CONVERGENCE OF ITERATIVE SCHEMES ON METRIC SPACES

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ABSTRACT. We analyze the convergence of iterative refinement processes on metric spaces, imposing the principle of contractivity to obtain convergence criteria. As a major result, we show that on Hadamard spaces a wide natural class of contractible barycentric subdivision schemes converges.

INTRODUCTION

Linear subdivision schemes have a short, yet eventful history in different areas of pure and applied mathematics. Topics in which these refinement methods appear include harmonic analysis, see e.g. [7], and the theory of functional equations, cf. [4]. A scheme of this type refines a data grid $x : \mathbb{Z}^s \to \mathbb{R}^n$ according to

$$Sx_i = \sum_{k \in \mathbb{Z}^s} a_{i-2k} x_k, \quad \text{where} \quad \#\operatorname{supp}(a) < \infty. \tag{1}$$

This actually amounts to 2^s linear subdivision rules, one for each residue class modulo $2\mathbb{Z}^s$. The scheme S is said to *converge* if for each $x \in \ell^\infty$ the powers of S acting on x approach a continuous limit $S^\infty x : \mathbb{R}^s \to \mathbb{R}^n$ uniformly in a sense that $\lim_m \sup_i \|S^\infty x(i/2^m) - S^m x_i\| = 0$. The convergence analysis of linear schemes with nonnegative *mask coefficients* a_k, its beginnings comprehensively described in [2], was further developed by Zhou in [11], who showed that these schemes converge as long as their mask's support has an appropriate shape and the property of *affine invariance* is fulfilled, meaning $\sum_k a_{i-2k} = 1$ for $i \in \mathbb{Z}^s$.

The hypothesis of affine invariance is natural in a sense that every convergent scheme is subject to it. Moreover, it allows to at least formally generalize equation (1) to arbitrary metric spaces:

$$Sx_i = \operatorname{argmin}\Big(\sum_{k \in \mathbb{Z}^s} a_{i-2k} d(x_k, \cdot)^2\Big). \tag{2}$$

This nonlinear refinement method, referred to as *barycentric subdivision*, has the advantage of being well-adapted to the metric structure of the underlying space. As a drawback, however, the existence and uniqueness of the minimizer in (2) is not always guaranteed, even if the weights are nonnegative. Nevertheless there is a nice class of metric spaces in which barycentric schemes with nonnegative masks are well-defined – namely the complete geodesic spaces of nonpositive Alexandrov curvature, known as *Hadamard spaces*, cf. [1].

As far as barycentric schemes on finite-dimensional Hadamard *manifolds* are concerned, at least smoothness properties are well-understood, cf. [5]. However, convergence analysis even in this setting remains to be developed. Convergence

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criteria known so far, see [10] for a univariate result, rely heavily on differentiability. This paper aims at a first understanding of barycentric schemes on general Hadamard spaces.

The article is organized as follows: In Section 1, we prove a convergence criterion valid on complete metric spaces based on contractivity. The second section handles the task of identifying contractive schemes. Section 3 uses the results obtained in the first two to establish convergence of contractive schemes on Hadamard spaces, see Theorem 10. Making use of this theorem and the contractivity criteria obtained in Section 2, we show that a barycentric scheme with nonnegative mask whose support is sufficiently well-behaved converges, cf. Corollary 11. We thus generalize corresponding linear results from [2].

1. Refinement schemes on metric spaces

This section establishes a theorem on complete metric spaces ensuring convergence of contractive schemes comparable to convergent ones. To begin with, we introduce some notions used throughout the paper:

Definition 1. Consider a metric space (X, d) and set $X_s = \{x \mid x : \mathbb{Z}^s \to X\}$. A *scheme* $S : X_s \to X_s$ is called *convergent on* $\Omega \subseteq X_s$ if for all $x \in \Omega$ there exists a continuous function $S^\infty x : \mathbb{R}^s \to X$ such that $d_\infty(S^\infty x(\cdot/2^n), S^n x) = \sup_j (S^\infty x(j/2^n), S^n x_j)$ tends to 0 as $n \to \infty$.

Moreover, the scheme is called *contractive* with respect to some nonnegative function $D : X_s \to \mathbb{R}_+$ if and only if

$$D(Sx) < \gamma D(x), \quad \text{where} \quad \gamma < 1.$$

We refer to D as a *contractivity function* for S. Unless specified otherwise, a contractive scheme is called *convergent* if and only if it has this property on the set $\Omega_D = \{x \in X_s \mid D(x) < \infty\}$.

