Combinatorial reciprocity theorems via geometry

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Combinatorial reciprocity relates two counting problems by evaluating a polynomial at positive and negative integers.

Main goal of this course is to develop combinatorial and geometric tools to understand combinatorial reciprocities.

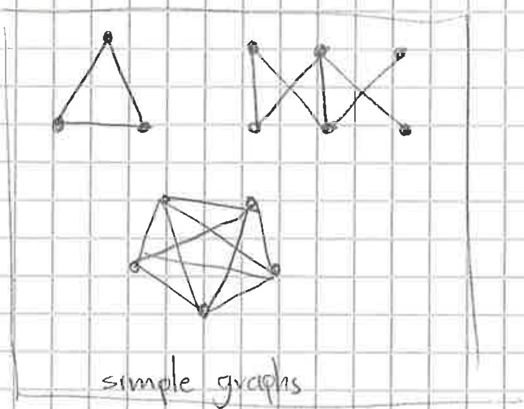
We will start by exploring some specific examples.

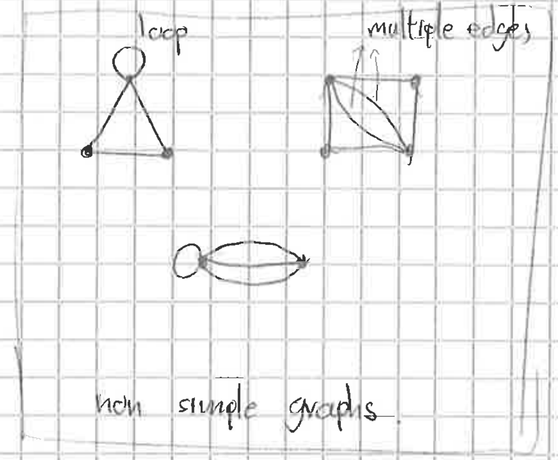
1 Graph colorings and acyclic orientations

A <u>graph</u> $G=(V,E)$ is a pair consisting of a finite set of nodes V and a collection E of unordered pairs of nodes, called edges.

A graph is called <u>simple</u> if it has no multiple edges or loops.

<u>Examples:</u>





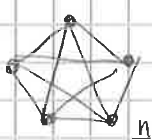
An <u>n-coloring</u> of a graph G is a map $c: V \to [n] := \{1,2,\dots,n\}$.

An n-coloring is called <u>proper</u> if no two nodes sharing an edge get assigned the same color, that is

$$c(u) \neq c(v) \quad \text{whenever} \quad uv \in E.$$

A graph G is <u>planar</u> if it can be drawn in the plane without edges crossing each other.

<u>Examples:</u>



<u>Four-color Theorem</u> Every planar map has a proper 4-coloring.

Famous result asked around 1852 by Francis Guthrie and shown with computer assistance 124 years later by Kenneth Appel and Wolfgang Haken.

George Birkhoff developed an interesting (but not yet successful) approach to proving the four-color theorem using chromatic polynomials.

Let

$$\chi_G(n) := \left| \{ c : V \to [n] \text{ proper } n\text{-coloring} \} \right|$$

Proposition (Birkhoff & Whitney) If G is a graph without loops, then $\chi_G(n)$ agrees with a polynomial of degree $|V|$ with integer coefficients.
If G has a loop, then $\chi_G(n) = 0$.

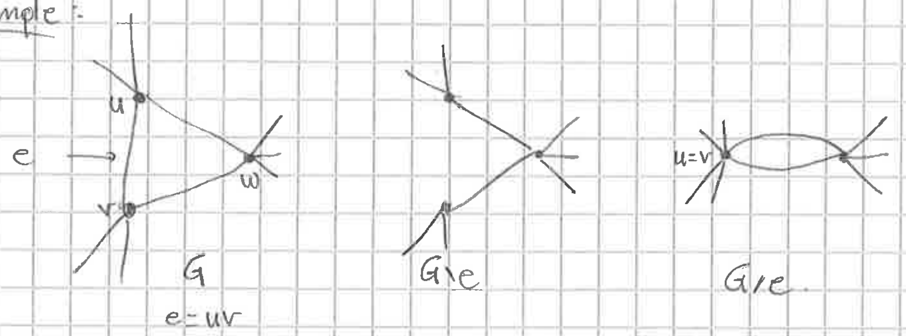
We call χ_G the chromatic polynomial of G.

