Course - Topics on combinatorics, algebra and geometry
24.11.2023
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Lecture 5

Last time : $q,t$-Catalan combinatorics
area, dinv, bounce and zeta map

Today : rational $q,t$-Catalan combinatorics
simultaneous $(a,b)$-core partitions
rational $(a,b)$-Dyck paths.
skew length statistic ($\ell$ = codinv)

- Simultaneous core partitions

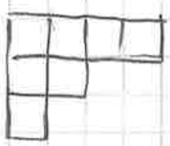
A partition of $n \in \mathbb{N}$ is a sequence $\lambda = (\lambda_1, \dots, \lambda_k)$ of positive integers such that

- $n = \lambda_1 + \dots + \lambda_k$
- $\lambda_1 \geq \lambda_2 \geq \dots \geq \lambda_k > 0$

Example $\lambda = (4,2,1)$ is a partition of 7

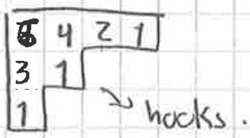
It is convenient to represent partitions by their Ferrers diagram

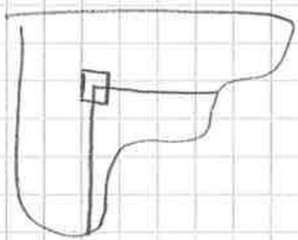
$\lambda = (4,2,1)$



$\rightarrow \lambda_i$ boxes in row $i$ (from top to bottom)

The hook number of a box is the number of boxes directly to the right or below the box including itself.



For $t \in \mathbb{N}$, a partition $\lambda$ is called a $t$-core if its Ferrers diagram has no hook equal to $t$.

The previous example is a 5-core, an 8-core, but not a 7-core.
A partition is called an $(a,b)$-core if it is simultaneously an $a$-core and a $b$-core

**Theorem (Anderson '02)**

If $a,b \in \mathbb{N}$ are relatively prime then the number of $(a,b)$-cores is equal to

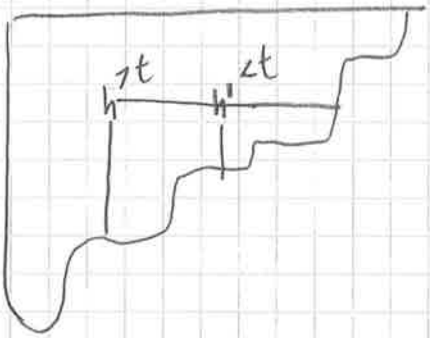
$$\frac{1}{a+b}\binom{a+b}{a}$$

<u>Exercise</u> Show that the number of $(a,b)$-cores is finite if and only if $a$ and $b$ are relatively prime.

In order to prove this theorem, the following lemma is useful.

<u>Lemma</u> Let $\lambda$ be a $t$-core. If $h > t$ is a hook in the Ferrers diagram of $\lambda$ the $h-t$ is also a hook.

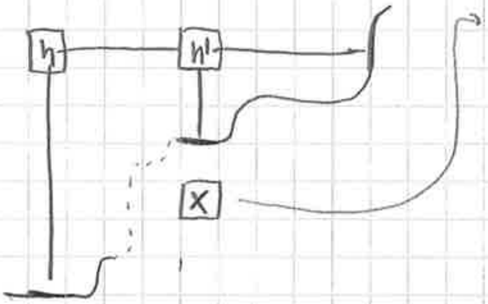
<u>Proof</u>



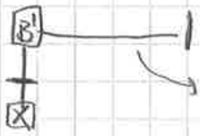

Let $B$ a box with hook number $h>t$

The box directly on the right of $B$ belongs to the diagram, otherwise there would be a box below $B$ with hook number $t$.

