Course - Topics on combinatorics, algebra and geometry
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Lecture 3

Last time : q-analogs and space of harmonics

Today : The space of diagonal harmonics
q,t-analogs
q,t-rational Catalan combinatorics

Briefly recall space of harmonics

$\mathbb{Q}[x_1, \dots, x_n]$ ring of polynomials in $x_1, \dots, x_n$

$f \in \mathbb{Q}[x_1, \dots, x_n]$ is symmetric if

$$f(x_1, \dots, x_n) = f(x_{\sigma(1)}, \dots, x_{\sigma(n)}) \quad \text{for } \forall\, \sigma \in S_n$$

The space of harmonics is

$$H_n = \{ h \in \mathbb{Q}[x_1, \dots, x_n] : (\partial f)(h) = 0 \text{ for every symmetric polynomial } f \text{ with no constant term} \}$$

notation $\partial x_i := \frac{\partial}{\partial x_i}$

$$= \{ h \in \mathbb{Q}[x_1, \dots, x_n] : \sum_{i=1}^{n} (\partial x_i)^k h = 0 \text{ for } k \geq 1 \}$$

$$= \mathrm{sp}\{ \text{Vandermonde determinant } \prod_{1 \leq i < j \leq n} (x_i - x_j) \text{ and all its partial derivatives of all orders} \}$$

And $\boxed{\dim H_n = n!}$

Example $n=2$
$$H_2 = \mathrm{sp}\{ \underset{H_2^0}{1}, \underset{H_2^1}{x_1 - x_2} \}$$
$$\dim H_2 = 2! = 2$$

Moreover, H_n can be decomposed into homogeneous components.

$$H_n = \bigoplus_{i=0}^{\binom{n}{2}} H_n^i \longrightarrow \text{homogeneous polynomials of degree } i$$

The q-Hilbert series of H_n is

$$\boxed{\mathrm{Hilb}_{H_n}(q) := \sum_{i=0}^{\binom{n}{2}} \dim(H_i)\, q^i}$$

In our example for $n=2$.

$$\mathrm{Hilb}_{H_2}(q) = \dim(H_0)\cdot q^0 + \dim(H_1)\cdot q^1$$
$$= 1 + q$$

In general

$$\boxed{\mathrm{Hilb}_{H_n}(q) = [n]_q!}$$

The space of diagonal harmonics

Let $\mathbb{Q}[x_1,\dots,x_n; y_1,\dots,y_n]$ be the ring of polynomial in two sets of variables $x_1,\dots,x_n$ and $y_1,\dots,y_n$.

A polynomial $f \in \mathbb{Q}[x_1,\dots,x_n; y_1,\dots,y_n]$ is called (diagonally) symmetric if

$$f(x_1,\dots,x_n,y_1,\dots,y_n) = f(x_{\sigma(1)},\dots,x_{\sigma(n)},y_{\sigma(1)},\dots,y_{\sigma(n)}) \quad \text{for all } \sigma \in S_n.$$

Examples $f_1 = x_1 + x_2$ and $f_2 = y_1 + y_2$ are symmetric for $n=2$

The space of diagonal harmonics is

$$DH_n := \{ h \in \mathbb{Q}[x_1,\dots,x_n,y_1,\dots,y_n] : (\partial f)(h) = 0 \text{ for every symmetric polynomial } f \text{ with no constant term} \}$$

$$= \{ h \in \mathbb{Q}[x_1,\dots,x_n,y_1,\dots,y_n] : \sum_{i=1}^{n} (\partial x_i)^r (\partial y_i)^s h = 0 \quad \text{for } r+s>0 \}$$

Example $n=2$

$$DH_2 = \{ h \in \mathbb{Q}[x_1,x_2,y_1,y_2] : \left((\partial x_1)^r(\partial y_1)^s + (\partial x_2)^r(\partial y_2)^s \right) h = 0 \quad \text{for } r+s>1 \}$$

In particular:

$$\left.\begin{array}{lll} r=1, s=0 \Rightarrow & (\partial x_1 + \partial x_2)\, h & = 0 \\ r=0, s=1 \Rightarrow & (\partial y_1 + \partial y_2)\, h & = 0 \\ r=1, s=1 \Rightarrow & (\partial x_1 \partial y_1 + \partial x_2 \partial y_2)\, h & = 0 \end{array}\right\} \overset{\text{Exercise}}{\Rightarrow} h \in \mathrm{sp}\{1, x_1 - x_2, y_1 - y_2\}$$

Therefore

$$DH_2 = sp\{1,\ x_1-x_2,\ y_1-y_2\}$$
$$\dim DH_2 = 3$$

In general

$$\boxed{\dim DH_n = (n+1)^{n-1}}$$

This was know as the $(n+1)^{n-1}$ conjecture a was proved by Haiman in 2002.

