COMBINATORIAL RECIPROCITY THEOREMS VIA GEOMETRY
Winter term 2020-2021, TU Graz
Cesar Ceballos, Lecture Notes.

Vorbesprechung 06.10.2020. 1/2

Course Website: available in my webpage (Teaching ⇒ Combinatorial Rec Thms)

Book: Combinatorial Reciprocity theorems: an invitation to enumerative geometric combinatorics.
By Matthias Beck and Raman Sanyal

Other literature: see course website.

The course consists of 3 university hours:

2h lecture	(90 min)	:	Tuesdays	9:00 - 10:30
1h exercises	(45 min)	:	Tuesdays	10:45 - 11:30

9:00-11:30

Place: Seminarraum 2 (Geometrie)

Grading: Oral exam
Active presentation of exercises will be taken into account.

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- What is combinatorial reciprocity?

$f(n) = \#$ of ..... of size n
$g(n) = \#$ of ..... of size n

$g(n) = \pm 1\, f(n)$

~~Also~~ examples related to:
- graphs
- posets
- hyperplane arrangements
- representation theory

<u>Example</u>: ① Assume that the DK Discrete Mathematics has n students. You want to form a committee of 2 student representatives

$\tilde{C}_2(n) = \#$ of such possible committees $= \frac{n(n-1)}{2}$

$C_2(n) = \#$ of possible committes with possible repetitions $= \frac{n(n-1)}{2} + n = \frac{n(n+1)}{2}$

Both are polynomials in n.

$$\tilde{C}(-n) = C(n)$$

- Committe of 3 or k student representatives without or with repetitions? (exercise) (done in class)

② Let G be a graph without loops or multiple edges

Let χ_G be its chromatic polynomial.

$\chi_G(n) = \#$ <u>proper colourings</u> of G with n colors
↓
color the vertices such that no two adjacent vertices have the same color.

<u>ex</u>:

G

$$\chi_G(n) = n(n-1)(n-2)$$

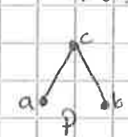
$$\chi_G(-1) = -6$$

Prop. $|\chi_G(-1)| =$ <u>acyclic orientations</u> of G
↓
orient the edges without forming cycles.

③ Let P be a poset (partially ordered set) on k elements

$\tilde{C}_P(n) = \#$ of strict order preserving maps from P to $[n]$

$C_P(n) =$ " " ― " " " " " "



$\tilde{C}_P(n) = 1^2 + 2^2 + \dots + (n-1)^2 = \frac{(n-1)n(2n-1)}{6}$

$C_P(n) = 1^2 + 2^2 + \dots + (n-1)^2 + n^2 = \frac{n(n+1)(2n+1)}{6}$

<u>Prop</u>: $\tilde{C}_P(-n) = (-1)^k C_P(n)$

<u>Exercise</u> Let $k \in \mathbb{N}$ fixed and $n \in \mathbb{N}$, let $F_k(n) = 1^k + \dots + n^k$.
- Show that $F_k(n)$ is a polynomial in n
- Show that $F_k(-n) = (-1)^{k+1} F_k(n-1)$