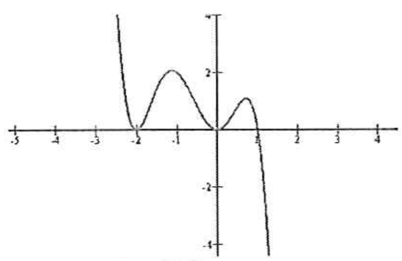
MATH 1300 A, Summer 2015 Name: ______________________

Midterm 1 Student number: ______________________

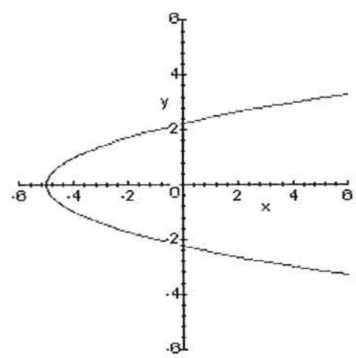
Solution

Show all work clearly and in order. Justify your answers whenever possible. No calculators, cell phones, head phones or books are allowed. You have 50 minutes.

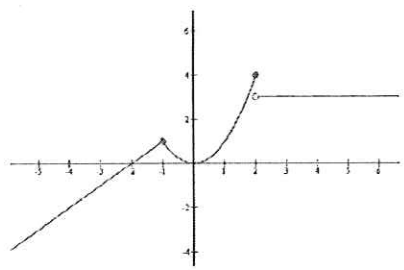
1. (*10 points*) For each of the following graphs determine if it is the graph of a function. Mark yes or not.



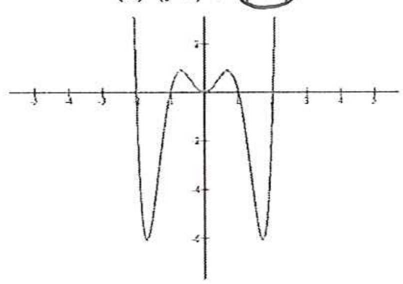
(a) (yes) or (not)



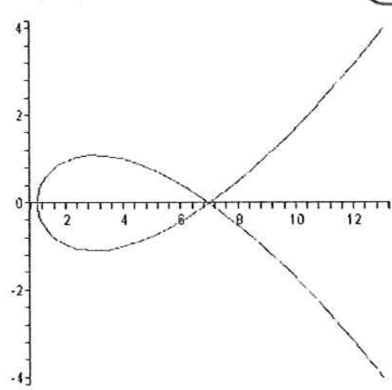
(b) (yes) or (not)



(c) (yes) or (not)



(d) (yes) or (not)



(e) (yes) or (not)

2. *(20 points)*

(a) Find the domain of the function $f(x) = \sqrt{\frac{x}{2-x}}$.

We need $\frac{x}{2-x} \geq 0$ and $x \neq 2$

		0		2	
x	$-$		$+$		$+$
$2-x$	$+$		$+$		$-$
	$-$		$+$		$-$

$\boxed{\text{Domain } f = [0,2)}$

(b) Let $g(x) = \frac{2x^2}{x^2+1}$ with domain the real numbers. Find the range of g.

Since $\frac{2x^2}{x^2+1} \geq 0$ and $\frac{2x^2}{x^2+1} < \frac{2x^2+2}{x^2+1} = 2$

the range of g is $\boxed{\text{Range } g = [0,2)}$

(c) Show that $f \circ g$ is defined, and find a formula for $(f \circ g)(x)$.

- The range $g = [0,2)$ is contained in the domain $f = [0,2)$.
  Therefore $f \circ g$ is defined.
- The formula for $(f \circ g)(x)$ is.

$$(f \circ g)(x) = f(g(x)) = \sqrt{\frac{\frac{2x^2}{x^2+1}}{2 - \frac{2x^2}{x^2+1}}} = \sqrt{\frac{\frac{\cancel{2}x^2}{\cancel{x^2+1}}}{\frac{\cancel{2}}{\cancel{x^2+1}}}} = \sqrt{x^2} = |x|.$$

$\Rightarrow \boxed{(f \circ g)(x) = |x|}$

3. (*20 points*)

(a) Explain why the function $p(x) = \frac{3x^2}{2x^2+1}$ with domain the real numbers is not one-to-one.

Note that $p(1) = p(-1) = 1$

Therefore, the function p is not one-to-one.

(b) Explain why the function $q(x) = \frac{3x^2}{2x^2+1}$ with domain $[0, \infty)$ is one-to-one.

$$\frac{3x_1^2}{2x_1^2+1} = \frac{3x_2^2}{2x_2^2+1}$$

$$3x_1^2(2x_2^2+1) = 3x_2^2(2x_1^2+1)$$

$$\cancel{6x_1^2x_2^2} + 3x_1^2 = \cancel{6x_2^2x_1^2} + 3x_2^2$$

$$3x_1^2 = 3x_2^2$$

$$x_1^2 = x_2^2$$

Since both $x_1, x_2 \in [0, \infty)$ are non-negative, the only solution to this equation is

$$x_1 = x_2.$$

Then, the function q is one-to-one.

