MATH 1300 A, Fall 2013
Solution Midterm 2

Show all work clearly and in order. Justify your answers whenever possible. No calculators, cellphones or books are allowed. You have 50 minutes.

1. (*10 points*) Use the definition of the derivative to find the derivative of $f(x) = x^2 + 1$.

$$\begin{aligned} f'(x) &= \lim_{h\to 0} \frac{f(x+h) - f(x)}{h} \\ &= \lim_{h\to 0} \frac{((x+h)^2+1) - (x^2+1)}{h} \\ &= \lim_{h\to 0} \frac{x^2 + 2xh + h^2 + 1 - x^2 - 1}{h} \\ &= \lim_{h\to 0} \frac{2xh + h^2}{h} \\ &= \lim_{h\to 0} (2x + h) \\ &= 2x \end{aligned}$$

2. (*15 points*) Find the second derivative of $\arcsin(2x^3)$

Let $f(x) = \arcsin(2x^3)$. We have

$$f'(x) = \frac{1}{\sqrt{1-(2x^3)^2}} \cdot 6x^2 = \frac{6x^2}{\sqrt{1-4x^6}}$$

Hence

$$\begin{aligned} f''(x) &= \frac{12x \cdot \sqrt{1-4x^6} - 6x^2 \cdot \frac{1}{2}(1-4x^6)^{-\frac{1}{2}}(-24x^5)}{1-4x^6} \\ &= \frac{12x(1-4x^6) - 3x^2(-24x^5)}{(1-4x^6)^{\frac{3}{2}}} \\ &= \frac{12x - 48x^7 + 72x^7}{(1-4x^6)^{\frac{3}{2}}} \\ &= \frac{12x + 24x^7}{(1-4x^6)^{\frac{3}{2}}} \end{aligned}$$

3. (*15 points*) Find the equation of the tangent line to the implicitly defined function $4x^3y^2 - 2x^2y^3 = 24$ at the point $(2, 1)$.

The equation of the tangent line to the graph of the function $y = f(x)$ implicitly defined by the formula above at the point $(x, y) = (2, 1)$ is given by

$$y = f'(2)(x-2) + f(2) = f'(2)(x-2) + 1$$

To find the implicit derivative of $y = f(x)$ in terms of (x, y) we differentiate both sides of the equation:

$$\frac{d}{dx}(4x^3y^2 - 2x^2y^3) = \frac{d}{dx}(24)$$

$$12x^2y^2 + 8x^3yy' - 4xy^3 - 6x^2y^2y' = 0$$

Replacing $(x, y) = (2, 1)$ we get

$$48 + 64y' - 8 - 24y' = 0$$

$$40y' + 40 = 0$$

$$y' = -1$$

Therefore, $f'(2) = -1$. Hence, the equation of the tangent line at $(x, y) = (2, 1)$ is

$$y = -(x-2) + 1 = -x + 3$$

4. (*15 points*) Find the critical points of the function $f(x) = (3x^2 - 3x - 6)^{\frac{2}{3}}$. Identify each of them as a local maximum, a local minimum or a non-local extremum.

The critical points of the function f are the values of x for which $f'(x)$ is equal to zero or does not exist.

$$\begin{aligned} f'(x) &= \frac{2}{3}(3x^2 - 3x - 6)^{-\frac{1}{3}} \cdot (6x - 3) \\ &= \frac{2(2x-1)}{(3x^2 - 3x - 6)^{\frac{1}{3}}} \\ &= \frac{2(2x-1)}{(3(x-2)(x+1))^{\frac{1}{3}}} \end{aligned}$$

Hence, the critical points are $x = \frac{1}{2}$, $x = 2$, and $x = -1$. Analyzing the sign of f' we get

$$\begin{array}{cccccccc} f': - - - - & (x=-1) & + + + + & (x=1/2) & - - - - & (x=2) & + + + + \\ \text{decreasing} & & \text{increasing} & & \text{decreasing} & & \text{increasing} \end{array}$$

So $x = -1$ is a local minimum, $x = 1/2$ is a local maximum, and $x = 2$ is a local minimum.

5. (*15 points*) Give an example of a continuos function which does not satisfy the conclusion of the Rolle's Theorem. Explain why.

The function $f(x) = |x|$ with domain $[-1, 1]$ does not satisfy the conclusion of the Rolle's Theorem. Although $f(-1) = f(1)$, there is no point between -1 and 1 where the derivative is equal to zero (or where the tangent line to the graph of f is horizontal). The Rolle's Theorem does not apply to this function because it is not differentiable at $x = 0$.

6. (*15 points*) Find the unique function f which satisfies $f'(x) = 3\sec^2 x$ and $f(0) = 3$.

The function f should be of the form

$$f(x) = 3\tan x + k$$

for some constant k. Since $f(0) = k = 3$, then

$$f(x) = 3\tan x + 3$$

7. (*15 points*) Sketch the graph of the function $f(x) = \frac{x^2-4}{x^2-1}$.

- The intercepts of the function are:
  x-intercepts: $x = 2$ and $x = -2$
  y-intercept: $y = f(0) = 4$

- The asymptotes are:
  vertical asymptotes: $x = 1$ and $x = -1$
  horizontal asymptote: $y = 1$. This is because

$$\lim_{x\to\infty} f(x) = \lim_{x\to-\infty} f(x) = 1.$$

- The first derivative is:

$$f'(x) = \frac{2x(x^2-1) - 2x(x^2-4)}{(x^2-1)^2} = \frac{6x}{(x^2-1)^2}$$

  The critical points of the function are: $x = 0$, $x = 1$ and $x = -1$. Analyzing the sign of the derivative we get

$$\begin{array}{ccccccc} f': - - - - & (x=-1) & - - - - & (x=0) & + + + + & (x=1) & + + + + \\ \text{decreasing} & & \text{decreasing} & & \text{increasing} & & \text{increasing} \end{array}$$

  So, $x = 0$ is a local minimum. Around the other two critical points there is no change of sign, so they are non-local extrema.

- Concavity and inflection points: The second derivative of the function f is

$$f''(x) = \frac{6(x^2-1)^2 - 6x \cdot 2(x^2-1)2x}{(x^2-1)^4} = \frac{6(x^2-1) - 24x^2}{(x^2-1)^3} = -\frac{6+18x^2}{(x^2-1)^3}$$

  The numerator of f'' is always positive. So, analyzing the sign of f'' we get:

$$\begin{array}{ccccc} f'': - - - - - - & (x=-1) & + + + + + + & (x=1) & - - - - - - \\ \text{concave down} & & \text{concave up} & & \text{concave down} \end{array}$$

Putting all information together, we deduce that the graph of the function is