Remark 1. Sometimes notation becomes more accessible if one views $S^n x$ not as an element of X_s, but rather as a function on the refined grid $S^n x : 2^{-n}\mathbb{Z}^s \to X$ through the natural identification $S^n x_j = S^n x(\frac{j}{2^n})$.

Theorem 1. *Let (X, d) be a complete metric space. Suppose a convergent iterative scheme $T : X_s \to X_s$ satisfies*

$$d_\infty(Tx, Ty) \le d_\infty(x, y). \tag{3}$$

Moreover, assume the scheme S is contractive with respect to D and

$$d_\infty(Tx, Sx) \le C \cdot D(x) \tag{4}$$

holds for any $x \in X_s$. Then S is convergent.

Proof. We set $f_n(y) := T^\infty(S^n x)(2^n y)$ and claim that this defines a Cauchy sequence in $(C(\mathbb{R}^s, X), d_\infty)$. Note first that given $n \in \mathbb{N}$ and $y \in \mathbb{R}^s$, by continuity of f_n respectively f_{n+1}, we find $j \in \mathbb{Z}^s$ and $m \in \mathbb{N}$ such that

$$d(f_r(y), f_r(2^{-m} j)) < C \cdot D(x)\gamma^n \quad \text{for } r = n, n+1. \tag{5}$$

Moreover, due to convergence of T, by multiplying both the numerator and the denominator of the number $j/2^m$ with a power of two if necessary we may assume m to be sufficiently large for

$$d(f_r(2^{-m} j), T^{m-r}(S^r x)_j) = d(T^\infty S^r x(2^{r-m} j), T^{m-r}(S^r x)_j) < C \cdot D(x)\gamma^n$$

to hold for $r = n, n+1$, in addition to (5). This together with (3) and (4) implies

$$\begin{aligned} d(f_n(y), f_{n+1}(y)) &\le d(f_n(y), f_n(2^{-m}j)) \\ &+ d(f_n(2^{-m}j), T^{m-n}S^n x_j) \\ &+ d(T^{m-n}S^n x_j, T^{m-n-1}S^{n+1}x_j) \\ &+ d(T^{m-n-1}S^{n+1}x_j, f_{n+1}(2^{-m}j)) \\ &+ d(f_{n+1}(2^{-m}j), f_{n+1}(y)) \\ &< 5C \cdot D(x)\gamma^n, \end{aligned}$$

showing that f_n is a Cauchy sequence. Since X is complete, we find a continuous $f : \mathbb{R}^s \to X$ with $f_n \to f$ uniformly. We claim that $S^n x$ converges to f in the sense of Definition 1. For $m \ge n$ and $j \in \mathbb{Z}^s$, we obtain the inequality

$$\begin{aligned} d(T^{m-n}S^n x_j, S^m x_j) &\le \sum_{k=n}^{m-1} d(T^{m-k}S^k x_j, T^{m-k-1}S^{k+1}x_j) \\ &\le \sum_{k=n}^{m-1} \gamma^k \cdot D(x)C \le \gamma^n \left(\frac{D(x)C}{1-\gamma}\right), \end{aligned}$$

which together with

$$d(f_n(2^{-m}j), S^m x_j) \le d(f_n(2^{-m}j), T^{m-n}S^n x_j) + d(T^{m-n}S^n x_j, S^m x_j)$$

establishes the claim. □

2. Recognition of Contractivity

This section is devoted to a simple criterion of contractivity, namely Proposition 5 below. Referring e.g. to [8] for details, we start by recalling some fundamental facts on Hadamard spaces. These complete geodesic spaces allow for a notion of nonpositive curvature in a sense that geodesic triangles are 'slim' compared to the Euclidean ones of the same edge lengths: The defining property of such a space X is the so called *Hadamard inequality*

$$d(z, x_{\frac{1}{2}})^2 \le \frac{1}{2}d(z, x_0)^2 + \frac{1}{2}d(z, x_1)^2 - \frac{1}{4}d(x_0, x_1)^2,$$

where $x_0, x_1, z \in X$, and $x_{\frac{1}{2}}$ denotes some *midpoint* of x_0 and x_1 satisfying $d(x_0, x_{\frac{1}{2}}) = d(x_{\frac{1}{2}}, x_1) = \frac{1}{2}d(x_0, x_1)$.

