Combinatorial reciprocity theorems via geometry

Cesar Ceballos

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(SS)

Last lecture : Stanley's reciprocity on simplicial cones

Today : - Ehrhart–Macdonald reciprocity

- Implications: reciprocity for order polynomials and chromatic polynomials.

Ehrhart Polynomial Reciprocity

Given a pointed rational cone $C \subseteq \mathbb{R}^{d+1}$ and a grading $a \in \mathbb{Z}^{d+1}$, recall that

$$h_C^a(n) := |\{m \in C \cap \mathbb{Z}^d : \langle a, m\rangle = n\}|$$

As a consequence of Stanley's reciprocity we got the following corollary.

Corollary Let $C \subseteq \mathbb{R}^{d+1}$ be a rational simplicial cone with generators $v_1, \dots, v_k$ and fundamental parallelepiped $\square = [0,1)v_1 + \dots + [0,1)v_k$. For a grading $a \in \mathbb{Z}^{d+1}$,

$$H_C^a(z) = \sum_{n \geq 0} h_C^a(z) z^n = \frac{H_\square^a(z)}{(1-z^{\langle a,v_1\rangle}) \cdots (1-z^{\langle a,v_k\rangle})}$$

and

$$H_C^a(\tfrac{1}{z}) = (-1)^k H_{C^\circ}^a(z)$$

The numerator $H_\square^a(z)$ counts the lattice points in $\square$ according to the grading a.

Now we are ready to prove the Ehrhart polynomial reciprocity for simplices.

Theorem (Ehrhart–Macdonald reciprocity for simplices)
Suppose $\Delta \subseteq \mathbb{R}^d$ is a lattice simplex. Then

(a) $\mathrm{ehr}_\Delta(n)$ agrees with a polynomial in n of degree $\dim(\Delta)$ and constant term 1.

(b) $(-1)^{\dim(\Delta)} \mathrm{ehr}_\Delta(-n) = |n\Delta^\circ \cap \mathbb{Z}^d| := \mathrm{ehr}_{\Delta^\circ}(n)$

Proof Let $\Delta = \mathrm{conv}\{v_1, \dots, v_{r+1}\}$ be an r-dimensional simplex in $\mathbb{R}^d$, and let $C = \mathrm{hom}(\Delta)$ with generators $w_i = (v_i, 1)$ for $i = 1, \dots, r+1$.
Let

$$\square = [0,1)w_1 + \dots + [0,1)w_{r+1}$$

be its associated fundamental parallelepiped.
Taking the grading $a = (0, \dots, 0, 1)$ we get

$$\mathrm{Ehr}_\Delta(z) = 1 + \sum_{n \geq 1} \mathrm{ehr}_\Delta(n) z^n = H_C^a(z) = \frac{H_\square^a(z)}{(1-z)^{r+1}} \qquad (*)$$

Where the numerator

$$h^*(z) := H_\square^a(z) = h_0^* + h_1^* z + \dots + h_r^* z^r$$

enumerates the lattice points in $\square$ according to their height.
In particular $\deg h^*(z) \leq r$ and $h^*(1) = |\square \cap \mathbb{Z}^{d+1}| \neq 0$.

Since $(*)$ is a rational generating function and $h^*(1) \neq 0$ then $ehr_\Delta(n)$ agrees with a polynomial in n of degree $r = \dim(\Delta)$.
The constant term is

$$ehr_\Delta(0) = H_c^a(0) = h^*(0) = 1.$$

This finishes the proof of (a).

For (b) we use the formal reciprocity for generating functions:

$$\sum_{n \geq 1} f(-n) z^n = -\sum_{n \geq 0} f(n) \left(\tfrac{1}{z}\right)^n$$

which in our case implies:

$$(-1)^r \sum_{n \geq 1} ehr_\Delta(-n) z^n = (-1)^{r+1} \sum_{n \geq 0} ehr_\Delta(n) \left(\tfrac{1}{z}\right)^n = (-1)^{r+1} Ehr_\Delta\left(\tfrac{1}{z}\right)$$

$$= Ehr_{\Delta^\circ}(z) \qquad \text{(by previous corollary!)}$$

$$:= \sum_{n \geq 1} ehr_{\Delta^\circ}(n) z^n$$

$\blacksquare$

The fact that here we get an open cone forces us to start this sum at $n=1$!

The previous theorems holds for lattice polytopes in general.

Theorem Let $P \subseteq \mathbb{R}^d$ be a lattice polytope. Then $ehr_P(n)$ agrees with a polynomial in n of degree $\dim(P)$ and

$$(-1)^{\dim(P)} ehr_P(-n) = |nP^\circ \cap \mathbb{Z}^d| := ehr_{P^\circ}(n).$$

Proof idea:
Consider a triangulation S of P (each cell is a simplex).

