Combinatorial Reciprocity Theorems via Geometry
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Lecture 10, 19.01.2021.

Last lecture : Generating functions

Today : - Generating function reciprocity
- Stanley's reciprocity for simplicial cones

- Generating function reciprocity

Recall from last time that every polynomial $f(n)$ of degree $\leq d$ can be expressed as

$$f(n) = \sum_{m=0}^{d} f^{(m)} \binom{n}{m}$$

and that its generating function is

$$F(z) = \sum_{n\geq 0} f(n) z^n = \sum_{m=0}^{d} f^{(m)} \frac{z^m}{(1-z)^{m+1}}$$

Now we want to evaluate the polynomial at negative integers using the machinery of generating functions. For this define

$$\boxed{F^{o}(z) := \sum_{n\geq 1} f(-n) z^n}$$

↳ there are good reasons to start at $n=1$!

Note that $f(-n)$ is also a polynomial in $n$, and so, $F^{o}(z)$ is also a rational generating function.

$$F^{o}(z) = \sum_{n\geq 1} \sum_{m=0}^{d} f^{(m)} (-1)^m \binom{n+m-1}{m} z^n \quad \text{(by binomial reciprocity)}$$

$$= \sum_{m=0}^{d} f^{(m)} (-1)^m \sum_{n\geq 1} \binom{n+m-1}{m} z^n$$

$$= \sum_{m=0}^{d} f^{(m)} (-1)^m \frac{z}{(1-z)^{m+1}} \quad \text{(by Exercise 1, Exercise Sheet 9)}$$

$$= -\sum_{m=0}^{d} f^{(m)} \frac{\left(\frac{1}{z}\right)^m}{\left(1-\frac{1}{z}\right)^{m+1}}$$

$$= -F\left(\tfrac{1}{z}\right)$$

$$\Rightarrow \boxed{F^{o}(z) = -F\left(\tfrac{1}{z}\right)}$$

Note that this is a formal reciprocity!
It does not tell us whether $f(-n)$ is a genuine counting function.

More generally, if $(f(n))_{n\geq 0}$ is a sequence satisfying a linear recurrence

$$c_0 f(n+d) + c_1 f(n+d-1) + \dots + c_d f(n) = 0$$

with $c_0, c_d \neq 0$ and initial values $f(0), f(1), \dots, f(d-1)$, its generating function is given by

$$\boxed{F(z) = \sum_{n\geq 0} f(n) z^n = \frac{P(z)}{c_0 + c_1 z + \dots + c_d z^d}}$$

for some polynomial of degree $< d$ that depends on the initial values.

We can run the recurrence "backwards" to determine $f(-1), f(-2), \dots, f(-n), \dots$

The sequence $\boxed{f^{\circ}(n) := f(-n)}$ satisfies the recurrence

$$c_d f^{\circ}(n+d) + c_{d-1} f^{\circ}(n+d-1) + \dots + c_0 f^{\circ}(n) = 0$$

And its associated generating function is given by

$$\boxed{F^{\circ}(z) := \sum_{n\geq 1} f^{\circ}(n) z^n = -\frac{z^d P(\frac{1}{z})}{c_0 z^d + c_1 z^{d-1} + \dots + c_d}}$$ (by Exercise 4.ii Exercisesheet 9)

As a consequence, we get the following theorem

<u>Theorem</u> (Generating function reciprocity)

Let $F(z) = \sum_{n\geq 0} f(n) z^n$ be a rational generating function. Then, $F^{\circ}(z) := \sum_{n\geq 1} f(-n) z^n$ is also rational. The two rational functions are related by

$$F^{\circ}(z) = -F\left(\frac{1}{z}\right).$$

<u>Proof</u>: 

$$-F\left(\frac{1}{z}\right) = -\frac{P(\frac{1}{z})}{c_0 + c_1 \frac{1}{z} + \dots + c_d (\frac{1}{z})^d}$$

multiply by $z^d$ in numerator and denominator

$$= -\frac{z^d P(\frac{1}{z})}{c_0 z^d + c_1 z^{d-1} + \dots + c_d}$$

$$= F^{\circ}(z) \quad \blacksquare$$

We will use this formal reciprocity to prove the Ehrhart polynomial reciprocity for polytopes.