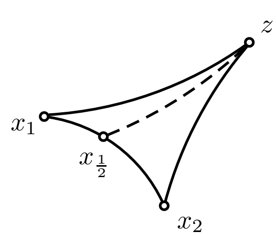

Figure 1. A 'slim' geodesic triangle.

Hadamard spaces for instance play an important role in the theory of cost-minimizing networks, see [3]. Topological examples are trees as well as euclidean Bruhat-Tits buildings. Notably, for a measure space M, and N Hadamard, the space of strongly measurable square-integrable functions $L^2(M, N)$ inherits the Hadamard property. It is remarkable that these spaces also occur as families of certain geometric and topological structures, such as spaces of Riemannian and

Kähler metrics or spaces of connections. The latter examples actually are generically infinite-dimensional Hadamard *manifolds*, see [6]. An instance of a finite-dimensional Hadamard manifold significant in applications is the space of symmetric positive definite matrices, which occurs in Diffusion Tensor Imaging.

We define the *barycenter* or *center of mass* of a L^2 probability measure μ on the Borel algebra of a Hadamard space by

$$b(\mu) = \operatorname{argmin} \int_X d(\cdot, x)^2 \mu(dx).$$

It is well-known that on Hadamard spaces, this barycenter exists and is unique, see [8]. Recall that a *coupling* of probability measures μ and ν is a measure on $X \times X$ satisfying $\pi(A \times X) = \mu(A)$ and $\pi(X \times A) = \nu(A)$ for all Borel sets A. We define the L^1-Wasserstein metric (see [9]) on the space of probability measures by

$$d^W(\mu, \nu) = \inf \Big\{ \int_{X \times X} d(x, y) \pi(dxdy) \mid \pi \text{ is a coupling of } \mu \text{ and } \nu \Big\}.$$

Proposition 2 ([8]). *On a Hadamard space, the barycenter b is Lipschitz-continuous as a map from the space of probability measures to X. More precisely,*

$$d(b(\mu), b(\nu)) \leq d^W(\mu, \nu).$$

We continue by adapting the theory of barycenters on Hadamard spaces to fit into the setup of nonlinear refinement schemes. Throughout the rest of the paper, we encounter situations in which two sets of data are given:

(i) A finite measure α on the discrete sigma algebra of $\mathbb{Z}^s$, whose *weights* $\alpha(\{i\}) = \alpha_i \geq 0$ satisfy $\sum_i \alpha_i = \alpha(\mathbb{Z}^s) = \sigma > 0$.

(ii) A multivariate sequence of *data points* $x : \mathbb{Z}^s \to X$.

In this setting, we consider the induced probability measure $\mu_x = \frac{1}{\sigma} x_* \alpha$ on $x(\mathbb{Z}^s)$ and define

$$\bar{x}_\alpha = b(\mu_x) = \operatorname{argmin}(\sum_i \alpha_i d(x_i, \cdot)^2).$$

This little instance of double-thinking allows us to apply convenient properties of the barycenter in the convergence analysis of refinement schemes based on formula (2). The involved inequalities to a certain extend resemble the corresponding Euclidean ones:

Corollary 3. *Let (X, d) be a Hadamard space. Suppose we are given weights $\alpha_i, \beta_i \in \mathbb{R}_{\geq 0}$ and data points $x_i, y_i \in X$, $i \in \mathbb{Z}$ such that $0 < \sigma = \sum_i \alpha_i < \infty$ and $0 < \tau = \sum_i \beta_i < \infty$. Moreover assume $\frac{1}{\sigma} x_* \alpha$, $\frac{1}{\tau} x_* \beta$ and $\frac{1}{\sigma} y_* \alpha$ constitute L^2-probability measures w.r.t. the metric d. Then the following inequalities hold good:*

$$d(\bar{x}_\alpha, \bar{y}_\alpha) \leq \frac{1}{\sigma} \sum_i \alpha_i \, d(x_i, y_i) \tag{6}$$

$$d(\bar{x}_\alpha, \bar{x}_\beta) \leq \frac{1}{\sigma\tau} \sum_{i,j} \alpha_i \beta_j \, d(x_i, x_j). \tag{7}$$