Proving the four-color theorem is equivalent to showing $\chi_G(4) > 0$ for all planar graphs G.

The proof of this proposition uses a deletion-contraction argument.

For $e \in E$, the deletion $G \setminus e := (V, E \setminus \{e\})$ is the result of removing the edge e from G.
The contraction G/e is the graph obtained by identifying the two nodes incident to e and removing e.

Example:



Proof of Proposition:

- If G has a loop $\to$ no proper coloring and $\chi_G(n) = 0$ ✓
- We proceed by induction on the number of edges.

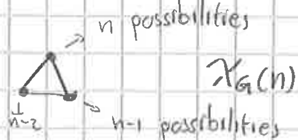
  If $|E| = 0$ then $\chi_G(n) = n^{|V|}$ ✓

  Assume $|E| \geq 1$ and let $e = uv \in E$ not be a loop. We claim that

$$\boxed{\chi_G(n) = \chi_{G \setminus e}(n) - \chi_{G/e}(n)} \implies \text{Deletion-Contraction Formula}$$

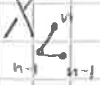
Indeed, a proper coloring of $G \setminus e$ fails to be a proper coloring of G when $c(u) = c(v)$; such colorings are exactly the proper colorings of G/e. ∎

polynomial of degree $|V|$ degree $|V|-1$

Example $G =$ 

$\chi_G(n) = n(n-1)(n-2)$

alternative: use deletion-contraction.

χ  $=$ χ  $-$ χ 

$$= n(n-1)(n-1) - n(n-1)$$

$$= n(n-1)(n-2)$$

Do the evalutions $\chi_G(-n)$ have a combinatorial meaning?
Richard Stanley 1973: yes!

An edge $uv \in E$ can be oriented in two possible directions $u \to v$ or $v \to u$.
An orientation ρ of G is an orientation of all its edges.
We denote by ρG the oriented graph.
A directed path in ρG is a sequence $v_0, v_1, \dots, v_s$ of distinct nodes such that $v_{i-1} \to v_i$ is a directed edge for $i = 1, \dots, s$.
If $v_s \to v_0$ is also a directed edge $v_0, v_1, \dots, v_s, v_0$ is called a directed cycle.

An orientation ρ of G is called acyclic if there are no directed cycles.

Each proper coloring of G induces a natural orientation ρ of G by orienting the edges from nodes with bigger color to nodes with smaller color:

$$u \to v \qquad \text{when} \qquad c(u) > c(v)$$

Prop: Let $c: V \to [n]$ be a proper coloring and ρ be the induced orientation of G. Then ρG is acyclic.

Proof Assume that $v_0 \to v_1 \to \dots \to v_s \to v_0$ is a cycle, then

$$c(v_0) > c(v_1) > \dots > c(v_s) > c(v_0) \quad \Rightarrow \lightning.$$

An orientation ρ and an n-coloring c of G are compatible if for every oriented edge

$$u \to v \quad \Rightarrow \quad c(u) \geq c(v)$$

The pair (ρ, c) is called strictly compatible if $c(u) > c(v)$ for every $u \to v$.

<u>Prop</u>: If (ρ, c) is strictly compatible, then c is a proper coloring and ρ is an acyclic orientation on G. In particular, $\chi_G(n)$ is the number of strictly compatible pairs (ρ, c), where c is a proper n-coloring.

<u>Proof</u> Since every edge is oriented then $c(u) < c(v)$ or $c(u) > c(v)$ for every $uv \in E$, so c is a proper coloring, and ρ is its induced acyclic orientation. $\square$

Our first combinatorial reciprocity theorem:

<u>Theorem</u> Let G be a finite graph on d nodes and $\chi_G(n)$ its chromatic polynomial. Then $(-1)^d \chi_G(-n)$ equals the number of compatible pairs (ρ, c) where c is an n-coloring and ρ is an acyclic orientation. In particular $(-1)^d \chi_G(-1)$ equals the number of acyclic orientations of G.

<u>Example</u>

G

$$\chi_G(n) = n(n-1)(n-2)$$

$$(-1)^3 \chi_G(-1) = 6 = \text{acyclic orientations of } G. \,!$$

$$(-1)^3 \chi_G(-n) = \;? \qquad \text{exercise.}$$

<u>Proof</u>
We will show that both expressions satisfy the same deletion-contraction recurrence with the same initial values.