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## Ehrhart polynomials

Consider a lattice polytope $P \subseteq \mathbb{R}^d$ and its Ehrhart function

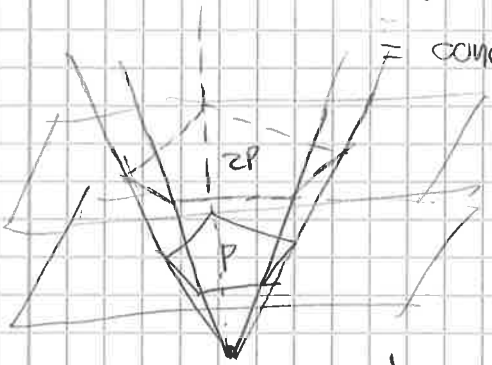
$$\boxed{\mathrm{ehr}_P(n) := |nP \cap \mathbb{Z}^d|}.$$

The Ehrhart series of $P$ is defined as

$$\boxed{\mathrm{Ehr}_P(z) := 1 + \sum_{n \geq 1} \mathrm{ehr}_P(n) z^n}$$

Our strategy to count lattice points in $nP$ is to consider the homogenization

$$\mathrm{hom}(P) = \{ (p,t) \in \mathbb{R}^{d+1} : t \geq 0,\ p \in tP \}$$
$$= \mathrm{cone}\{ (v,1) : v \in \mathrm{vert}(P) \}$$




$\rightarrow$ copy of $nP$ at height $n$

$$\boxed{\mathrm{Ehr}_P(z) = \sum_{n\geq 0} \left( \begin{array}{c}\#\text{lattice points in } \mathrm{hom}(P) \\ \text{at height } n\end{array} \right) z^n}$$

Given a set $S \subseteq \mathbb{R}^{d+1}$, we define the integer point transform

$$\boxed{\sigma_S(z_1, \dots, z_{d+1}) := \sum_{m \in S \cap \mathbb{Z}^{d+1}} z^m}$$

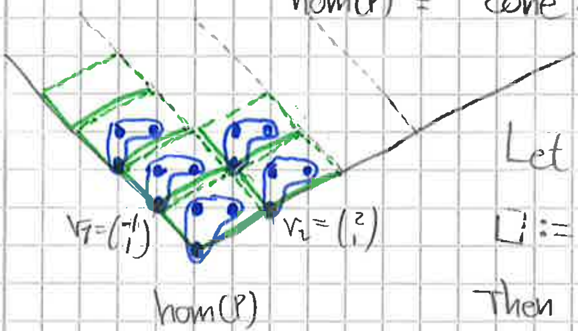
where $z^m := z_1^{m_1} z_2^{m_2} \dots z_{d+1}^{m_{d+1}}$

Thus

$$\boxed{\mathrm{Ehr}_P(z) = \sigma_{\mathrm{hom}(P)}(1, 1, \dots, 1, z)}$$

Example Let $P = [-1, 2] \subseteq \mathbb{R}^1$ be a segment and

$$\mathrm{hom}(P) = \mathrm{cone}\left\{ \underset{v_1}{\begin{pmatrix}-1\\1\end{pmatrix}}, \underset{v_2}{\begin{pmatrix}2\\1\end{pmatrix}} \right\} \subseteq \mathbb{R}^2$$




Let $\square$ be the half open parallelepiped

$$\square := [0,1) v_1 + [0,1) v_2$$

Then

$$\mathrm{hom}(P) = \biguplus_{j_1, j_2 \geq 0} (j_1 v_1 + j_2 v_2 + \square)$$

is the disjoint union of half open parallelepipeds. Moreover, every integer point in $\mathrm{hom}(P)$ is of the form

$$j_1 v_1 + j_2 v_2 + q$$

for some $j_1, j_2 \geq 0$ and some integer point in $\square$.