Proof. Set $u = x \times y, v = x \times x : \mathbb{Z}^2 \to X \times X$, i.e. $u(i,j) = (x_i, y_j)$ and $v(i,j) = (x_i, x_j)$ for $i, j \in \mathbb{Z}$. Moreover, define measures μ and ν on $\mathbb{Z}^2$ via $\mu(i,j) = \delta_{ij} \alpha_i$, where δ_{ij} denotes the Kronecker delta, and $\nu(i,j) = \alpha_i \beta_j$. Let $\pi_i^X : X \times X \to X$ and $\pi_i : \mathbb{Z} \times \mathbb{Z} \to \mathbb{Z}$ denote the projections on the i-th components for $i = 1, 2$. Then, using the fact that for any pair of functions $f_1, f_2 : \mathbb{Z} \to X$ one has $\pi_i^X \circ (f_1 \times f_2) = f_i \circ \pi_i$, it is easy to verify that $\frac{1}{\sigma} u_* \mu$ is a coupling of $\frac{1}{\sigma} x_* \alpha$ and $\frac{1}{\sigma} y_* \alpha$. Similarly, one shows that $\frac{1}{\sigma\tau} v_* \nu$ is a coupling of $\frac{1}{\sigma} x_* \alpha$ and $\frac{1}{\tau} x_* \beta$. Thus the statement directly follows from Proposition 2. $\square$

Lemma 4. *Suppose for $i = 1, \dots, n$ we are given nonnegative weights α_i, β_i and control points $x_i \in X$ such that $\sum_i \alpha_i = \sum_i \beta_i = \sigma > 0$. Then there exist a positive integer r, nonnegative weights γ_j and control points $y_j, z_j \in \{x_1, \dots, x_n\}$, $j = 1, \dots, r$, such that*

$$\sum_j \gamma_j d(y_j, \cdot)^2 = \sum_i \alpha_i d(x_i, \cdot)^2 \quad \textit{and} \quad \sum_j \gamma_j d(z_j, \cdot)^2 = \sum_i \beta_i d(x_i, \cdot)^2.$$

Moreover, we may require $\gamma_1 = \min(\alpha_1, \beta_1)$.

Proof. We use induction over n to prove the statement. Note first that w.l.o.g. we may assume $\alpha_1 = \min(\alpha_1, \beta_1) =: \gamma_1$. Set $k = \max\{\ell \mid \beta_1 - \sum_{i=1}^{\ell} \alpha_i \geq 0\}$ and $\delta = \beta_1 - \sum_{i=1}^{k} \alpha_i \leq 0$. In case $k = n$, we have $\sigma = \beta_1$ and $\beta_\ell = 0$ for $\ell > 1$. The γ_i, y_i and z_i are then given by

i	1	$\dots$	n
γ	α_1	$\dots$	α_n
y	x_1	$\dots$	x_n
z	x_1	$\dots$	x_1

If $k < n$, we conclude that $\tilde{\sigma} = \sigma - \beta_1 > 0$ and $\delta < \alpha_{k+1}$. Similar to the above, the first $k+1$ members of γ, y and z are given by

i	1	$\dots$	k	$k+1$
γ	α_1	$\dots$	α_k	δ
y	x_1	$\dots$	x_k	x_{k+1}
z	x_1	$\dots$	x_1	x_1

.

The remaining weights $\tilde{\alpha}$, $\tilde{\beta}$ and data points $\tilde{x}$ defined by

i	1	$\dots$	k	$k+1$	$\dots$	$n-1$
$\tilde{\alpha}$	0	$\dots$	$\alpha_{k+1} - \delta$	α_{k+2}	$\dots$	α_n
$\tilde{\beta}$	β_2	$\dots$	β_{k+1}	β_{k+2}	$\dots$	β_n
$\tilde{x}$	x_2	$\dots$	x_{k+1}	x_{k+2}	$\dots$	x_n

(8)

fulfill $\tilde{\sigma} = \sigma - \beta_1 = \sum_i \tilde{\alpha}_i = \sum_i \tilde{\beta}_i$, so we obtain the number $r \geq k+1$ and the weights γ_i and data points y_i, z_i, $i = k+2, \dots, r$ by applying the induction hypothesis to (8). □

We are now in a position to prove our central contractivity criterion:

Proposition 5. *Suppose α_i, β_i and x_i are given as in Lemma 4, and set $D(x) = \max_{k,\ell} d(x_k, x_\ell)$. Then the following contractivity property holds true:*

$$d(\bar{x}_\alpha, \bar{x}_\beta) \leq (1 - \max_i(\min(\alpha_i, \beta_i))/\sigma) D(x).$$

