d\Delta_{3-1}$, that give rise to all the possible set of positions satisfying the conditions of the theorem. For simplicity, we will call $A, B, ..., H$ the d vertices of the simplex $3\Delta_{d-1}$. The Minkowski sums of the simplices in a fine mixed subdivision of $3\Delta_{d-1}$ have three components: one equal to $AB...H$, and the other two components are composed just by one letter each one; these last two letters determine the position of the simplices and they are shown on the left hand side of the triangles in Figure 16. Note that a set of three pairs of letters determine positions of simplices satisfying the conditions of the theorem if and only if there is not one letter that is in the three pairs. Figure 16 shows all the possible sets of three pairs which give rise to positions of simplices satisfying conditions of the theorem (up to relabeling), it also shows dual systems C^* with such sets as adjacent colors of the vertices. Let $C = (C^*)^*$ be an acyclic system of $3\Delta_{d-1}$, and let S be a fine mixed subdivision of $3\Delta_{d-1}$ having system of colors equal to C. Therefore, the positions of the simplices of S are precisely the positions showed in Figure 16 (Theorem 6.1). $\square$

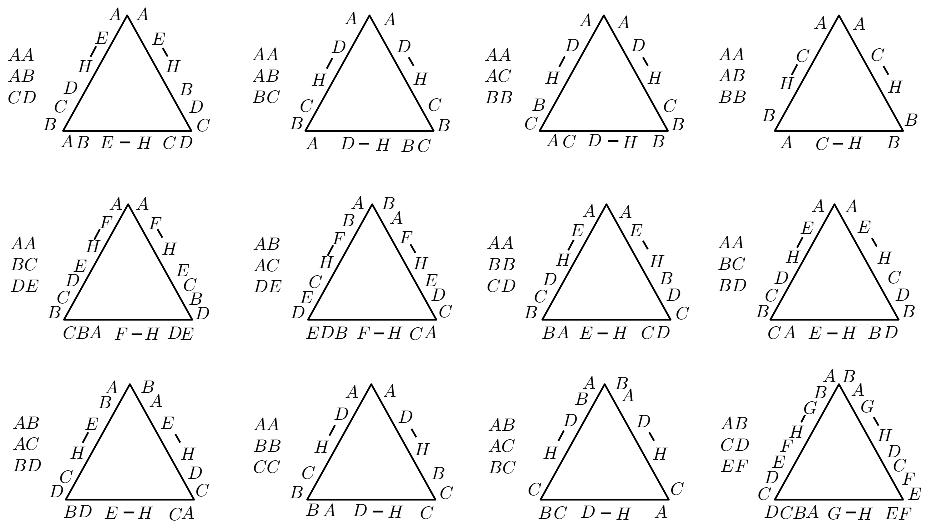


Figure 16: Possible positions of the simplices in a fine mixed subdivision of $3\Delta_{d-1}$ and dual acyclic systems (not unique) which generate such positions.

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