Combinatorial reciprocity theorems via geometry
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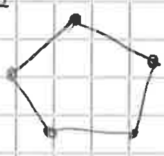
During the course we have seen several examples of combinatorial reciprocities and some proofs.

Today: We will start a general geometric approach to combinatorial reciprocities.

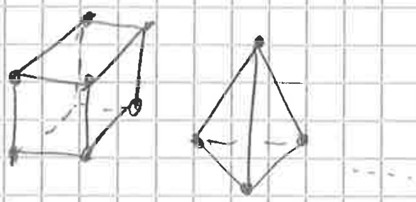
- Towards Ehrhart polynomials in higher dimensions

We say that $P \subseteq \mathbb{R}^d$ is a (convex) polytope if it is the convex hull of finitely many points. If these points have integer coordinates, we say that $P$ is a lattice polytope.

Examples



Dimension 2:
Polytopes = convex polygons



Dimension 3

We define the Ehrhart polynomials

$$\mathrm{ehr}_P(n) := |nP \cap \mathbb{Z}^d| \qquad \mathrm{ehr}_{P^\circ}(n) := |nP^\circ \cap \mathbb{Z}^d|$$

where $P^\circ$ is the relative interior of $P$. Our objective is the following result:

**Theorem**
Let $P \subseteq \mathbb{R}^d$ be a lattice polytope. Then,

i) For positive integers $n$, $\mathrm{ehr}_P(n)$ agrees with a polynomial in $n$ of degree $\dim P$, and constant term 1. (Ehrhart's Theorem)

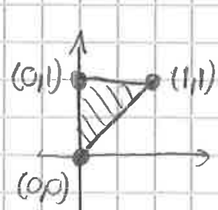
ii) Evaluating this polynomial at negative integers we obtain (Ehrhart-Macdonald reciprocity)

$$(-1)^{\dim P}\, \mathrm{ehr}_P(-n) = \mathrm{ehr}_{P^\circ}(n)$$

In order to prove this result we will need to develop several tools from geometry and will also use the concept of generating functions.

We will expend some time introducing these preliminaries.

## A motivating example




Let $\Delta := \{ (x_1, x_2) \in \mathbb{R}^2 : 0 \leq x_1 \leq x_2 \leq 1 \}$

We have previously computed $ehr_\Delta(n)$.

It counts multisubsets of $[n+1]$ consisting of 2 elements with possible repetitions. Therefore

$$ehr_\Delta(n) = \binom{n+1+2-1}{2} = \binom{n+2}{2}$$

Here is an alternative way to compute this polynomial using generating functions.

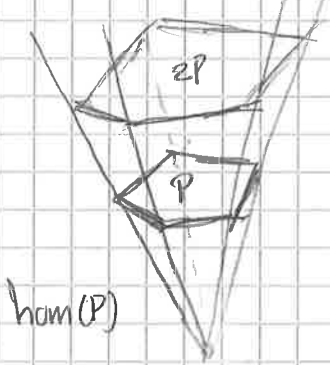
Define the <u>Ehrhart series</u> of P as the generating function

$$\boxed{Ehr_P(z) := 1 + \sum_{n \geq 1} ehr_P(n)\, z^n}$$

This is a power series in the variable $z$.

Define the <u>homogenization</u> of P as:

$$hom(P) := \{ (p,t) \in \mathbb{R}^{d+1} : t \geq 0,\ p \in tP \}$$
$$= cone\{ (v,1) : v \in vertex(P) \}$$




→ A copy of nP at height n.

→ lift P one dimension higher at height 1.

Therefore

$$Ehr_P(z) = \sum_{n \geq 0} \#(\text{lattice points in } hom(P) \text{ at height } n)\, z^n$$

For $S \subseteq \mathbb{R}^{d+1}$, we define the <u>integer point transform</u>

$$\boxed{\sigma_S(z_1, z_2, \ldots, z_{d+1}) := \sum_{m \in S \cap \mathbb{Z}^{d+1}} \mathbf{z}^m}$$

where

$$\mathbf{z}^m := z_1^{m_1} z_2^{m_2} \ldots z_{d+1}^{m_{d+1}}$$

Thus,

$$\boxed{Ehr_P(z) = \sigma_{hom(P)}(1, \ldots, 1, z)}$$

Going back to our example:

$$\Delta = \operatorname{conv}\{(0,0),(0,1),(1,1)\}$$

and

$$\operatorname{hom}(\Delta) = \mathbb{R}_{\geq 0}\underbrace{\begin{pmatrix}0\\0\\1\end{pmatrix}}_{v_1} + \mathbb{R}_{\geq 0}\underbrace{\begin{pmatrix}0\\1\\1\end{pmatrix}}_{v_2} + \mathbb{R}_{\geq 0}\underbrace{\begin{pmatrix}1\\1\\1\end{pmatrix}}_{v_3}$$

Since $v_1, v_2, v_3$ form a lattice basis of $\mathbb{Z}^3$ ( every point can be uniquely expressed as an integral linear combination of $v_1, v_2, v_3$),

then every lattice point in $\operatorname{hom}(\Delta)$ can be uniquely written as a non-negative integral linear combination of $v_1, v_2, v_3$.