(c) Find a formula for $y = q^{-1}(x)$.

$$y = \frac{3x^2}{2x^2+1}$$

$$y(2x^2+1) = 3x^2$$

$$y = 3x^2 - 2yx^2$$

$$y = (3-2y)x^2$$

$$\frac{y}{3-2y} = x^2$$

Since x is in the domain $[0, \infty)$ of q, the only solution to this equation is

$$x = \sqrt{\frac{y}{3-2y}}$$

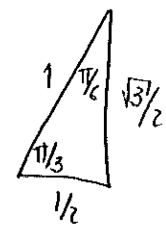
replacing y's by x's

$$\boxed{q^{-1}(x) = \sqrt{\frac{x}{3-2x}}}$$

4. (*20 points*) Evaluate each of the following:

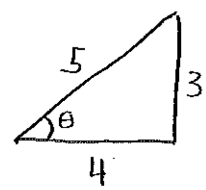
(a) $\cos(\frac{2\pi}{3})$

$$\cos\left(\frac{2\pi}{3}\right) = -\cos\left(\pi - \frac{2\pi}{3}\right) = -\cos\left(\frac{\pi}{3}\right) = \boxed{-\frac{1}{2}}$$



(b) $\cos(\text{arccot}\,\frac{4}{3})$

$$\cos\left(\text{arccot}\,\frac{4}{3}\right) = \boxed{\frac{4}{5}}$$



$\cot\theta = \frac{4}{3}$

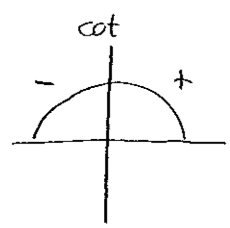
$\cos\theta = \frac{4}{5}$

(c) $\text{arccot}(-\sqrt{3})$

$\text{arccot}(\sqrt{3}) = \pi/6$

since $\text{arccot}\,x \in (0, \pi)$

$\Rightarrow \text{arccot}(-\sqrt{3}) = \pi - \pi/6 = \boxed{5\pi/6}$

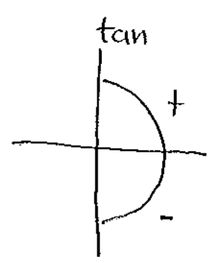


(c) $\arctan(-\sqrt{3})$

$\arctan(\sqrt{3}) = \pi/3$

since $\arctan x \in (-\pi/2, \pi/2)$

$\Rightarrow \arctan(-\sqrt{3}) = \boxed{-\pi/3}$



5. (*20 points*) Find the following limits:

(a) $\lim_{x\to\infty} \frac{4-\cos x}{x^2+1} =$

$$3 \le 4-\cos x \le 5$$

$$\frac{3}{x^2+1} \le \frac{4-\cos x}{x^2+1} \le \frac{5}{x^2+1}$$

Applying the Pinching Theorem

$$\lim_{x\to\infty} \frac{3}{x^2+1} = \lim_{x\to\infty} \frac{5}{x^2+1} = 0 \quad \Rightarrow \quad \lim_{x\to\infty} \frac{4-\cos x}{x^2+1} = \boxed{0}$$

(b) $\lim_{x\to\infty} x\sin(\frac{1}{x}) =$

$$\lim_{x\to\infty} x\sin(\tfrac{1}{x}) = \lim_{x\to\infty} \frac{\sin(\frac{1}{x})}{\frac{1}{x}} = \lim_{\theta\to 0} \frac{\sin\theta}{\theta} = \boxed{1}$$

(c) $\lim_{x\to 3} \frac{\sin(x^2-2x-3)}{\sqrt{x+1}-2} =$ this is of the form $\frac{0}{0}$

$$= \lim_{x\to 3} \underbrace{\frac{\sin(x^2-2x-3)}{x^2-2x-3}}_{1} \cdot \frac{x^2-2x-3}{\sqrt{x+1}-2} = \lim_{x\to 3} \frac{x^2-2x-3}{\sqrt{x+1}-2}$$

$$= \lim_{x\to 3} \frac{x^2-2x-3}{\sqrt{x+1}-2} \cdot \frac{\sqrt{x+1}+2}{\sqrt{x+1}+2} = \lim_{x\to 3} \frac{\cancel{(x-3)}(x+1)(\sqrt{x+1}+2)}{\cancel{x-3}} = 4\cdot(2+2)$$

$$= \boxed{16}$$

6. (*10 points*) Is the function f below continuous?

$$f(x) = \begin{cases} \frac{\sin(\pi x)}{1-x} & \text{for } x \neq 1 \\ \pi & \text{for } x = 1 \end{cases}$$

We need to check if $f(1) = \pi = \lim_{x \to 1} \frac{\sin(\pi x)}{1-x}$

$$\lim_{x \to 1} \frac{\sin(\pi x)}{1-x} = \lim_{\tilde{x} \to 0} \frac{\sin(\pi(1-\tilde{x}))}{\tilde{x}}$$ taking $\tilde{x} = 1-x$, $x = 1-\tilde{x}$

$$= \lim_{\tilde{x} \to 0} \frac{\sin(\pi - \pi\tilde{x})}{\tilde{x}}$$

$$= \lim_{\tilde{x} \to 0} \frac{\sin(\pi\tilde{x})}{\tilde{x}}$$

$$= \lim_{\tilde{x} \to 0} \underbrace{\frac{\sin(\cancel{\pi}\tilde{x})}{\cancel{\pi}\tilde{x}}}_{1} \cdot \pi$$

$$= \boxed{\pi}$$

Since $\lim_{x \to 1} f(x) = f(1)$, then $\boxed{\text{the function is continuos}}$