Therefore

$$\sigma_{\mathrm{hom}(P)}(\mathbf{z}) = \sum_{\substack{j_1,j_2\geq 0\\ q\in\square}} \mathbf{z}^{j_1 v_1 + j_2 v_2 + q}$$

$$= \left(\sum_{j_1\geq 0} \mathbf{z}^{j_1 v_1}\right)\left(\sum_{j_2\geq 0} \mathbf{z}^{j_2 v_2}\right)\left(\sum_{q\in\square} \mathbf{z}^{q}\right)$$

$$= \left(\frac{1}{1-\mathbf{z}^{v_1}}\right)\left(\frac{1}{1-\mathbf{z}^{v_2}}\right)\left(\sum_{q\in\square} \mathbf{z}^{q}\right)$$

$$= \frac{\sigma_{\square}(\mathbf{z})}{(1-\mathbf{z}^{v_1})(1-\mathbf{z}^{v_2})}$$

Evaluating at $\mathbf{z}=(1,z)$ we obtain

$$\mathrm{Ehr}_P(z) = \sigma_{\mathrm{hom}(P)}(1,z) = \frac{1+2z}{(1-z)^2}$$

Expanding the numerator in the $\binom{n}{m}$-basis

$$1+2z = \sum_{m=0}^{d} f^{(m)} z^m (1-z)^{d-m} = \underbrace{1}_{f^{(0)}}\cdot(1-z) + \underbrace{3}_{f^{(1)}}\cdot z \quad , d=1,$$

we recover the coefficients $f^{(m)}$ of

$$\mathrm{ehr}_P(n) = \sum_{m=0}^{d} f^{(m)}\binom{n}{m} = \binom{n}{0} + 3\binom{n}{1} = 1+3n. \quad \blacksquare$$

The computations in this example can be extended to any polytope $P$ which is a $(d-1)$-dimensional simplex. Its homogenization is a pointed simplicial cone in $\mathbb{R}^d$.

Let $C\subseteq\mathbb{R}^d$ be a pointed, rational $d$-dimensional cone. A <u>grading</u> of $C$ is a vector $a\in\mathbb{Z}^d$ such that $\langle a,p\rangle > 0$ for all $p\in C\setminus\{0\}$. For a grading $a\in\mathbb{Z}^d$ define the <u>Hilbert function</u> of $C$ as

$$\boxed{h_C^a(n) := \left|\{m\in C\cap\mathbb{Z}^d : \langle a,m\rangle = n\}\right|}$$

The <u>Hilbert series</u> of $C$ with respect to $a$ is

$$\boxed{H_C^a(z) := 1 + \sum_{n\geq 1} h_C^a(n) z^n}$$

<u>Proposition</u> $\boxed{H_C^a(z) = \sigma_C(z^{a_1},\dots,z^{a_d})}$ (only one single variable $z$ not a tuple)

<u>Proof</u> $\sigma_C(z^{a_1},\dots,z^{a_d}) = \sum_{m\in C\cap\mathbb{Z}^d} z^{a_1m_1+\dots+a_dm_d} = 1 + \sum_{m\in C\cap\mathbb{Z}^d\setminus\{0\}} z^{\langle a,m\rangle}$

$$= 1 + \sum_{n\geq 0} h_C^a(n) z^n \quad \blacksquare$$

**Theorem** (Simplicial version of Stanley's reciprocity)

For linearly independent vectors $v_1, v_2, \dots, v_K \in \mathbb{Z}^d$ and $1 \le m \le K$. Define the two half open cones

$$\hat{C} := \mathbb{R}_{\geq 0} v_1 + \dots + \mathbb{R}_{\geq 0} v_{m-1} + \mathbb{R}_{>0} v_m + \dots + \mathbb{R}_{>0} v_K,$$

$$\check{C} := \mathbb{R}_{>0} v_1 + \dots + \mathbb{R}_{>0} v_{m-1} + \mathbb{R}_{\geq 0} v_m + \dots + \mathbb{R}_{\geq 0} v_K.$$

Then $\sigma_{\hat{C}}(z), \sigma_{\check{C}}(z)$ are rational generating functions in $z_1, \dots, z_d$ which are related by

$$\sigma_{\check{C}}\left(\frac{1}{z}\right) = (-1)^K \sigma_{\hat{C}}(z),$$

where $\frac{1}{z} = \left(\frac{1}{z_1}, \dots, \frac{1}{z_d}\right)$.

