1 Volume of permutahedra and parking functions

The purpose of the project is to understand a known connection between the volume of a permutahedron and the number of parking functions based on Postnikov's work [1].

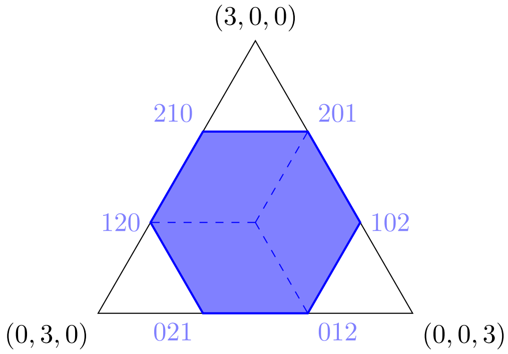
For $x_1, \dots, x_n \in \mathbb{R}$ define the permutahedron as the convex hull of all permutations of $(x_1, \dots, x_n)$:

$$P_n(x_1, \dots, x_n) ;= \operatorname{conv}\{(x_{i_1}, \dots, x_{i_n}) : \{i_1, \dots, i_n\} = [n]\} \subseteq \mathbb{R}^n$$

Since the sum of the coordinates is constant, then $P_n(x_1, \dots, x_n)$ is an $(n-1)$-dimensional polytope.

As an example, the permutahedron $P_3(2,1,0)$ is the two dimensional polytope (an hexagon):

$$P_3(2,1,0) = \operatorname{conv}\{(2,1,0), (2,0,1), (1,2,0), (1,0,2), (0,2,1), (0,1,2)\}.$$

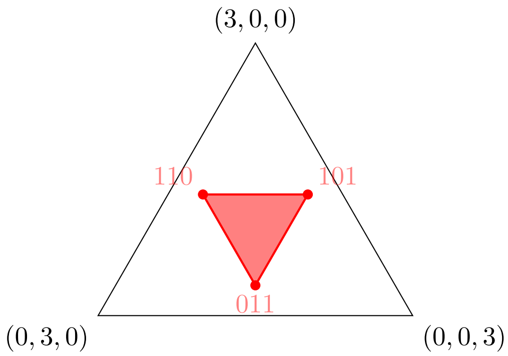


In [1, Corollary 11.5], Postnikov proved that the volume of the permutahedron $P_n(n-1, n-2, \dots, 0)$ is equal to n^{n-2}, the number of parking functions. In our example, $n = 3$ and the volume of $P_3(2,1,0)$, area of the blue hexagon, is equal to $3^1 = 3$. See also Theorem 9.3, Example 9.7 and Example 11.4 in [1].

Equivalently, Postnikov shows that the volume of $P_n(n-1, n-2, \dots, 0)$ counts the number of integer points of

$$P_n(n-1, n-2, \dots, 0) - \Delta_{[n]} = P_n(n-2, n-2, \dots, 0).$$

For our running example with $n = 3$, this polytope is shown in red below. The number of lattice points is equal to 3.



References

[1] Alexander Postnikov. Permutohedra, associahedra, and beyond. *Int. Math. Res. Not. IMRN*, (6):1026–1106, 2009.