MATH 1300 A, Fall 2013
Solution Midterm 3

Show all work clearly and in order. Justify your answers whenever possible. No calculators, cellphones or books are allowed. You have 50 minutes.

1. (*20 points*) Evaluate the following limits:

(a) $\lim_{x\to 0} \frac{x - \sin x}{x - \tan x}$

When we evaluate $x = 0$ we get $\frac{0}{0}$, so we can apply L'Hopital's rule:

$$\lim_{x\to 0} \frac{x - \sin x}{x - \tan x} = \lim_{x\to 0} \frac{1 - \cos x}{1 - \sec^2 x}$$

Simplifying we get:

$$\lim_{x\to 0} \frac{x - \sin x}{x - \tan x} = \lim_{x\to 0} \frac{1 - \cos x}{\frac{\cos^2 x - 1}{\cos^2 x}} = \lim_{x\to 0} -\frac{\cos^2 x}{\cos x + 1} = -\frac{1}{2}$$

(b) $\lim_{x\to \frac{\pi}{2}} \frac{\tan(3x)}{\tan(5x)}$

Before applying L'Hopital's rule it is convenient to simplify this limit:

$$\lim_{x\to \frac{\pi}{2}} \frac{\tan(3x)}{\tan(5x)} = \lim_{x\to \frac{\pi}{2}} \frac{\sin(3x)}{\sin(5x)} \cdot \frac{\cos(5x)}{\cos(3x)} = \frac{-1}{1} \lim_{x\to \frac{\pi}{2}} \frac{\cos(5x)}{\cos(3x)} = -\lim_{x\to \frac{\pi}{2}} \frac{\cos(5x)}{\cos(3x)}$$

When we evaluate $x = \frac{\pi}{2}$ we get $\frac{0}{0}$, so we can apply L'Hopital's rule:

$$-\lim_{x\to \frac{\pi}{2}} \frac{\cos(5x)}{\cos(3x)} = -\lim_{x\to \frac{\pi}{2}} \frac{-\sin(5x)\cdot 5}{-\sin(3x)\cdot 3} = -\frac{-5}{3} = \frac{5}{3}$$

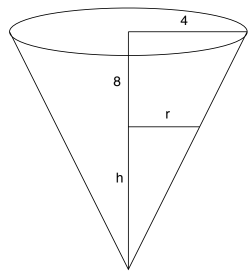
(c) $\lim_{x\to 1} \frac{x^3 + x^2 - 2x + 1}{x^2 - 1}$

When we evaluate $x = 1$ we get $\frac{1}{0}$. In particular, we can not apply L'Hopital's rule. In fact, this limit does not exist. The two side limits are:

$$\lim_{x\to 1^-} \frac{x^3 + x^2 - 2x + 1}{x^2 - 1} = -\infty \qquad \lim_{x\to 1^+} \frac{x^3 + x^2 - 2x + 1}{x^2 - 1} = \infty$$

(d) $\lim_{x\to 0} \frac{\sin x}{x} = 1$

2. (*20 points*) Water is leaking out of a conical cup of height 8 centimeters and radius 4 centimeters. Find the rate of change of the height of water in the cup at the instant that the cup is full, if the volume is decreasing at a constant rate of 8 cubic centimeters per second.



$$V = \frac{1}{3}\pi r^2 h$$

$$\frac{r}{4} = \frac{h}{8} \quad \Rightarrow \quad r = \frac{h}{2}$$

We are given that $\frac{dV}{dt} = 8$ cm^3/sec, and are asked to find $\frac{dh}{dt}$ when $h = 8$ cm. Replacing the formula for the radius r in the formula for the volume we get

$$V = \frac{1}{12}\pi h^3$$

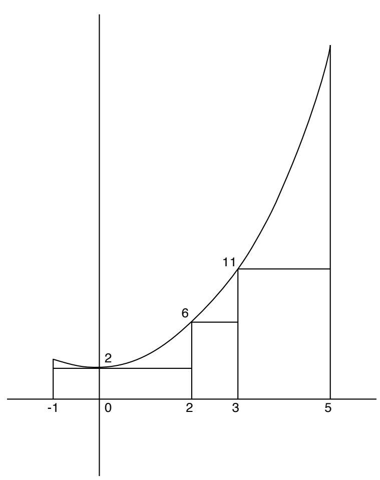
Taking the derivative with respect to time we get

$$8 = \frac{1}{4}\pi h^2 \frac{dh}{dt}$$

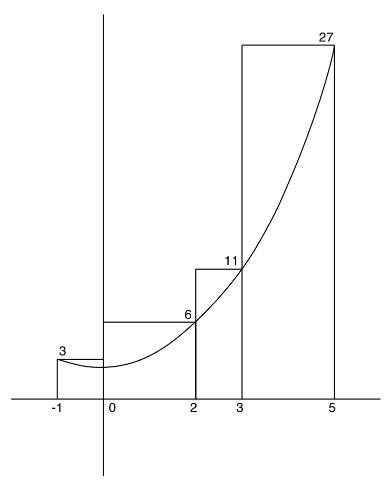
When $h = 8$ we get

$$8 = \frac{1}{4}\pi 8^2 \frac{dh}{dt} \quad \Rightarrow \quad \frac{dh}{dt} = \frac{1}{2\pi} \text{ cm/sec}$$

3. (*15 points*) Let $f(x) = x^2 + 2$ and $P = \{-1, 0, 2, 3, 5\}$ be a partition of the interval $[-1, 5]$. Find the lower Riemann sum $L(P, f)$ and the upper Riemann sum $U(P, f)$.