In particular, for $m=1$

$$\check{C} = C := \text{cone}\{v_1, \dots, v_K\}$$

$$\hat{C} = C^0 \quad \text{the relative interior of } C$$

**Corollary** Under the same assumptions as above.

$$\sigma_C\left(\frac{1}{z}\right) = (-1)^K \sigma_{C^0}(z).$$

Proof of theorem. We use a tiling argument. Define the fundamental parallelepiped of $\hat{C}$ as

$$\hat{\square} := [0,1) v_1 + \dots + [0,1) v_{m-1} + (0,1] v_m + \dots + (0,1] v_K$$

We can tile $\hat{C}$ by translates of $\hat{\square}$:

$$\hat{C} = \biguplus_{j_1, \dots, j_K \geq 0} (j_1 v_1 + \dots + j_K v_K + \hat{\square})$$

Therefore

$$\sigma_{\hat{C}}(z) = \left(\sum_{j_1 \geq 0} z^{j_1 v_1}\right) \cdots \left(\sum_{j_K \geq 0} z^{j_K v_K}\right) \sigma_{\hat{\square}}(z)$$

$$\boxed{\sigma_{\hat{C}}(z) = \frac{\sigma_{\hat{\square}}(z)}{(1-z^{v_1}) \cdots (1-z^{v_K})}}$$

Similarly for the fundamental parallelepiped of $\check{C}$:

$$\check{\square} := (0,1] v_1 + \dots + (0,1] v_{m-1} + [0,1) v_m + \dots + [0,1) v_K$$

We get:

$$\boxed{\sigma_{\check{C}}(z) = \frac{\sigma_{\check{\square}}(z)}{(1-z^{v_1}) \cdots (1-z^{v_K})}}$$

Now $x \in \check{\square}$ iff $v_1 + \dots + v_k - x \in \hat{\square}$. Thus

$$\hat{\square} = v_1 + \dots + v_k - \check{\square} .$$

In terms of generating functions this reads as

$$\sigma_{\hat{\square}}(z) = z^{v_1 + \dots + v_k} \sigma_{\check{\square}}\left(\tfrac{1}{z}\right)$$

This yields

$$\sigma_{\check{C}}\left(\tfrac{1}{z}\right) = \frac{\sigma_{\check{\square}}\left(\frac{1}{z}\right)}{(1-z^{-v_1})\cdots(1-z^{-v_k})} = \frac{z^{-v_1-\dots-v_k}\sigma_{\hat{\square}}(z)}{(1-z^{-v_1})\cdots(1-z^{-v_k})}$$

$$= (-1)^k \frac{\sigma_{\hat{\square}}(z)}{(1-z^{v_1})\cdots(1-z^{v_k})} = (-1)^k \sigma_{\hat{C}}(z)$$

∎

For the extreme case $m=1$, $\check{C} = C$ and $\hat{C} = C^\circ$, we get the following:

**Corollary** Let $C \subseteq \mathbb{R}^{d+1}$ be a rational simplicial cone with generators $v_1, \dots, v_k$ and fundamental parallelepiped $\Pi = [0,1)v_1 + \dots + [0,1)v_k$. For a grading $a \in \mathbb{Z}^{d+1}$,

$$H_C^a(z) = \sum_{n \geq 0} h_C^a(n) z^n = \frac{H_\Pi^a(z)}{(1-z^{\langle a, v_1\rangle})\cdots(1-z^{\langle a, v_k\rangle})}$$

and

$$H_C^a\left(\tfrac{1}{z}\right) = (-1)^k H_{C^\circ}^a(z) .$$