Course: Topics on Combinatorics, algebra and geometry

12.1.2024

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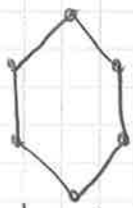
Lecture 8

Last time: Posets and lattices
Incident algebra
Zeta function and Möbius function

Today: Some nice families of lattices:
- Weak order
- Tamari lattice.
- lattice of partitions
- lattice of non-crossing partitions

Weak order

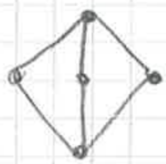


\# elements $= n!$

Tamari lattice

\# elements $= C_n = \frac{1}{n+1}\binom{2n}{n}$

Lattice of partitions



$\mu(\hat{0},\hat{1}) = (-1)^{n-1}(n-1)!$

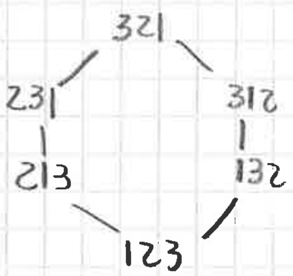
Lattice of noncrossing partitions

$\mu(\hat{0},\hat{1}) = (-1)^{n-1} C_{n-1}$

Nice zeta polynomial: Fuß-Catalan numbers

• <u>The weak order on permutations of $[n]$</u>

Fix $n \in \mathbb{N}$ and $[n] = \{1,2,\dots,n\}$



Elements of the poset = permutations $i_1 i_2 \dots i_n$ of $[n]$

Cover relations:

$$\dots ab \dots \lessdot \dots ba \dots \quad \text{for } a<b$$

Known fact: this defines a lattice.

Question: what is the join of

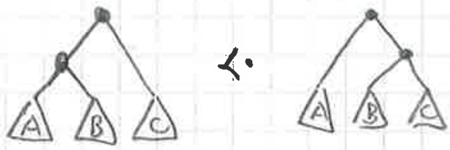
$$2314 \vee 1342 = ?$$

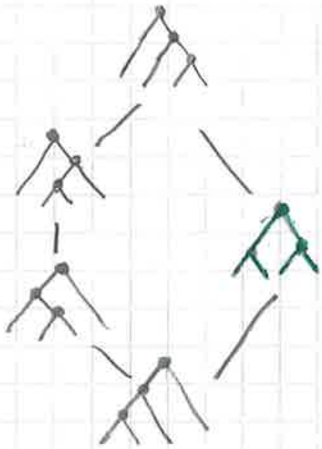
- <u>The Tamari lattice.</u> Fix $n \in \mathbb{N}$

Several equivalent descriptions:

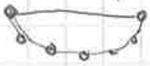
(i) On plane binary trees with n internal nodes (each node has two children).

Cover relations: tree rotation

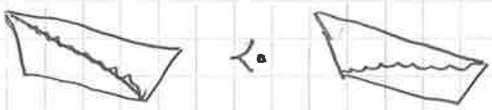


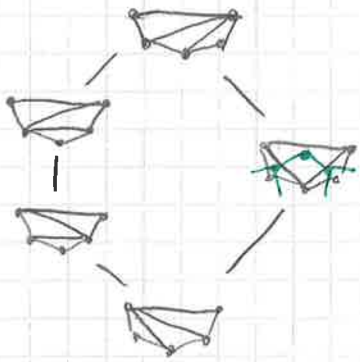


(ii) On triangulations of a convex $(n+2)$-gon



Cover relations: increasing slope diagonal flips



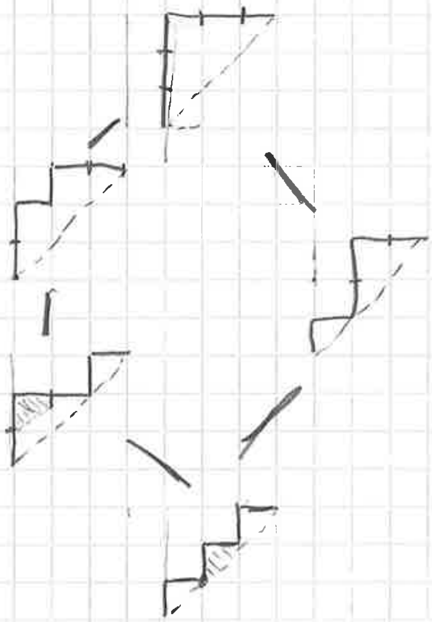


(iii) On Dyck paths on an $n \times n$ square.

Cover relations: for each valley EN

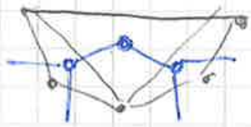
laser in direction of the diagonal until hitting the path again

"slide" P to the left.

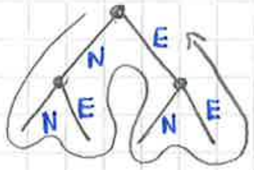


Poset isomorphisms :

(i) $\leftrightarrow$ (ii) dual tree of the triangulation



(i) $\to$ (iii)



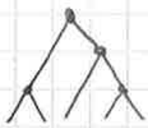
read labels in preorder

NENNEE

label edges N or E on the right according to their direction

Known fact : The Tamari poset is a lattice.

Question : what is the meet of

 $\wedge$  ?

 $\wedge$  ?