We can count integer points in nP by counting lattice points in each dilated simplex and taking care of the points that are overcounted:

$$ehr_P(n) = \sum_{F \in S} \clubsuit\, ehr_F(n) \qquad \clubsuit = \begin{cases} 0 & \text{if } F \text{ is a boundary face} \\ (-1)^{\dim P - \dim F} & \text{otherwise} \end{cases}$$

$$= \sum_{\substack{F \in S \\ F \not\subseteq \partial P}} (-1)^{\dim P - \dim F} ehr_F(n)$$

This is a polynomial in n of degree $\dim P$. Evaluating at $-n$:
$ehr_F(-n) = (-1)^{\dim F} ehr_{F^\circ}(n)$ and so

$$(-1)^P ehr_P(-n) = \sum_{\substack{F \in S \\ F \not\subseteq \partial P}} ehr_{F^\circ}(n) = |nP^\circ \cap \mathbb{Z}^d| \qquad \blacksquare$$

<u>Exercise</u> show that the constant term of $ehr_P(n)$ is $ehr_P(0) = \chi(P) = 1$.

- Order Polynomials Revisited

Given a finite poset $(\Pi, \leq)$ we defined

$$\Omega_\Pi(n) = |\{\phi : \Pi \to [n] \text{ order preserving}\}|,$$

$$\Omega^\circ_\Pi(n) = |\{\phi : \Pi \to [n] \text{ strictly order preserving}\}|.$$

We showed that these two counting functions are polynomials in n related by

$$\boxed{(-1)^{|\Pi|}\Omega^\circ_\Pi(-n) = \Omega_\Pi(n)}$$

This result is straightforward consequence of Ehrhart-Macdonald reciprocity.

To see this we define the order polytope of Π as

$$O_\Pi := \left\{ \phi \in \mathbb{R}^{|\Pi|} : \begin{array}{ll} 0 \leq \phi(p) \leq 1 & \text{for all } p \in \Pi \\ \phi(a) \leq \phi(b) & \text{for all } a \leq b \end{array} \right\}.$$

Proposition For a finite poset Π,

$$\Omega_\Pi(n) = \mathrm{ehr}_{O_\Pi}(n-1).$$

Proof For a map $\phi : \Pi \to [n]$ let $f : \Pi \to [n]$ defined by

$$\phi(p) = f(p) + 1.$$

Then, ϕ is order preserving iff $f \in (n-1)O_\Pi \cap \mathbb{Z}^{|\Pi|}$. ∎

Theorem $\boxed{(-1)^{|\Pi|}\Omega^\circ_\Pi(-n) = \Omega_\Pi(n)}$

Proof Similarly as above, lattice points in $(n+1)O^\circ_\Pi$ are in bijection with strictly order preserving maps $\phi : \Pi \to [n]$.

Therefore

$$\boxed{\Omega^\circ_\Pi(n) = \mathrm{ehr}_{O^\circ_\Pi}(n+1)}$$

By Ehrhart–Macdonald reciprocity:

$$\Omega^\circ_\Pi(-n) = \mathrm{ehr}_{O^\circ_\Pi}(-n+1) = (-1)^{|\Pi|}\mathrm{ehr}_{O_\Pi}(n-1) = (-1)^{|\Pi|}\Omega_\Pi(n)$$ ∎

Chromatic Polynomials Revisited

Given a finite graph $G=(V,E)$, an *n-coloring* of G is a map $c: V \to [n]$.
An n-coloring is called *proper* if $c(u) \neq c(v)$ whenever $uv \in E$.

The counting function

$$\chi_G(n) := \left| \{ c: V \to [n] \text{ proper n-coloring} \} \right|$$

is called the *chromatic polynomial* of G.

An orientation ρ and an n-coloring c of G are *compatible* if

$$c(u) \geq c(v) \quad \text{for every oriented edge } u \to v.$$

We proved the following the following theorem.

Theorem Let G be a finite graph on d nodes. Then

$$(-1)^d \chi_G(-n) = \# \text{compatible pairs } (\rho, c) \text{ of acyclic orientation } \rho \text{ and n-coloring } c \text{ of } G. \quad (*)$$

In particular, $(-1)^d \chi_G(-1) = \#$ acyclic orientations of G.