Let $h'$ be the biggest hook smaller than $t$ in the row of $B$, and $B'$ be the box having this hook.




let $X$ be the box below $B'$ such that



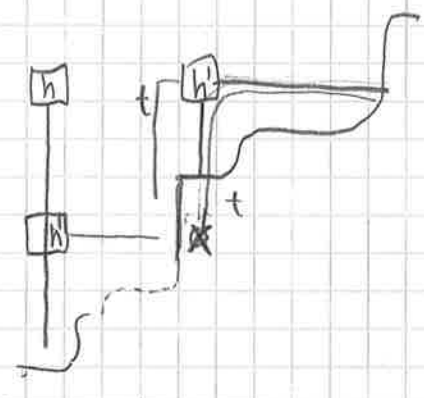

length of this hook $= t$

Claim: the box directly on the left of $X$ belongs to the Ferrers diagram

Otherwise the hook number of the box directly on the left of $B'$ would be $\leq t$.

Let $B''$ be the box below $B$ to the left of $X$

Claim: hook of $B''$ is $h-t$.

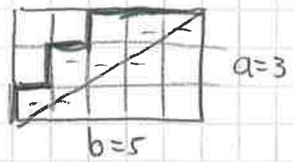



<u>Corollary</u> A partition $\lambda$ is a $t$-core iff its Ferrers diagram has not hooks divisible by $t$.

## • Anderson's bijection with $(a,b)$-Dyck paths

Let $a,b \in \mathbb{N}$ be relatively prime , $a<b$

An $(a,b)$-Dyck path is a lattice path from $(0,0)$ to $(b,a)$ using north and east steps that stays weakly above the diagonal.

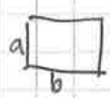



__Exercise__ : Show that if $(a,b)=1$ (relatively prime) then

- the number of $(a,b)$-Dyck paths is equal to
$$\frac{1}{a+b}\binom{a+b}{a}$$
- the expression
$$\frac{1}{[a+b]_q}\begin{bmatrix} a+b \\ a \end{bmatrix}_q$$
is a polynomial in $q$.

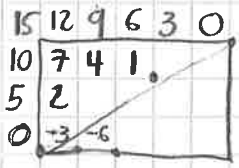
Hint : use $[a]_q\begin{bmatrix} a+b \\ a \end{bmatrix}_q = [a+b]_q\begin{bmatrix} a+b-1 \\ a-1 \end{bmatrix}$

and the fact that $[a]_q$ and $[a+b]_q$ have no roots in common.

We add labels to the boxes in the $a \times b$ box  as follows

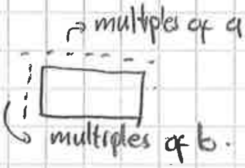


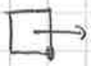
$a=3$
$b=5$




- start with 0 in the left-bottom corner
- every time you move up add $b$
- every time you move right substract $a$

The boxes above the diagonal are positive
The boxes below the diagonal are negative




 The label is measuring how "far" you are from the diagonal.

__Anderson's Bijection__

4 implies that $1=4-3$ is there.


$(3,5)$-core $\qquad$ $(3,5)$-Dyck path

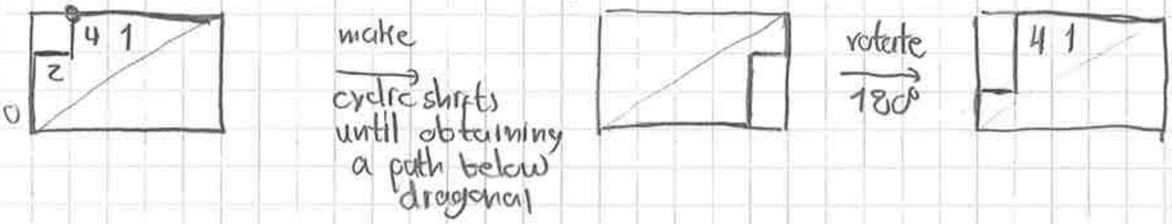
Take the $(a,b)$-Dyck path whose labels below are the hooks of the first column.

The result is an $(a,b)$-Dyck path because of the Lemma. ∎

Conjugation : A nice involution on (a,b)-cores and (a,b)-Dyck paths.

