Course: Topics on combinatorics, algebra and geometry
15.12.2023
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Lecture 7

So far in the course:
- Three nice combinatorial sequences
  $n!$, $C_n = \frac{1}{n+1}\binom{2n}{n}$, $(n+1)^{n-1}$
- Motivated from diagonal harmonic spaces in representation theory
- Beautiful combinatorics
  - q,t-catalan
  - zeta map
  - a rational generalizations.

Goal for the rest of the course: Explore further nice connections to combinatorics and geometry.
- Posets / lattices
- Polytopes
- Hyperplane arrangements

Today: Partially ordered sets.

- Partially ordered set (Poset)

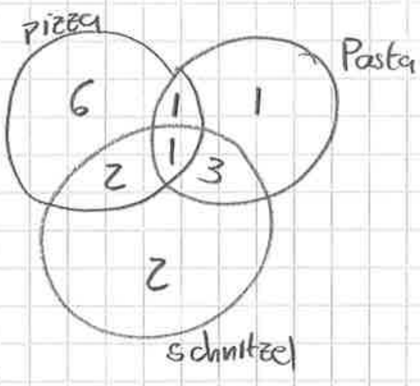
- A warming up problem

A survey about food preferences to a group of students gave the following resuls:

- 10 students like pizza
- 8 " " schnitzel
- 6 " " pasta

- 3 like pizza & schnitzel
- 4 schnitzel & pasta
- 2 pasta & pizza

- 1 likes all three



How many students participated in the survey?

Answer: $1 + (1+2+3) + (1+2+6) = 16. = (10+8+6) - (3+4+2) + 1$

This is an instance of the inclusion-exclusion principle.
An elegant generalization is the Möbius inversion formula for posets.

- **Posets**

A <u>poset P</u> is a set with a binary relation $\leq$ satisfying

i) For $t \in P$, $t \leq t$ (reflexivity)

ii) If $s \leq t$ and $t \leq s$, then $s = t$ (antisymmetry)

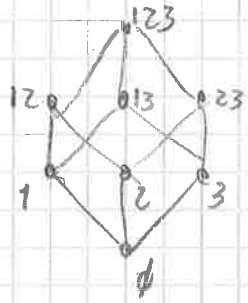
iii) If $s \leq t$ and $t \leq u$, the $s \leq u$ (transitivity)

<u>Examples</u> (1) For $n \in \mathbb{N}$, the set $[n]$ with the usual order

$n=$

(2) The boolean poset $B_n = 2^{[n]}$ of all subsets of $[n]$ ordered by containment:

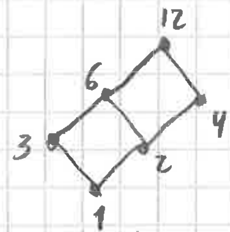
$S \leq T$ in B_n if $S \subseteq T$.



$n=3$

(3) The set D_n of all divisors of n where

$i \leq j$ in D_n if i divides j



$n=12$

We usually draw the Hasse diagram of P, a graph whose vertices are the elements of P and edges are the cover relations $x \lessdot y$ putting x below y.

- **Lattices**

There is an important class of posets known as lattices.

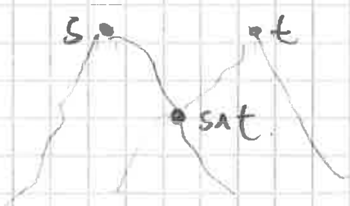
A poset P is called a <u>lattice</u> if for every $s, t \in P$

(i) The set of elements greater than or equal to s and t has a minimum element which we denote by

$s \vee t$ (the <u>join</u>)

ii) The set of element less than or equal to s and t has a maximum element which we denote by

$s \wedge t$ (the <u>meet</u>)



The following poset is not a lattice 

The other three previous examples are lattices

(1) $([n], \leq)$

$$a \vee b = \max\{a,b\}$$
$$a \wedge b = \min\{a,b\}.$$

(2) $(2^{[n]}, \subseteq)$

$$S \vee T = S \cup T \qquad \text{union} \qquad \{1,2\} \vee \{2,3\} = \{1,2,3\}$$
$$S \wedge T = S \cap T \qquad \text{intersection} \qquad \{1,2\} \wedge \{2,3\} = \{2\}$$

(3) $(D_n, \text{divisibility})$

$$a \vee b = \mathrm{lcm}(a,b) \qquad \text{least common multiple} \qquad 6 \vee 4 = 12$$
$$a \wedge b = \gcd(a,b) \qquad \text{greatest common divisor} \qquad 6 \wedge 4 = 2$$

- <u>The incident algebra</u>

Let P be a finite poset. Fix $\mathbb{K}$ a field.

Denote by $\mathrm{Int}(P)$ the set of intervals of P : $[s,t]$ for $s \leq t$

$$[s,t] = \{u \in P : s \leq u \leq t\}$$

The <u>incident algebra</u> $I(P)$ of P over $\mathbb{K}$ is the $\mathbb{K}$-algebra of all functions ($\mathbb{K}$-algebra: vector space with multiplication)

$$f : \mathrm{Int}(P) \to \mathbb{K}$$

where the multiplication is defined by

$$\boxed{fg(s,u) = \sum_{s \leq t \leq u} f(s,t)\, g(t,u)}$$

This is an associative algebra with (two sided) identity δ defined by

$$\boxed{\delta(s,t) = \begin{cases} 1, & \text{if } s = t \\ 0, & \text{otherwise} \end{cases}}$$

There are two important functions in $I(P)$: the zeta function
the Möbius function