Proof. W.l.o.g. $\min(\alpha_1, \beta_1) = \max_i(\min(\alpha_i, \beta_i))$. Using Lemma 4, we find γ, y and z with $\bar{x}_\alpha = \bar{y}_\gamma$, $\bar{x}_\beta = \bar{z}_\gamma$, $\gamma_1 = \min(\alpha_1, \beta_1)$ and $z_1 = y_1$. Now by inequality (6) of Corollary 3, we may estimate

$$d(\bar{x}_\alpha, \bar{x}_\beta) = d(\bar{y}_\gamma, \bar{z}_\gamma) \leq \sum_j (\gamma_j/\sigma) d(y_j, z_j) \leq (1 - \alpha_1/\sigma) \max_{k,\ell} d(x_k, x_\ell),$$

which proves the statement. □

Remark 2. As a direct consequence of (7) we obtain

$$d(\bar{x}_\alpha, \bar{x}_\beta) \leq (1 - \frac{1}{\sigma^2} \sum_i \alpha_i \beta_i) \max_{k,\ell} d(x_k, x_\ell),$$

which gives a different (sometimes larger, sometimes smaller) contractivity constant. Thus Proposition 5 should also be regarded as a means to quantify the speed of convergence, see Proposition 13 below.

3. Contractivity as a convergence criterion

This chapter is devoted to the convergence analysis of barycentric schemes with nonnegative masks on Hadamard spaces, on which they are well-defined. As a central result, see Theorem 10, we show how contractivity leads to convergence. Throughout the section, nonnegativity of the mask will be assumed for any occurring barycentric scheme.

Definition 2. Let $S : X_s \to X_s$ and $T : X_t \to X_t$ be barycentric subdivision schemes with masks a and b, respectively. Then we define the *tensor product* of S and T to be the scheme $S \otimes T : X_{s+t} \to X_{s+t}$ whose mask is given by $c_{(i,j)} = a_i b_j$.

The next proposition establishes a class of contractive schemes on Hadamard spaces, including the ones generating splines of arbitrary degree. Remarkably this result and its linear counterpart, see [2], are equally powerful.

Proposition 6. *Suppose the mask of the scheme S acting on data from a Hadamard space is supported on a convex set $\tilde{\Omega}$ for which there exists $x_0 \in \Omega$ such that $\Omega = \tilde{\Omega} - x_0$ is balanced. With the definitions*

$$\rho_\Omega(\xi) = \inf\{\lambda \geq 0 \mid \xi \in \lambda\Omega\}, \quad \xi \in \mathbb{R}^s$$

(the Minkowski functional *of Ω) and*

$$D_\Omega(x) = \sup_{\rho_\Omega(i-j)<2} d(x_i, x_j), \quad x \in X_s,$$

we have

$$D_\Omega(Sx) \leq \gamma D_\Omega(x),$$

where $\gamma = \min(\gamma_1, \gamma_2)$, with

$$\gamma_1 = 1 - \min_{\rho_\Omega(i-j)<2} \max_{k\in\mathbb{Z}^s} \min(a_{i-2k}, a_{j-2k})$$

$$\gamma_2 = 1 - \min_{\rho_\Omega(i-j)<2} \sum_{k\in\mathbb{Z}^s} a_{i-2k} a_{j-2k}.$$

In particular, if for each $i, j \in \mathbb{Z}^s$ with $\rho_\Omega(i-j) < 2$ one finds $k \in \mathbb{Z}^s$ such that $i - 2k \in \operatorname{supp}(a)$ and $j - 2k \in \operatorname{supp}(a)$, then $\gamma < 1$ and hence S is contractive w.r.t. D_Ω.

Proof. Note first that $y \in \tilde{\Omega} \implies \rho_\Omega(y - x_0) \leq 1$. Introducing sequences α^i by letting $\alpha^i_k = a_{i-2k}$, we have $Sx_i = \bar{x}_{\alpha^i}$ for $i \in \mathbb{Z}^s$. Thus, with the notation

$$\eta_{ij} = \max\left(\max_{k\in\mathbb{Z}^s} \min(a_{i-2k}, a_{j-2k}), \sum_{k\in\mathbb{Z}^s} a_{i-2k} a_{j-2k}\right),$$