Lower Riemann Summ



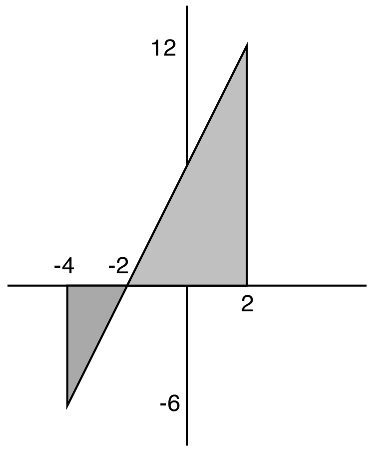
Upper Riemann Sum

$$L(P, f) = 2 \times 1 + 2 \times 2 + 6 \times 1 + 11 \times 2 = 34$$

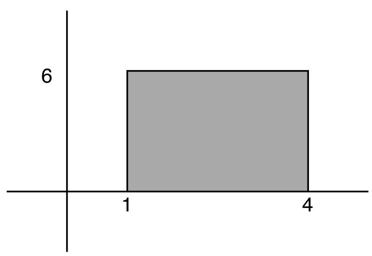
$$U(P, f) = 3 \times 1 + 6 \times 2 + 11 \times 1 + 27 \times 2 = 80$$

4. (*20 points*) Determine the value of each definite integral. In each case sketch the corresponding area. Do not use the fundamental theorem of calculus.

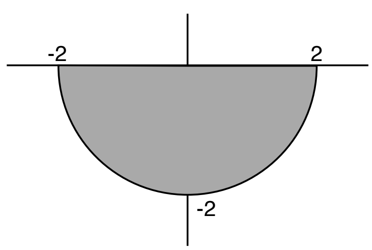
(a) $\int_{-4}^{2}(3x+6)\ dx = \frac{4\times 12}{2} - \frac{2\times 6}{2} = 24 - 6 = 18$



(b) $\int_{1}^{4} 6\ dx = 3 \times 6 = 18$



(c) $\int_{-2}^{2} -\sqrt{4-x^2}\ dx = -\frac{1}{2}\pi 2^2 = -2\pi$

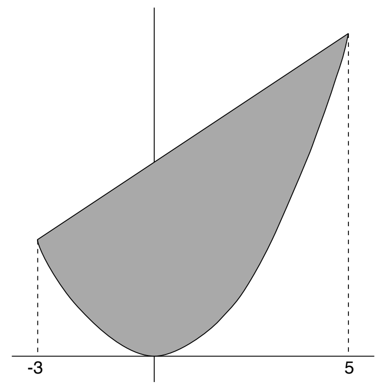


(d) $\int_{-\pi}^{\pi} x^4 \sin(\frac{1}{x^3})\ dx = 0$

because the function is odd.

5. (*15 points*) Find the area of the region between the graphs of $f(x) = x^2$ and $g(x) = 2x+15$. The intersection between the graphs of the two functions is attained when

$$x^2 - 2x - 15 = (x+3)(x-5) = 0$$



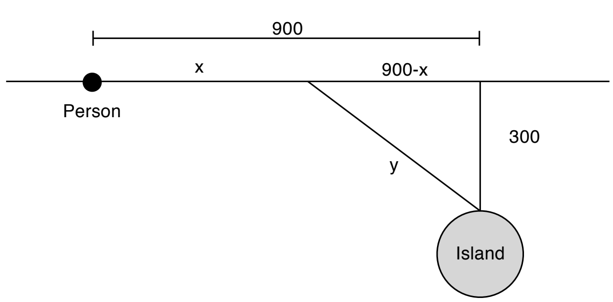
So the area of the region between the graphs is equal to

$$\int_{-3}^{5} (g(x) - f(x))dx = \int_{-3}^{5} (2x + 15 - x^2)dx = x^2 + 15x - \frac{1}{3}x^3 \Big|_{-3}^{5}$$

$$= (5^2 + 15(5) - \frac{1}{3}5^3) - ((-3)^2 + 15(-3) - \frac{1}{3}(-3)^3) = 127 - \frac{125}{3}$$

$$= \frac{256}{3}$$

6. (*15 points*) An island is located 300 meters south of a straight east-west beach. A person on the beach 900 meters west of the island wants to go to the island. If this person runs at 5 km/hr and swims at 4 km/hr, what is is the shortest possible time this person needs to go to the island?



The time $T(x)$ that the person needs to go to the island is given by the equation

$$T(x) = \frac{x}{5000} + \frac{y}{4000} = \frac{x}{5000} + \frac{\sqrt{(900-x)^2 + 300^2}}{4000}$$

The derivative of T is

$$T'(x) = \frac{1}{5000} + \frac{1}{4000} \cdot \frac{-2(900-x)}{2\sqrt{(900-x)^2+300^2}} = \frac{1}{20000} \cdot \frac{4\sqrt{(900-x)^2+300^2} - 5(900-x)}{\sqrt{(900-x)^2+300^2}}$$

This fraction is equal to zero only when the numerator is equal to zero:

$$4\sqrt{(900-x)^2+300^2} - 5(900-x) = 0$$

$$4\sqrt{(900-x)^2+300^2} = 5(900-x)$$

$$16(900-x)^2 + 16 \cdot 300^2 = 25(900-x)^2$$

$$16 \cdot 300^2 = 9(900-x)^2$$

$$4^2 \cdot 100^2 = (900-x)^2$$

$$400 = (900-x)$$

$$500 = x$$

Therefore the shortest possible time the person needs is attained when $x = 500$, which implies $y = 500$. The shortest time is then equal to:

$$T(500) = \frac{500}{5000} + \frac{500}{4000} = \frac{9}{40} \text{ hours} = \frac{9}{40} \cdot 60 \text{ minutes} = 13.5 \text{ minutes}$$