<u>Remark</u> : The Tamari lattice Tam(n) is important from the point of view of diagonal harmonics.

elements of Tam(n) = C_n = dimension of the space of alternants in diagonal harmonics (two sets of variables $x_1, \dots, x_n, y_1, \dots, y_n$)

intervals of Tam(n) = $\frac{2}{n(n+1)}\binom{4n+1}{n-1}$ (Chapoton 2006) Conjectured observed by Haiman 1994 to be the dimension of the space of alternants in trivariate diagonal harmonics (three sets of variables $x_1, \dots, x_n, y_1, \dots, y_n, z_1, \dots, z_n$)

This conjecture is widely open.

- <u>The lattice of partitions P_n</u>

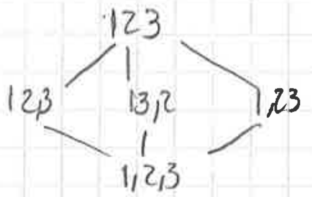
Elements are partitions $\{B_1, \dots, B_k\}$ of $[n]$,
that means $[n] = B_1 \sqcup B_2 \sqcup \dots \sqcup B_k$ a disjoint union into blocks.

They are ordered by refinement:

$$\{B_1, \dots, B_k\} \preceq \{B_1', \dots, B_\ell'\}$$

if each block B_i' is the union of some B_j's

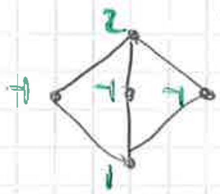
<u>n=3</u>:



→ Top has just one block $[n]$

→ Bottom has n singleton blocks.

The Möbius function $\mu_{P_3}(\hat{0}, \hat{1})$ is $2 = 2!$



2×1

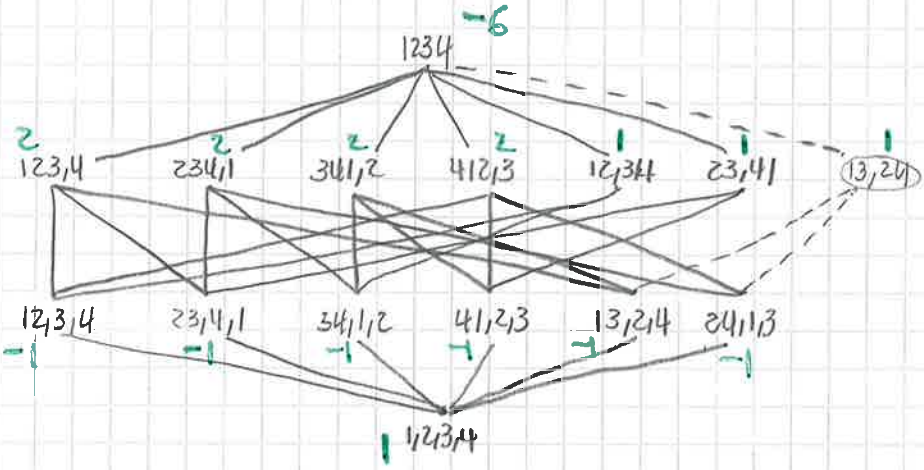
$-3 \times t$

$1 \times t^2$

Characteristic polynomial:

$$\chi_{P_3}(t) = t^2 - 3t + 2 = (t-1)(t-2)$$

<u>n=4</u>



-6×1

$11 \times t$

$-6 \times t^2$

$1 \times t^3$

$\mu_{P_4}(\hat{0}, \hat{1}) = -6 = -3!$

$$\chi_{P_4}(t) = t^3 - 6t^2 + 11t - 6 = (t-1)(t^2 - 5t + 6) = (t-1)(t-2)(t-3)$$

Exercise: Show that

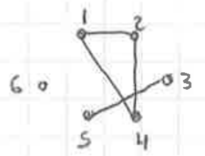
$$\boxed{\mu_{P_n}(\hat{0}, \hat{1}) = (-1)^{n-1}(n-1)!}$$

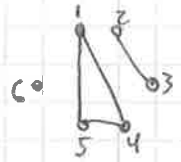
Moreover

$$\boxed{\chi_{P_n}(t) = (t-1)(t-2)\cdots(t-n+1)}$$

- The lattice of noncrossing partitions NC_n

A partition $\{B_1, \dots, B_k\}$ of $[n]$ is called noncrossing if the blocks do not "cross"

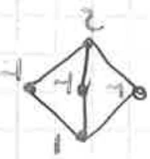




124, 35, 6   145, 23, 6

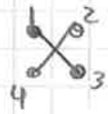
crosses X   Noncrossing ✓

We denote by NC_n the lattice of noncrossing partitions ordered by refinement.



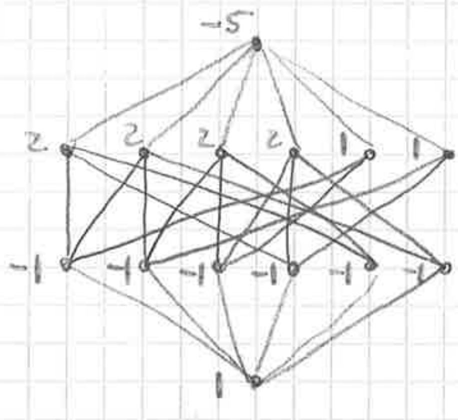
For $n=3$ every partition is noncrossing and $P_3 = NP_3$

For $n=4$ there is one partition that is crossing:



13, 24

Removing this element and the dashed lines in the previous figure of P_4 we obtain NP_4



Exercise 1.4 ii from Lecture 1 asked to prove

$$\boxed{\mu_{NC_n}(\hat{0}, \hat{1}) = (-1)^{n-1} C_{n-1}}$$

Matthias Müller presented a proof using Weiner's Theorem.

Next time: we will see an alternative proof computing the zeta function.