This also follows from Ehrhart-Macdonald reciprocity.
For this, recall that every acyclic orientation ρ of G can be viewed as a poset $\Pi(\rho G)$:

$$u \to v \text{ in } \rho G \iff u \geq v \text{ in } \Pi(\rho G).$$

Each proper n-coloring of G induces an orientation of G by orienting the edges from bigger to smaller color. Thus, we deduced

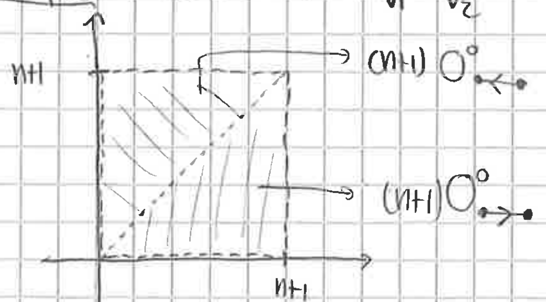
$$\boxed{\chi_G(n) = \sum_{\rho} \Omega^{\circ}_{\Pi(\rho G)}(n)} \quad \text{(sum of strictly order polynomials)}$$

where the sum is over all acyclic orientations of G.

Therefore

$$\boxed{\chi_G(n) = \sum_{\rho} \mathrm{ehr}_{O^{\circ}_{\Pi(\rho G)}}(n+1)}.$$

Example: Let $G = \overset{\bullet}{v_1} - \overset{\bullet}{v_2}$



In general, $\chi_G(n)$ counts integer lattice points in

$$(0, n+1)^V = \{ c \in \mathbb{R}^V : 0 < c(v) < n+1 \text{ for } v \in V \}$$

minus those points where $c(u) = c(v)$ for some edge $uv \in E$.

The order polytopes $O_{\Pi(\rho G)}$ form a subdivision of $[0,1]^V$.

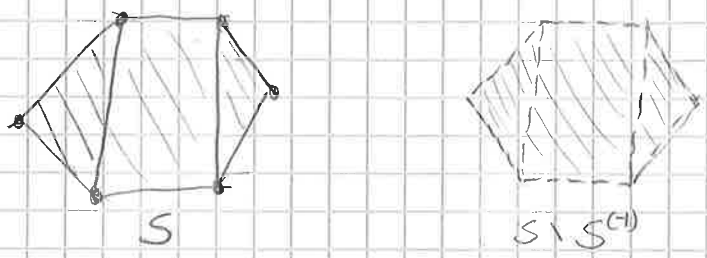
This motivates the following construction: For a polyhedral complex S, we write $S^{(-1)}$ for the subcomplex of faces $F \in S$ of dimension $\leq \dim S - 1$.

Define $\mathrm{ehr}_S^{[1]}(n)$ to be the number of lattice points in the n-th dilate of $|S| \setminus |S^{(-1)}|$, that is

$|S| = \bigcup_{F \in S} F$

$$\mathrm{ehr}_S^{[1]}(n) := \mathrm{ehr}_S(n) - \mathrm{ehr}_{S^{(-1)}}(n) = \sum_{\substack{F \in S \\ \dim F = \dim S}} \mathrm{ehr}_{F^\circ}(n)$$

Example



Proposition Let S be a pure complex of lattice polytopes. Then

$$(-1)^{\dim S} \mathrm{ehr}_S^{[1]}(-n)$$

is the number of lattice points in $n|S|$, each counted with multiplicity equal to the number of maximal cells containing the point.

Proof: By Ehrhart-Macdonald reciprocity

$$(-1)^{\dim S} \mathrm{ehr}_S^{[1]}(-n) = \sum_{\substack{F \in S \\ \dim F = \dim S}} \mathrm{ehr}_F(n)$$

each point in $|S| = \bigcup_{F \in S} F$ is counted with multiplicity equal to the number of maximal cells containing that point. ∎

Proof of Theorem (*) Consider the polyhedral subdivision

$$S := \{ O_{\pi(\rho G)} : \rho \text{ acyclic} \} \quad \text{of} \quad [0,1]^V.$$

Then $$\chi_G(n) = \mathrm{ehr}_S^{[1]}(n+1).$$

Therefore

$$(-1)^{|V|} \chi_G(-n) = (-1)^{|V|} \mathrm{ehr}_S^{[1]}(-n+1)$$

which counts the number of lattice points in $(n-1)[0,1]^V = [0, n-1]^V$, each weighted by the number of order polytopes $O_{\pi(\rho G)}$ containing the point. This is equal to the number of n-colorings c of G, each weighted by the number of compatible acyclic orientations. ∎

Finally, you are invited to read a geometric proof of the reciprocity for flows on graphs in the book (Section 7.6) (Not needed for the oral exam).