Proposition 5 and the subsequent remark imply

$$\begin{aligned} D_\Omega(Sx) &= \sup_{\rho_\Omega(i-j)<2} d(\bar{x}_{\alpha^i}, \bar{x}_{\alpha^j}) \\ &\leq \sup_{\rho_\Omega(i-j)<2} [(1-\eta_{ij}) \max_{\rho_\Omega(i-2k-x_0), \rho_\Omega(i-2\ell-x_0)\leq 1} d(x_k, x_\ell)]. \end{aligned} \tag{9}$$

By convexity of Ω, its Minkowski functional is subadditive. Therefore $\rho_\Omega(i-j) < 2$ together with $\rho_\Omega(i - 2\ell - x_0) \leq 1$ and $\rho_\Omega(j - 2k - x_0) \leq 1$ implies

$$\rho_\Omega(k-\ell) \leq \rho_\Omega(k - \frac{1}{2}(j + x_0)) + \rho_\Omega(\frac{1}{2}(\ell + x_0) - i) + \rho_\Omega(\frac{1}{2}(i-j)) < 2.$$

Combining this with (9), we obtain $D_\Omega(Sx) \leq \gamma D_\Omega(x)$ as required. □

Proposition 6 provides us with a contractivity criterion that solely depends on the structure of the mask's support. Thus every linear scheme seen to be contractive using the linear version of the above proposition possesses a contractive barycentric analogue. In particular, this applies to the class of schemes identified in chapter 3 of [2], see Corollary 7 below. Recall that a *centered zonotope* is defined as $Z(X) = \{Xu \mid u \in \mathbb{R}^n,\ \|u\|_\infty \leq 1\}$ with $X \in \mathbb{Z}^{s\times n}$, $n > s$. $Z(X)$ is called *unimodular* if and only if each $s \times s$ – minor of X has determinant -1, 0, or 1, and $\operatorname{rank}(X) = s$.

Corollary 7. *Suppose the barycentric scheme S possesses a mask whose support is an integer quad with edges of length at least 2, or $supp(a) = Z(X) \cap \mathbb{Z}^s$ with $Z(X)$ unimodular. Then S is contractive w.r.t. some contractivity function D_Ω.*

Proof. This is a direct consequence of the proofs of Theorems 3.3 and 3.4 of [2], and Proposition 6. □

Up to now it is not clear how Theorem 1 is of value in detecting convergent schemes. It would be desirable to have some kind of convergent model scheme to compare a given contractive scheme with, leading to an implication

$$\boxed{\text{Contractivity} \Longrightarrow \text{Convergence.}}$$

For contractivity functions of the form D_Ω with Ω balanced and convex, cf. Proposition 6, we are able to provide such a model scheme (see Lemma 9 below).

Lemma 8. *Suppose Ω and Ω' are bounded, balanced and convex subsets of $\mathbb{R}^s$ having nonempty interior. Then there exists a constant $C > 0$ such that*

$$D_{\Omega'} \leq C \cdot D_\Omega.$$

Proof. Choose $x_1, \ldots, x_n \in \mathbb{R}^s$ such that $2(\Omega')^\circ \subseteq \bigcup_{i=1}^n (x_i + 2\Omega^\circ)$. Thus, given $i, j \in \mathbb{Z}^s$ with $\rho_{\Omega'}(i-j) < 2 \Leftrightarrow i - j \in 2(\Omega')^\circ$, one finds $i = y_0, y_1, \ldots, y_{m-1}, y_m = j$ with $y_k - y_{k-1} \in 2\Omega^\circ \Leftrightarrow \rho_\Omega(y_k - y_{k-1}) < 2$, such that $m \leq n$. It is plain to show that as a consequence, $D_{\Omega'} \leq n \cdot D_\Omega$. □

The next lemma identifies linear B-spline subdivision as a model scheme suitable for our convergence analysis.

Lemma 9. *Let $\tilde{T}$ denote the univariate linear B-spline subdivision scheme defined by $a_0 = 1$ and $a_{-1} = a_1 = \frac{1}{2}$. Then the barycentric analogue of $T = \tilde{T} \otimes \cdots \otimes \tilde{T}$ converges on any Hadamard space. Moreover, $d_\infty(Tx, Ty) \leq d_\infty(x, y)$ for all $x, y \in X_s$.*

Proof. We begin by proving convergence. Define $D_1 = D_{\mathbb{D}_\infty}$, where $\mathbb{D}_\infty$ denotes the closed unit disk with respect to the maximum norm. By Corollary 7, $D_1(Tx) \leq \gamma D_1(x)$, with $\gamma < 1$. For $n \in \mathbb{N}_0$, define $f_n : \mathbb{R}^s \to X$ as follows:

1. For $t \in \mathbb{R}$, set $\varphi_0(t) = \max\{1 - |t|, 0\}$ and define $\varphi(t_1, \ldots, t_s) = \prod_i \varphi_0(t_i)$.
2. Set $f_n(\zeta) = \operatorname{argmin}(\sum_k \varphi(2^{n-1}\zeta - k) d(\,\cdot\,, T^{n-1}x_k)^2)$.

