Exercise 1

The Catalan number $c_n = \frac{1}{n+1}\binom{2n}{n}$ counts the number of elements of the following families:

(i) rooted plane binary trees with n internal nodes

(ii) Dyck paths inside an $n \times n$ rectangle.

(iii) 312-avoiding permutations

(iv) non-crossing partitions of $[n]$

(v) non-crossing perfect matchings of $[n]$

In Lecture 1, we discussed maps between the family of rooted plane binary trees and the other families. Show that these maps are bijections. Describe bijections between the family of Dyck paths and the other families.

Exercise 2

The Fuß-Catalan number $c_n(m) = \frac{1}{mn+1}\binom{(m+1)n}{n}$ counts the number of elements of the following families:

(i) rooted plane m-ary trees with n internal nodes
(each node has $m+1$ children)

(ii) m-Dyck paths inside an $mn \times n$ rectangle
(lattice paths using north and east steps from $(0,0)$ to (mn, n) that stay weakly above the main diagonal)

(iii) 312-avoiding Stirling m-permutations
(a multi-permuations of $\{1^m, \dots, n^m\}$ that avoids the patterns 212 and 312)

(iv) non-crossing m-divisible partitions of $[mn]$
(non-crossing partitions of $[mn]$ whose block sizes are divisible by m)

(v) non-crossing m-matchings of $[mn]$
(non-crossing partitions with blocks of size m)

Extend the bijections from Exercise 1 to bijections between these families. Show that their number of elements is counted by the Fuß-Catalan number.

Exercise 3

Let $(a_1, \dots, a_n)$ be a sequence of positive integers $a_i \in [n]$, and the let $b_1 \leq \cdots \leq b_n$ be the increasing rearrangement of $(a_1, \dots, a_n)$.

(i) Show that $(a_1, \dots, a_n)$ is a parking function if and only if $b_i \leq i$ for all i.

(ii) Show that the set of parking functions is closed under the action of the symmetric group (that acts by permuting the entries of the tuple $(a_1, \dots, a_n)$).

(iii) A labeled Dyck path in an $n \times n$ rectangle is a Dyck path whose vertical steps are labeled with the numbers $1, \dots, n$, such that the labels increase from bottom to top along columns. Show that parking functions of length n are in bijection with labeled Dyck paths in an $n \times n$ rectangle.

Exercise 4 (Optional)

Given a finite poset P (a partially ordered set) with a unique minimal element $\hat{0}$, the Möbius function

$$\mu : P \to \mathbb{Z}$$

is recursively defined by $\mu(\hat{0}) = 1$ and $\sum_{x \leq y} \mu(x) = 0$ for all $y \in P$.
The poset NC_{n+1} is the poset of non-crossing partitions of $[n+1]$ ordered by refinement.

(i) Show that the number of maximal chains of NC_{n+1} is the number of parking functions $(n+1)^{n-1}$.

(ii) Let $\hat{1} \in \mathrm{NC}_{n+1}$ be the maximal element. Show that $(-1)^n \mu(\hat{1}) = \frac{1}{n+1}\binom{2n}{n}$, the nth Catalan number.