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Topics in Combinatorics, Algebra and Geometry
Potential Final Project A

Winter Semester
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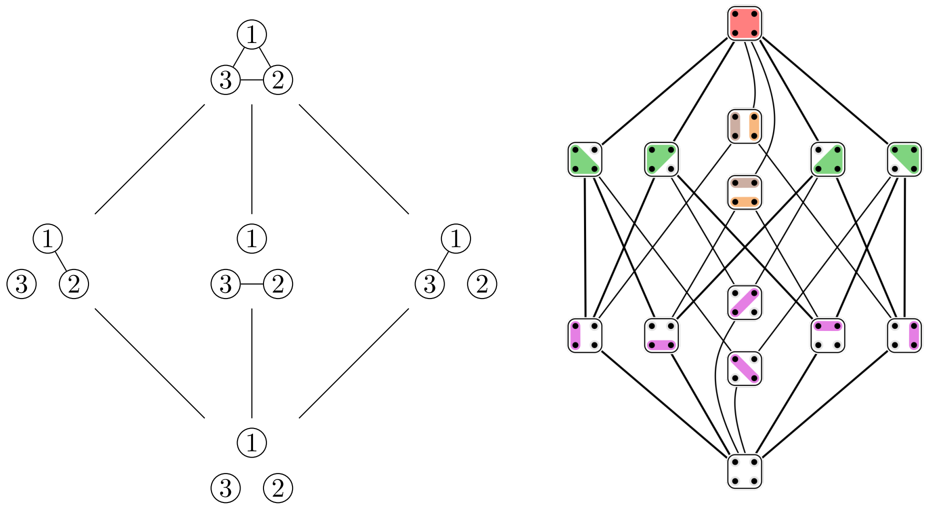
1 The lattice of non-crossing partitions

The collection of noncrossing partitions of $[n+1]$ ordered by refinement forms a poset called the lattice of noncrossing partitions NC_{n+1} (go up in the poset by gluing blocks together). The following are examples of NC_3 and NC_4.



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This project regards generalizations of the following known enumerative properties of these lattices:

1. The number of maximal chains of NC_{n+1} is $(n+1)^{n-1}$, the number of parking functions [11, Corollary 3.3], see also [13, Theorem 3.1].

2. The Möbius function of the top element is up to sign the nth Catalan number

$$(-1)^n \mu(\hat{1}) = \frac{1}{n+1}\binom{2n}{n}.$$

   This result is due to Kreweras [12], see also [5].

This has been generalized for m-divisible non-crossing partitions and for other Coxeter groups; see [1] and the references therein, in particular [3, 2, 14]. I propose to generalize this further in the context of signature Catalan combinatorics [8], using the geometry/combinatorics of ν-associahedra [9, 10].

- The noncrossing partition lattice was generalized to any finite real reflection group by Brady and Watt [6] and Bessis [4]. In [3, Corollary 4.3], the Möbius function of any interval $[u, v]$ is calculated in terms of falling chains in $[u, v]$. If u is the minimal element and v is the maximum, one gets the Möbius number of the noncrossing partition lattice. Using results of [7], they relate it to the number of positive clusters of the generalized associahedron of the corresponding type.

- The lattice of m-divisible non-crossing partitions was first studied by Edelman in [11]. A generalization denoted $NC^k(W)$ for finite Coxeter groups is due to Armstrong [2]. The number of maximal chains is $n!(kh)^n/|W|$ where h is the Coxeter number and n is the rank of W, see [2, Corollary 3.6.10 and Theorem 3.6.9]. For an analog result related to the Möbius function see [2, Theorem 3.7.7]

References

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