- The <u>zeta function</u> ζ is defined by

$$\boxed{\zeta(t,u) = 1 \quad \text{for all } t \leq u \text{ in } P}$$

Therefore

$$\zeta^2(s,u) = \sum_{s \leq t \leq u} \underbrace{\zeta(s,t)}_{1} \underbrace{\zeta(t,u)}_{1} = \sum_{s \leq t \leq u} 1$$

$$= |[s,u]| \qquad \text{number of elements in the interval } [s,u]$$

More generally

$$\boxed{\begin{aligned} \zeta^k(s,u) &= \sum_{s=s_0 \leq s_1 \leq \dots \leq s_k = u} 1 \\ &= \#\ \text{multi-chains of length } k \text{ from } s \text{ to } u \end{aligned}}$$

with possible repetitions

<u>Similarly</u>

$$\boxed{(\zeta - 1)(s,u) = \begin{cases} 1, & \text{if } s < u \\ 0, & \text{if } s = u \end{cases}}$$

and

$$\boxed{(\zeta-1)^k(s,u) = \#\ \text{of chains } s = s_0 < s_1 < \dots < s_k = u \text{ of length } k}$$

- The <u>Möbius function</u> is the inverse of the zeta function. :

$$\mu \zeta = \delta \qquad (\text{Also } \zeta \mu = \delta)$$

This is equivalent to

$$\boxed{\mu(s,s) = 1}$$

$$\sum_{s \leq u \leq t} \mu(s,u) = 0 \qquad \text{for } s < t$$

equivalent to

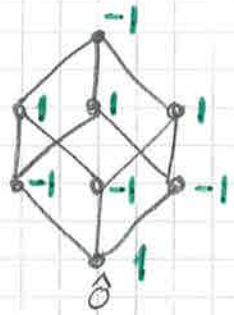
$$\boxed{\mu(s,t) = -\sum_{s \leq u < t} \mu(s,u)}$$

sum of $\mu(s,u)$ here should be zero

Examples

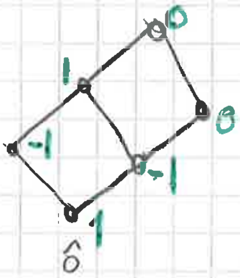
(1) 

$\mu(1,3) = 0$
$\mu(1,2) = -1$ because $-1+1 = 0$
$\mu(1,1) = 1$

(2) 

$\mu(\hat{0}, *) = ?$

Exercise: Boolean poset B_n : $\mu(S,T) = (-1)^{|T \setminus S|}$

(3) 

$\mu(\hat{0}, *) = ?$

Exercise: Poset D_n of divisors of n

$$\mu(r,s) = \begin{cases} (-1)^t, & \text{if } s/r \text{ is a product of } t \text{ distinct primes} \\ 0 & \text{otherwise} \end{cases}$$

So $\mu(r,s)$ is the classical number-theoretic Möbius function $\mu(s/r)$

- The Möbius inversion formula

Theorem Let P be a finite poset and $f, g : P \to \mathbb{K}$. Then

$$g(t) = \sum_{s \le t} f(s) \quad \text{for all } t \in P$$

if and only if

$$f(t) = \sum_{s \le t} g(s)\, \mu(s,t) \quad \text{for all } t \in P$$

<u>Proof</u> Let $f: P \to \mathbb{K}$ and $\xi \in I(P)$

The incident algebra $I(P)$ acts (on the right) on the vector space of function from $P \to \mathbb{K}$ by

$$(f\xi)(t) = \sum_{s \le t} f(s)\,\xi(s,t)$$

The Möbius inversion formula is equivalent to

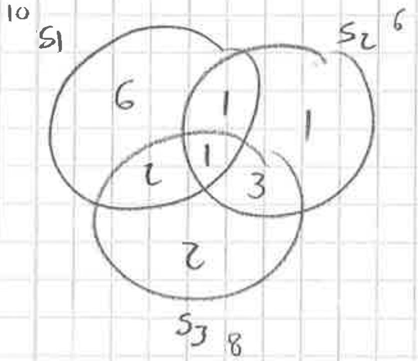
$$f \cdot Z = g \iff f = g\mu$$

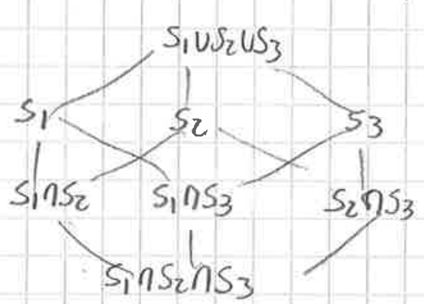
which follows from $Z^{-1} = \mu$ ∎

• Back to the warming up pizza-schnitzel-pasta problem (inclusion-exclusion)

Let $S_1, \dots, S_n$ be finite sets and P be the poset whose element are all intersections and the total union $S_1 \cup \dots \cup S_n$, ordered by inclussion

$g(S_1) = 10$
$f(S_1) = 6$





$g(S_1 \cup S_2 \cup S_3) = \boxed{16}$
$f(S_1 \cup S_2 \cup S_3) = 0$

For $T \in P$, let $g(T) = |T|$ be the number of elements in T.

We want to compute $g(\hat{1})$.

Let $f(T)$ be the number of elements of T that belong to no $T' < T$ in P

<u>Therefore</u>
$$g(T) = \sum_{T' \le T} f(T')$$

Using Möbius inversion

$$0 = f(\hat{1}) = \sum_{T \in P} g(T)\,\mu(T,\hat{1})$$

$$\Rightarrow \quad g(\hat{1}) = -\sum_{T < \hat{1}} |T|\,\mu(T,\hat{1})$$

In our example: $g(\hat{1}) = (10+6+8) - (2+3+4) + (1) = \boxed{16}$