Let $\overline{\chi}_G(n) = (-1)^d \chi_G(-n)$. Recall that $\chi_G(n) = \chi_{G\setminus e}(n) - \chi_{G/e}(n)$.
Therefore

$$\boxed{\overline{\chi}_G(n) = \overline{\chi}_{G\setminus e}(n) + \overline{\chi}_{G/e}(n)}$$

Let $\widetilde{\chi}_G(n) :=$ # compatible pairs (ρ, c) of an acyclic orientation and an n-coloring of G.

<u>Claim</u>: $$\boxed{\widetilde{\chi}_G(n) = \widetilde{\chi}_{G\setminus e}(n) + \widetilde{\chi}_{G/e}(n)}$$

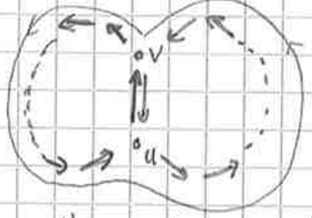
We prove this claim as follows. First, we prove that every compatible pair (ρ, c) of $G\setminus e$ can be obtained by restricting a compatible pair of G; second, we prove that every compatible pair of $G\setminus e$ can be obtained once or twice this way, and these that are obtained twice are counted by $\widetilde{\chi}_{G/e}(n)$.

<u>First part for the proof of the claim: Every</u> compatible pair (ρ, c) of $G\setminus e$ can be obtained by restricting a compatible pair of G.

For this, we show that at least one of the two possible orientations of $e = uv \in E$ completes (ρ, c) to a compatible pair of G.

- If $c(u) > c(v)$ then orienting $u \to v$ completes (ρ, c) to a compatible pair of G, otherwise there would be an oriented cycle:  and $c(u) > c(v) \geq \dots \geq c(u)$ ↯
- Similarly, if $c(v) > c(u)$ then orienting $v \to u$ works.

• If $c(u) = c(v)$ then at least one of the two possible orientations $u \to v$ or $v \to u$ completes (ρ, c) to a compatible orientation of G. Otherwise there would be two oriented cycles:



But the exterior of this picture would then be an oriented cycle for the orientation ρ of $G\setminus e$, which is a contradiction.

Second part for the proof of the claim: Every compatible pair (ρ, c) of $G\setminus e$ is obtained once or twice by restricting compatible pairs of G. Those that are obtained twice are counted by $\tilde{\chi}_{G/e}(n)$.

As we have seen in the previous part, every compatible pair (ρ, c) of $G\setminus e$ can be obtained by restricting a compatible pair of G. There are one or two possible compatible pair extensions corresponding to the two possible orientations of $e = uv \in E$.

Assume that the two orientations $u \to v$ and $v \to u$ extend (ρ, c) to a compatible pair of G. Then, by compatibility, we have

$$c(u) \geq c(v) \quad \text{and} \quad c(v) \geq c(u).$$

Therefore u and v have the same color: $c(u) = c(v)$.
This means that u and v can be identified in $G\setminus e$, and the orientation ρ of $G\setminus e$ is still acyclic after this identification. Otherwise, it would have a cycle, which together with one of the orientations $u \to v$ or $v \to u$, prior identification, would form a cycle in the orientation $\rho + u \to v$ or $\rho + v \to u$ of G. This contradicts the fact that both $\rho + u \to v$ and $\rho + v \to u$ are compatible with c on G.

As a consequence, the number of compatible pairs (ρ, c) of $G\setminus e$ that are obtained twice by restricting a compatible pair of G equals to $\tilde{\chi}_{G/e}(n)$.

Therefore

$$\tilde{\chi}_G(n) = \tilde{\chi}_{G\setminus e}(n) + \tilde{\chi}_{G/e}(n)$$

This finishes the proof of our claim.
Now, $\bar{\chi}_G(n)$ and $\tilde{\chi}_G(n)$ satisfy the same recurrence relation. To prove that they are equal we need to show that they coincide in the base case where G has no edges.
In this case,

$$\bar{\chi}_G(n) = (-1)^d \chi_G(-n) = (-1)^d (-n)^d = n^d$$

and

$$\tilde{\chi}_G(n) = n^d$$

∎