Moreover, DH_n can be decompomposed into homogeneus components

$$DH_n = \bigoplus_{i,j} DH_n^{i,j}$$

Where $DH_n^{i,j}$ consists of the polynomial of degree i and j in the variables x and y, respectively.

In our example:

$$DH_2 = DH_2^{0,0} \oplus DH_2^{1,0} \oplus DH_2^{0,1}$$
$$= sp\{1\} \oplus sp\{x_1-x_2\} \oplus sp\{y_1-y_2\}.$$

The <u>q,t-Hilbert series</u> of DH_n is

$$\boxed{\mathrm{Hilb}_{DH_n}(q,t) := \sum_{i,j} \dim(DH_n^{i,j})\, q^i t^j}$$

<u>For $n=2$</u>:

$$\boxed{\mathrm{Hilb}_{DH_2}(q,t) = 1 + q + t}$$

This is counting parking functions with some weights / statatistics on q and t.

We want to understand what these statistics are combinatorially!

<u>Remarkable Known Facts</u>

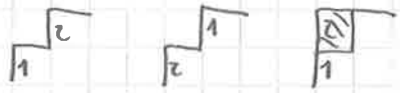
- $\mathrm{Hilb}_{DH_n}(q,0) = [n]_q!$
- $q^{\binom{n}{2}}\mathrm{Hilb}_{DH_n}(q,q^{-1}) = [n+1]_q^{\,n-1}$
- $\mathrm{Hilb}_{DH_n}(q,1) = \sum_{P\in \mathrm{Park}(n)} q^{\mathrm{area}(P)}$

→ Contains the q-analogs from Lecture 2.!

one statistic! what is the other?

Example $n=2$ $\quad \mathrm{Hilb}_{DH_2}(q,t) = 1+q+t$.

- $\mathrm{Hilb}_{DH_2}(q,0) = 1+q = [2]_q!$
- $q^{\binom{2}{2}}\mathrm{Hilb}_{DH_2}(q,q^{-1}) = q(1+q+q^{-1})$
$$= q+q^2+1$$
$$= [3]_q^1$$
- $\mathrm{Hilb}_{DH_2}(q,1) = 1+q+1$
$$= 2+q = \sum_{P\in \mathrm{Park}(2)} q^{\mathrm{area}(P)}$$



1 $\quad$ 1 $\quad$ q

Exercise Verify these equalities for $n=3$

In order to verify this one needs to compute DH_3 explicitly. This is not an easy task but can be done using a result by Mark Haiman called Operator Theorem.

Haiman's Operator Theorem

For $p>0$, define the polarization operator as the partial differential operator
$$E_p = \sum_{i=1}^{n} y_i(\partial x_i)^p$$

Example $n=2, p=1$
$$E_1 = y_1\partial x_1 + y_2\partial x_2$$

This can be applied to any other polynomial, for example
$$\left.\begin{aligned} E_1(x_1) &= y_1 \\ E_1(x_2) &= y_2 \end{aligned}\right\} \quad E_1(x_1-x_2) = y_1-y_2$$

Operator Theorem (Haiman 2002, conjectured by Haiman 1994)

The space DH_n is the smallest space containing the Vandermond determinant $\prod_{i<j}(x_i-x_j)$ that is closed under polarizations E_p for $p\geq 1$ and ∂x_i for all i.

↳ Using properties of the Hilber scheme $\mathrm{Hilb}^n(\mathbb{C}^2)$.

<u>Example $n=2$</u>

Vandermond determinant : $\Delta = x_1 - x_2$

Polarizations $E_1\Delta = y_1 - y_2$

$E_p\Delta = 0$ for $p>1$

Derivatives $\partial x_1 \Delta = 1$ higher derivatives $=0$

$\partial x_2 \Delta = -1$

Therefore.

$$DH_2 = sp\{x_1 - x_2,\ y_1 - y_2,\ 1\}$$

<u>Exercise</u>. Use Haiman operator theorem to:

i) Compute DH_3 explicity

ii) Show that $\dim(DH_3) = 4^2 = 16$ $\Rightarrow$ # parking functions.

iii) Show that

$$\boxed{Hilb_{DH_3}(q,t) = 1 + 2q + 2t + 2q^2 + 3qt + 2t^2 + q^3 + q^2t + qt^2 + t^3}$$

t^3	1			
t^2	2	1		
t	2	3	1	
1	1	2	2	1
	1	q	q^2	q^3

$\Rightarrow$ Symmetric in q,t

What about Catalan numbers?