Therefore

$$\sigma_{\operatorname{hom}(\Delta)}(\mathbf{z}) = \sum_{m \in \operatorname{hom}(\Delta) \cap \mathbb{Z}^3} \mathbf{z}^m$$

$$= \sum_{k_1,k_2,k_3 \geq 0} \mathbf{z}^{k_1v_1+k_2v_2+k_3v_3}$$

$$= \frac{1}{(1-\mathbf{z}^{v_1})(1-\mathbf{z}^{v_2})(1-\mathbf{z}^{v_3})}$$

$$= \frac{1}{(1-z_3)(1-z_2z_3)(1-z_1z_2z_3)}$$

Thus

$$\operatorname{Ehr}_\Delta(z) = \sigma_{\operatorname{hom}(\Delta)}(1,1,z) = \frac{1}{(1-z)^3}$$

$$= (1+z+z^2+z^3+\dots)^3$$

$$= \sum_{n\geq 0} a_n z^n = \sum_{n \geq 0} \binom{n+2}{2} z^n$$

$a_n = \#$ compositions $(\alpha_1,\alpha_2,\alpha_3)$ of $n$
i.e. triples $\alpha_1,\alpha_2,\alpha_3 \geq 0$ whose sum equals $n$


$$\Rightarrow a_n = \binom{n+2}{2}$$

As a consequence

$$\operatorname{ehr}_\Delta(n) = \binom{n+2}{2}$$

as we have deduced before.

Exercise: Let $\Delta$ be the convex hull of the origin and the d unit vectors in $\mathbb{R}^d$. Show that

$$\mathrm{ehr}_\Delta(n) = \binom{n+d}{d}$$

More generally, show that $\mathrm{ehr}_\Delta(n) = \binom{n+d}{d}$ for every unimodular simplex $\Delta$ in $\mathbb{R}^d$.

$\Delta = \mathrm{conv}\{v_0, v_1, \dots, v_d\} \subseteq \mathbb{R}^d$ is called unimodular if $v_1 - v_0, v_2 - v_0, \dots, v_d - v_0$ form a $\mathbb{Z}$-basis for $\mathbb{Z}^d$.

Before calculating Ehrhart series for general simplices and polytopes we will introduce some preliminaries:

- Polyhedral geometry
- Generating functions

- Polyhedral Geometry

Inequalities and polyhedra

A polyhedron $Q \subseteq \mathbb{R}^d$ is the set of solutions to a system of finitely many inequalities:

$$\begin{aligned}
a_{1,1}x_1 + \dots + a_{1,d}x_d &\leq b_1 \\
a_{2,1}x_1 + \dots + a_{2,d}x_d &\leq b_2 \\
&\vdots \\
a_{K,1}x_1 + \dots + a_{K,d}x_d &\leq b_K
\end{aligned}$$

for some $a_{ij}, b_i \in \mathbb{R}$ for $1 \leq i \leq K$ and $1 \leq j \leq d$.

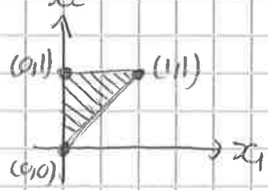
In compact form we can write

$$Q = \{\mathbf{x} \in \mathbb{R}^d : A\mathbf{x} \leq \mathbf{b}\}$$

Where $A \in \mathbb{R}^{K \times d}$ is the matrix of coefficients and $b \in \mathbb{R}^K$ collects the right hand sides.

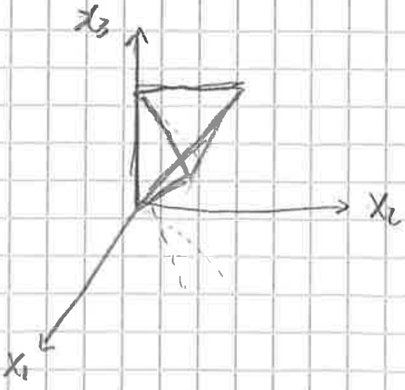
Examples

- $\begin{bmatrix} -1 & 0 \\ 1 & -1 \\ 0 & 1 \end{bmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \leq \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$




$x_1 \geq 0$
$x_1 \leq x_2$
$x_2 \leq 1$

- $$\begin{bmatrix} -1 & 0 & 0 \\ 1 & -1 & \\ & 1 & -1 \\ & & 1 \end{bmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} \le \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}$$




$x_1 \geq 0$
$x_1 \le x_2$
$x_2 \le x_3$
$x_3 \le 1$

We call $Q$ a rational polyhedron if $A$ and $b$ can be chosen over the rational numbers.

The set of solutions $H \subset \mathbb{R}^d$ to a single equation is called an affine hyperplane

$$H := \{ x \in \mathbb{R}^d : \langle a, x \rangle = b \}$$

for some $a \in \mathbb{R}^d \setminus \{0\}$ and $b \in \mathbb{R}$. Here $\langle \, , \rangle$ denotes the standard inner product on $\mathbb{R}^d$.