- Conjugation on (a,b)-cores


- Conjugation on (a,b)-Dyck paths




Exercise  Show that conjugation on (a,b)-cores corresponds to conjugation on (a,b)-Dyck paths under Anderson's bijection.

The skew length statistic  ( Based on Armstrong-Hanusa-Jones '14 )

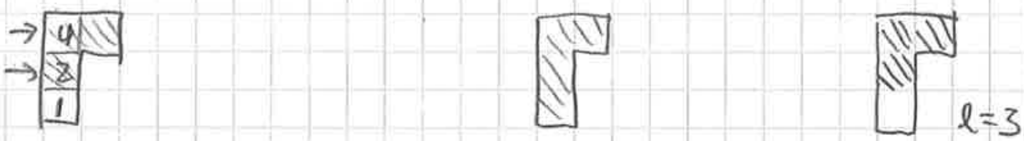
Given an (a,b)-core $\lambda$, we define.

a-rows : rows with largest hooks of each residue mod a in first column

b-boundary : boxes with hooks less than b.

example

$a=3$
$b=5$




a-rows : 1(4) = 1 mod (3)
2

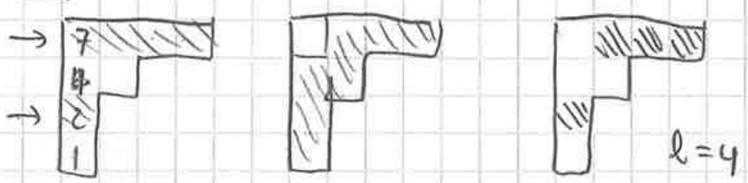
b-boundary
hooks < 5

$sl = 3$

skew length := # boxes in both the a-rows and b-boundary =: $sl$

Other example

$a=3$
$b=5$




a-rows    b-boundary    $sl = 4$

coskew length = $sl' := \frac{(a-1)(b-1)}{2} - sl$

$\frac{(a-1)(b-1)}{2}$ = # boxes in a×b above diagonal.

The length $\ell(\lambda)$ = # rows

What happens to skew length under conjugation. or changing a,b ? b,a ?

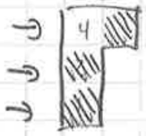
**Theorem** (C.-Dexter-Hanusa, Xin)
(i) The skew length is preserved under conjugation
(ii) The skew length is independent of the order of a and b.

**Example** $a=3$, $b=5$

(i)  $sl=3$ $\xrightarrow{\text{conjugate}}$  $sl=3$ same skew length!

a-rows and b-boundary — a-rows and b-boundary

(ii) $b=3$ $a=5$

 $sl=3$ Same skew length as above!

a-rows and b-boundary

The following results were conjectured by Armstrong-Hanusa-Jones and follow from Mellit's rational shuffle theorem.

**Theorem** For a,b coprime.

(i) $$\frac{1}{[a+b]_q}\begin{bmatrix} a+b \\ a \end{bmatrix}_q = \sum_{\lambda\ (a,b)\text{-core}} q^{\ell(\lambda)+sl(\lambda)}$$

(ii) $$\sum_{\lambda} q^{\ell(\lambda)} t^{sl'(\lambda)} = \sum_{\lambda} t^{\ell(\lambda)} q^{sl'(\lambda)}$$ (symmetry!)

**Exercise** - Verify these equalities for $a=3, b=5$ (compute explicitly)
- Is there a simple proof of (i)?
- A combinatorial proof of (ii) is a big open problem.

**Note:** Under Anderson's bijection $(a,b)$-core $\lambda \longleftrightarrow$ $(a,b)$-Dyck path $\pi$

$$\boxed{\begin{aligned} \ell(\lambda) &= \operatorname{area}(\pi) \\ sl'(\lambda) &= \operatorname{dinv}(\pi) \end{aligned}}$$ $\longrightarrow$ generalized dinv for rational Dyck paths (= dinv in classical case)