<u>The sub-space of alternants</u>

We denote by $\sigma = (i,j)$ the permutation that exchanges i and j

For <u>example</u> for $n=3$ $(1,2) = [2,1,3]$

$(1,3) = [3,2,1]$

We say that $h \in DH_n$ is an <u>alternant</u> if

$$\boxed{(i,j)\circ h = -h.} \quad \text{for all } i \neq j$$

We denote by Alt_n the <u>subspace of alternants</u> of DH_n.

Example $n=2$

$$\boxed{\begin{aligned} DH_2 &= sp\{1, x_1-x_2, y_1-y_2\} \\ Alt_2 &= sp\{x_1-x_2, y_1-y_2\}\end{aligned}}$$

Indeed

$$(1,2)\circ(x_1-x_2) = x_2-x_1 = -(x_1-x_2)$$

$$(1,2)\circ(y_1-y_2) = y_2-y_1 = -(y_1-y_2)$$

but $\quad (1,2)\circ(1) \neq -1$

In this case, we have

$$\boxed{\dim Alt_2 = 2} \qquad \boxed{\dim DH_2 = 3}$$

In general

$$\boxed{\dim Alt_n = C_n = \frac{1}{n+1}\binom{2n}{n}}$$ the nth Catalan number !

$$\boxed{\dim DH_n = (n+1)^{n-1}}$$ Number of parking functions !

Similarly as above, Alt_n decomposes into homogeneous components:

$$\boxed{Alt_n = \bigoplus_{i,j} Alt_n^{i,j}}$$

homogeneous component of degrees i and j in the variable x and y, respectively.

The q,t-Hilbert series of Alt_n is

$$Hilb_{Alt_n}(q,t) = \sum_{i,j} \dim(Alt_n^{i,j})\, q^i t^j$$

Example $n=2$

$$\boxed{\begin{aligned} Alt_2 &= Alt_2^{1,0} \oplus Alt_2^{0,1} \\ &= sp\{x_1-x_2\} \oplus sp\{y_1-y_2\}.\end{aligned}}$$

$$\boxed{Hilb_{Alt_2} = q + t}$$ → Counts Catalan objects
What are the statistics ?

Remarkable known facts

- $q^{\binom{n}{2}} Hilb_{Alt_n}(q,q^{-1}) = \frac{1}{[n+1]_q}\begin{bmatrix} 2n \\ n\end{bmatrix}_q$ → Contains the q-analogs from lecture 2. !
- $Hilb_{Alt_n}(q,1) = \sum_{\pi\in Dyck(n)} q^{area(\pi)}$ → one statistic. what is the other.

Example $n=2$ $\qquad \mathrm{Hilb}_{\mathrm{Alt}_2}(q,t) = q+t$.

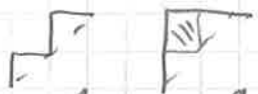
- $q^{\binom{2}{2}} \mathrm{Hilb}_{\mathrm{Alt}_2}(q,q^{-1}) = q(q+q^{-1})$

$$= q^2+1$$

$$= \frac{1}{[3]_q}\begin{bmatrix}4\\2\end{bmatrix}_q = \frac{1+q+q^2+q^3}{1+q}$$

- $\mathrm{Hilb}_{\mathrm{Alt}_2}(q,1) = q+1$

$$= \sum_{\pi \in \mathrm{Dyck}(2)} q^{\mathrm{area}(\pi)}$$



1 $\qquad$ q

Exercise Verify these equalities for $n=3$.

The q,t-Catalan number (polynomial) is defined as

$$\boxed{C_n(q,t) = \mathrm{Hilb}_{\mathrm{Alt}_n}(q,t)}$$

By construction. It is a symmetric polynomial in q,t.

Exercise: i) Compute Alt_3 explicity

ii) show that $\boxed{C_3(q,t) = q^3+q^2t+qt+qt^2+t^3}$

The purpose of next lecture is to provide combinatorial models for the q,t-Catalan polynomials.

$$\boxed{C_n(q,t) = \sum_{\pi\in \mathrm{Dyck}(n)} q^{\mathrm{area}(\pi)} t^{\mathrm{dinv}(\pi)}}$$

$\to$ dinv statistic and more...