This function is continuous since the center of mass depends continuously on the weights. Moreover, $f_r(j/2^r) = T^r x_j$ for each $j \in \mathbb{Z}^s$ by construction of φ. Suppose $\zeta \in \mathbb{R}^s$ is contained in some dyadic cube $Q = \prod[k_i 2^{-r+1}, (k_i + 1)2^{-r+1}]$, where $k_i \in \mathbb{Z}$. Clearly $\varphi \equiv 0$ outside $\{\xi \in \mathbb{R}^s \mid \|\xi\|_\infty < 1\}$, from which we conclude that

$$f_r(\zeta) = \operatorname{argmin} \sum_{v \in V(Q)} \varphi(2^{r-1}(\zeta - v)) d(\cdot, T^{r-1}x(v))^2,$$

where $V(Q)$ denotes the vertex set of Q. Applying the inequality (7), we obtain

$$\begin{aligned} \max_{v \in V(Q)} d(T^{r-1}x(v), f_r(\zeta)) &\leq \max_{v,w \in V(Q)} d(T^{r-1}x(v), T^{r-1}x(w)) \\ &\leq D_1(T^{r-1}x) \leq \gamma^{r-1} D_1(x). \end{aligned} \tag{10}$$

Certainly, every dyadic cube of edge length 2^{-n} shares a vertex with a dyadic cube of edge length 2^{-n+1}. Together with (10) applied to $r = n, n+1$, this implies $d(f_n(\zeta), f_{n+1}(\zeta)) < 2\gamma^{n-1} D_1(x)$. It is straightforward to show that $f := \lim_n f_n$ is a uniform limit of $S^n x$.

The second statement is an easy consequence of inequality (6). Indeed, for $i \in \mathbb{Z}^s$ we have

$$d(Tx_i, Ty_i) \leq \sum_{k \in \mathbb{Z}^s} a_{i-2k} d(x_k, y_k) = \begin{cases} d(x_\ell, y_\ell), & \text{if } i = 2\ell \\ \frac{1}{2}(d(x_\ell, y_\ell) + d(x_{\ell+1}, y_{\ell+1}), & \text{if } i = 2\ell + 1. \end{cases}$$

□

Now all the ingredients of our main theorem on the convergence of subdivision schemes on Hadamard spaces are at hand, and we proceed to

Theorem 10. *Suppose the barycentric scheme S acting on data from a Hadamard space is contractive with respect to some D_Ω, with Ω bounded, balanced, convex and having nonempty interior. Then S converges.*

Proof. Throughout the proof let T denote the s-fold tensor product of the barycentric analogue of the linear B-spline scheme, whose mask a was defined Lemma 9.

Choose $\Omega' \subset \mathbb{R}^s$ bounded, balanced, and convex such that $\operatorname{supp}(a) \subset \Omega'$ and $\mathbb{D}_\infty(0,1) \subset \Omega$. Applying (7), for $i \in \mathbb{Z}^s$ one obtains

$$d(Sx_i, Tx_i) \leq \sum_{k,\ell} b_{i-2k} a_{i-2\ell} d(x_k, x_\ell). \tag{11}$$

Note that the summands in (11) are nonzero only if $\rho_{\Omega'}(i-2k) \leq 1$ and $\|i-2\ell\|_\infty \leq 1 \Longrightarrow \rho_{\Omega'}(i-2\ell) \leq 1$, which implies $\rho_{\Omega'}(k-\ell) = \frac{1}{2}(\rho_{\Omega'}(i-2k) + \rho_{\Omega'}(i-2\ell)) \leq 1$. This together with (11) gives $d_\infty(Sx, Tx) \leq D_{\Omega'}(x)$. By Lemma 8 we find $C > 0$ such that $D_{\Omega'}(x) \leq C \cdot D_\Omega(x)$ for $x \in X_s$. