Course: Topics on combinatorics, algebra and geometry
19.1.2024
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Lecture 9

Last time:
- Weak order & Tamari lattice
- Lattice of partitions & noncrossing partitions

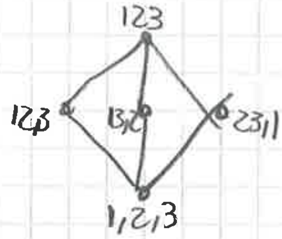
Today:
- Lattice of noncrossing partitions
  - multichains
  - Möbius function
- Polytopes
  - Permutahedron
  - Associahedron

- The lattice of noncrossing partitions

Let NC_n be the lattice of noncrossing partions of $[n]$ ordered by refinement.

NC_2 :

NC_3 :



We are interested in counting the number of multichains.

$$\boxed{x_1 \le x_2 \le \ldots \le x_k}$$

in NC_n. For $k=1$, we get $|NC_n| = C_n = \frac{1}{n+1}\binom{2n}{n}$

In general, this is equal to

$$\boxed{\begin{aligned} Z(NC_n, k+1) &:= Z^{k+1}_{NC_n}(\hat{0},\hat{1}) \\ &= \#\text{ multichains } \hat{0} = x_0 \le x_1 \le \ldots \le x_k \le x_{k+1} = \hat{1} \end{aligned}}$$

which is a polynomial in k (Exercise 7.4).

Evaluating, this polynomial at $k=-2$ we recover the Möbius function

$$\boxed{\mu_{NC_n}(\hat{0},\hat{1}) = Z^{-1}_{NC_n}(\hat{0},\hat{1})}$$

noncombinatorial — combinatorial — generalization — bijective

Theorem [Kreweras '72, Poupard '72, Edelman '80 '82]

The number of multichains $x_1 \le x_2 \le \ldots \le x_k$ in the noncrossing partition lattice NC_n is given by the Fuß-Catalan number

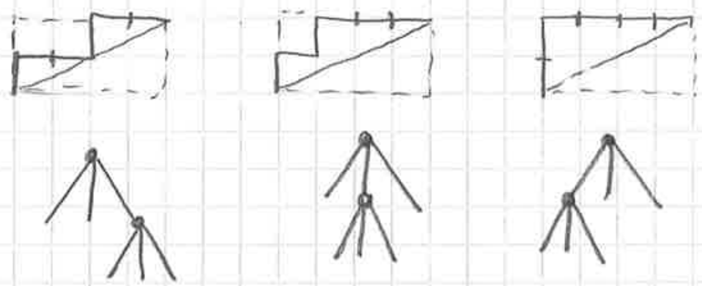
$$\frac{1}{kn+1}\binom{(k+1)n}{n} \qquad (*)$$

For $k=1$, $(*) = C_n = |NC_n|$ ✓

In general, $(*)$ counts the number of (Exercise 1.2)

- lattice paths from $(0,0)$ to (kn,n) that stay weakly above the diagonal of the $kn \times n$ rectangle
- rooted plane $(k+1)$-ary trees with n internal nodes (each internal node has $k+1$ children)

For $n=2, k=2$: $\quad \frac{1}{4+1}\binom{3\times 2}{2} = \frac{1}{5}\,\frac{6\times 5}{2\times 1} = 3.$

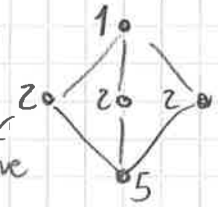


Intervals in NC_2:

12	1,2	1,2
12	12	1,2

For $n=3, k=2$: $\quad \frac{1}{6+1}\binom{3\times 3}{3} = \frac{1}{7}\,\frac{9\times 8\times 7}{3\times 2\times 1} = 12$

Intervals in NC_3:



\# elements above

$1+2+2+2+5 = 12$

Proof (Based on Edelman '82)

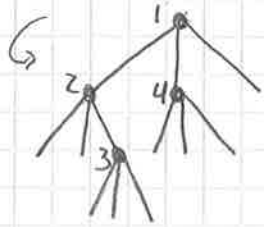
We present a bijection

rooted plane $(k+1)$-ary trees T $\longleftrightarrow$ Multichains in NC_n: $\hat{0} = X_0 \leq X_1 \leq \dots \leq X_k \leq X_{k+1} = \hat{1}$

Let T be a $(k+1)$-ary tree. Label internal nodes in preorder

For $i = 0, 1, \dots, k+1$

Define T_i be the the subgraph obtained from T by removing all child edges of each node except the last i edges



T

$i=0$ remove all edges

$i=1$ remove all but the last

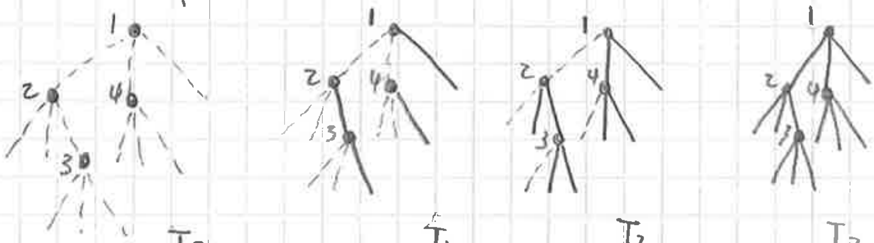
$i=2$ remove all but the last two

$\vdots$

$i=k+1$ Dont remove anything

Let X_i be the partition of $[n]$ whose blocks are the labels of the connected components of T_i

In our example:



T_0 T_1 T_2 T_3

$X_0 = 1,2,3,4$ $X_1 = 1,23,4$ $X_2 = 14,23$ $X_3 = 1234$

Claim X_i is a noncrossing partition for $i = 0,1,\dots,k+1$ (Exercise)

Moreover: $\hat{0} = X_0 \leq X_1 \leq \dots \leq X_{k+1} = \hat{1}$

and the map $T \to \hat{0} \leq X_0 \leq X_1 \leq \dots \leq X_{k+1} = \hat{1}$ is a bijection.

The inverse:

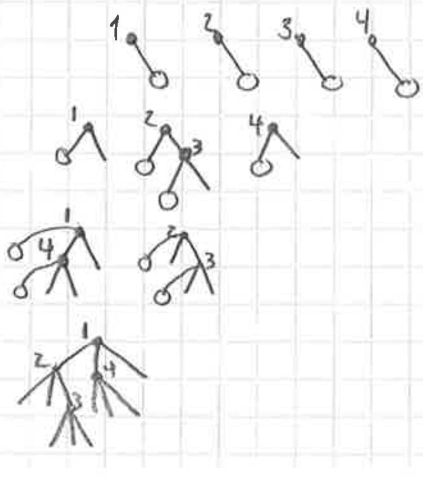
$X_0 = 1,2,3,4$

$X_1 = 1,23,4$

$X_2 = 14,23$

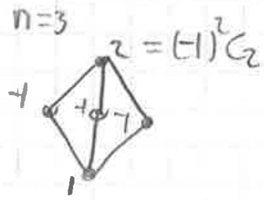
unique! glue here

$X_3 = 1234$



{ add an edge to each node
Glue together the blocks that merge.

repeat the process



Corollary

$$\boxed{\mu_{NC_n}(\hat{0},\hat{1}) = (-1)^{n-1} C_{n-1}}$$

Proof We showed that

$$\boxed{Z^{k+1}_{NC_n}(\hat{0},\hat{1}) = \frac{1}{kn+1}\binom{(k+1)n}{n}}$$

Evaluating at $k=-2$, we get

$$\mu_{NC_n}(\hat{0},\hat{1}) = Z^{-1}_{NC_n}(\hat{0},\hat{1}) = \frac{1}{-2n+1}\binom{-n}{n}$$

$$= \frac{1}{\cancel{-(2n-1)}} \frac{(-n)(-n-1)(-n-2)\dots(\cancel{-n}-n+1)}{n(n-1)(n-2)\dots(1)}$$

$$= (-1)^{n-1}\frac{1}{n}\binom{2n-2}{n-1}$$

$$= (-1)^{n-1} C_{n-1}$$

∎

- ## Geometry

Two remarkable polytopes

Permutahedron	Associahedron
	
Edge graph = Hasse diagram of the weak order	Edge graph = Hasse diagram of the Tamari lattice

$\Rightarrow$ A much richer structure than just the poset.

- ## The permutahedron

Let $n \in \mathbb{N}$. The permutahedron $\mathrm{Perm}(n)$ is the convex hull of all permutations of $[n]$.

$$\boxed{\mathrm{Perm}(n) = \mathrm{conv}\,\{(i_1, \dots, i_n) \;:\; \{i_1, \dots, i_n\} = [n]\}} \subseteq \mathbb{R}^n$$

Since the sum of the coordinates of all vertices is constant $= 1+2+\dots+n$ then $\mathrm{Perm}(n)$ lies in an $(n-1)$-dimensional affine space in $\mathbb{R}^n$.

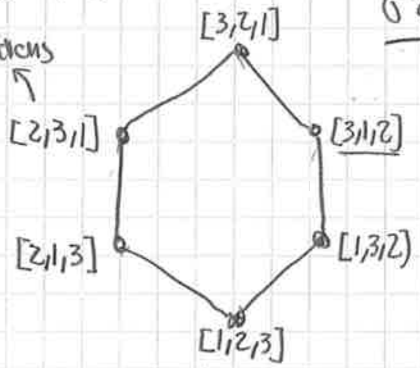
$$\boxed{\dim(\mathrm{Perm}(n)) = n-1}$$

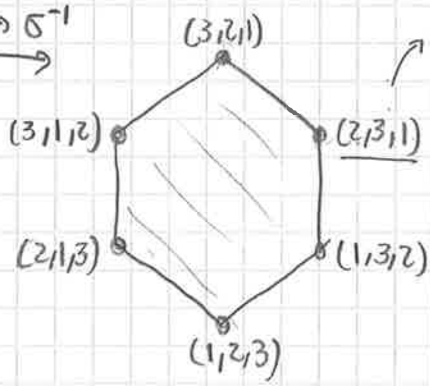
Example $n=3$

permutations

$\sigma \leftrightarrow \sigma^{-1}$

coordinates





Weak order of permutations of [3]

Perm(3)

Cover relations:

$$[\dots \overset{i}{a} \overset{i+1}{b} \dots] \lessdot [\dots \overset{i}{b} \overset{i+1}{a} \dots]$$

swap consecutive positions

Edges:

$$(\dots \overset{a}{i} \dots \overset{b}{i+1} \dots) - (\dots \overset{a}{i+1} \dots \overset{b}{i} \dots)$$

swap consecutive values

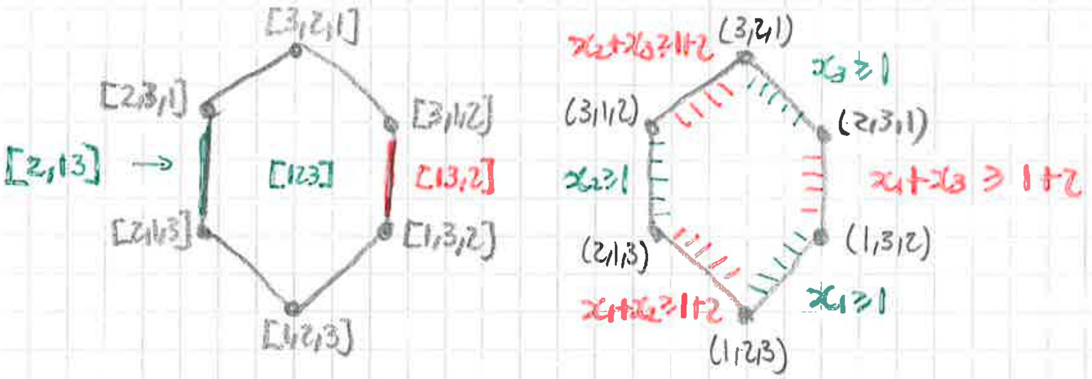
- Defining inequalities

Perm(n) lies in the affine space

$$x_1 + \dots + x_n = 1 + \dots + n.$$

And satisfies the following inequalities for each nonempty proper subset $A \subset [n]$

$$\boxed{\sum_{a \in A} x_a \geq 1 + 2 + \dots + |A|}$$



- Characterization of the faces

The faces are in correspondence with ordered partitions $[B_1, \dots, B_k]$ of $[n]$

