Exercise 1

Let $\pi \in \mathrm{Dyck}(n)$ be a Dyck path of size n, and $a(\pi) = (a_1, \ldots, a_n)$ be its area vector. Show that the two following definitions of the dinv statistic coincide:

$$\mathrm{dinv}(\pi) = |\{(i,j) : i < j \text{ and either } a_i = a_j \text{ or } a_i = a_j + 1\}|$$

$$\mathrm{dinv}(\pi) = |\{B \text{ box above } \pi : \mathrm{arm}(B) = \mathrm{leg}(B) \text{ or } \mathrm{arm}(B) = \mathrm{leg}(B) + 1\}|$$

Exercise 2

Let $\zeta :, \mathrm{Dyck}(n) \to \mathrm{Dyck}(n)$ be the zeta map defined in Lecture 4. Show that:

(i) ζ is a bijection

(ii) $\mathrm{area}(\pi) = \mathrm{bounce}(\zeta(\pi))$

(iii) $\mathrm{dinv}(\pi) = \mathrm{area}(\zeta(\pi))$

Exercise 3

Verify the following equality (calculate explicitly) for $n \leq 4$:

$$\frac{1}{[n+1]_q} \begin{bmatrix} 2n \\ n \end{bmatrix}_q = \sum_{\pi \in \mathrm{Dyck}(n)} q^{\mathrm{area}(\pi) + \mathrm{codinv}(\pi)},$$

where $\mathrm{codinv}(\pi) := \binom{n}{2} - \mathrm{dinv}(\pi)$.