If $b=0$, we say that $H$ is a linear hyperplane. This is equivalent to $0 \in H$.

We also define the closed half spaces associated with $H$ as:

$$H^{\geq} := \{ x \in \mathbb{R}^d : \langle a, x \rangle \geq b \},$$
$$H^{\leq} := \{ x \in \mathbb{R}^d : \langle a, x \rangle \leq b \}.$$

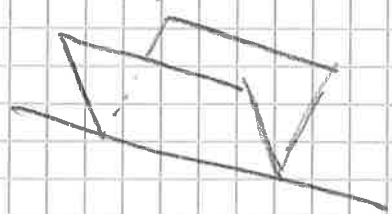
Hence, a polyhedron $Q$ is the intersection of finitely many closed half spaces:

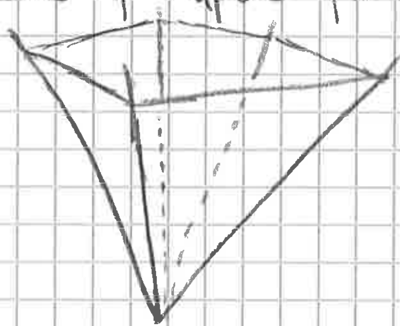
$$Q = H_1^{\leq} \cap \dots \cap H_K^{\leq} = \{ x \in \mathbb{R}^d : \langle a_i, x \rangle \leq b_i \ , \ 1 \leq i \leq K \}$$

An affine subspace is the intersection $L = H_1 \cap \dots \cap H_K$ of finitely many affine hyperplanes. A polyhedron $Q$ is proper if it is not an affine subspace.

One special type of polyhedra we use are polyhedral cones. A polyhedron $C \subseteq \mathbb{R}^d$ is a polyhedral cone if $\mu p \in C$ for any $p \in C$ and $\mu \geq 0$.

Examples





**Proposition** A polyhedron $Q \subseteq \mathbb{R}^d$ is a polyhedral cone if and only if it is of the form

$$Q = \{ x \in \mathbb{R}^d : Ax \leq 0 \}$$

for some matrix $A \in \mathbb{R}^{k \times d}$.

**Proof**: Exercise.

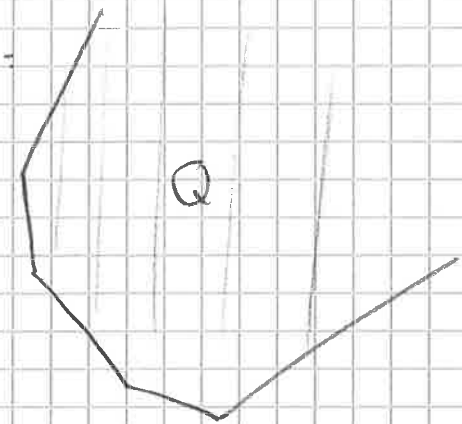
Except for $C = \{0\}$, polyhedral cones are examples of unbounded polyhedra.

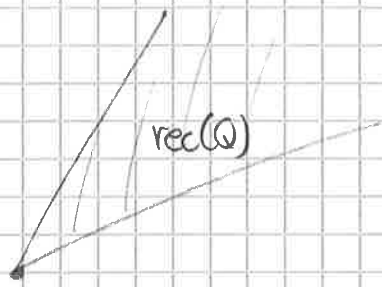
The recession cone $\mathrm{rec}(Q)$ of a polyhedron $Q$ is defined as:

$$\mathrm{rec}(Q) := \{ u \in \mathbb{R}^d : p + \mathbb{R}_{\geq 0} u \subseteq Q \text{ for some } p \in Q \}$$

Here $\mathbb{R}_{\geq} u$ denotes the set $\{ \lambda u : \lambda \geq 0 \}$.

**Example**:




**Exercise**: show that $\mathrm{rec}(Q)$ is a polyhedral cone, and that $p + \mathrm{rec}(Q) \subseteq Q$ for all $p \in Q$.

**Proposition** A nonempty polyhedron $Q \subseteq \mathbb{R}^d$ is bounded if and only if $\mathrm{rec}(Q) = \{0\}$.

**Proof** exercise.

The lineality space $\mathrm{lineal}(Q)$ is the inclusion-maximal linear subspace $L \subseteq \mathbb{R}^d$ such that $p + L \subseteq Q$ for some $p \in Q$.

**Proposition** Let $Q = \{ x \in \mathbb{R}^d : Ax \leq b \}$ be a nonempty polyhedron. Then

$$\mathrm{lineal}(Q) = \mathrm{rec}(Q) \cap (-\mathrm{rec}(Q)) = \{ x \in \mathbb{R}^d : Ax = 0 \}$$

In particular, $p + \mathrm{lineal}(Q) \subseteq Q$ for all $p \in Q$.

**Proof** Exercise.