A face $[B_1, \dots, B_k]$ is contained in a face $[B_1', \dots, B_\ell']$ if each B_i is contained in some B_j' (B refines B')

In particular, the vertices of $[B_1, \dots, B_k]$ are obtained by all possible permutation of the elements inside the blocks

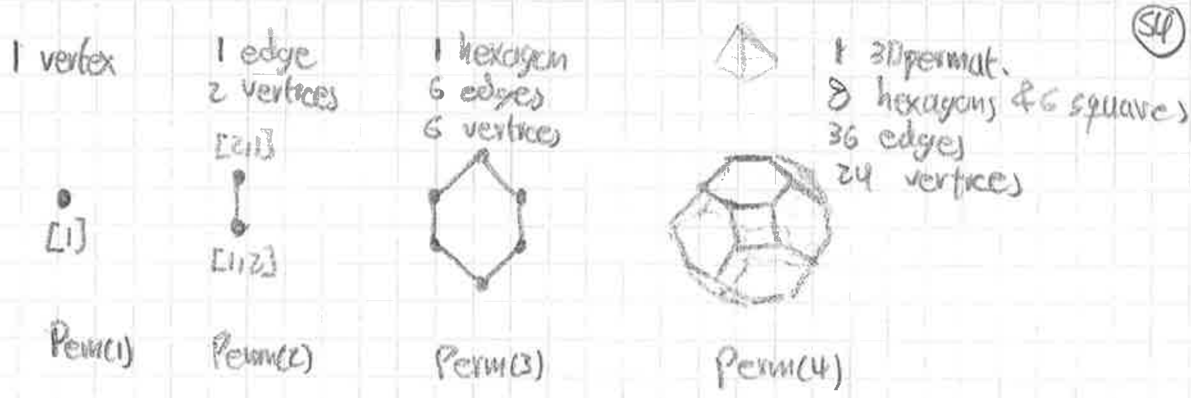
For instace: the face $[2,13]$ (an edge) contains vertices

$[2,1,3]$
$[2,3,1]$

The dimension of $[B_1, \dots, B_k]$ is $n-k$.
And this face is combinatorially the product of smaller permutohedra

$$\text{Perm}(|B_1|) \times \dots \times \text{Perm}(|B_k|)$$

- Many nice applications / properties



- <u>Inverting power series</u> (Ardila - Aguiar '17+ '23)

The number of faces of Perm(n) has nice applications in the context of formal power series:

If
$$\left(1+a_1x+a_2\frac{x^2}{2!}+a_3\frac{x^3}{3!}+\cdots\right)\left(1+b_1x+b_2\frac{x^2}{2!}+\cdots\right)=1$$

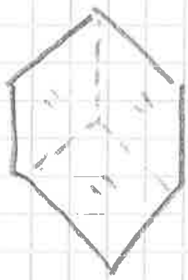
then
$$b_1=-a_1$$
$$b_2=-a_2+2a_1a_1$$
$$b_3=-a_3+6a_2a_1+6a_1a_1a_1$$
$$b_4=-a_4+(8a_3a_1+6a_2a_2)-36a_2a_1a_1+24a_1a_1a_1a_1$$
$$\vdots$$

Each face $[A_1,\dots,A_k]$ contributes a term

$$(-1)^k a_{|A_1|}a_{|A_2|}\cdots a_{|A_k|}$$

to that expression!

- <u>The volume of permutahedra</u> (Postnikov '09)



volume=3

Volume $(\mathrm{Perm}(n)) = n^{n-2}$ parking functions
[Postnikov]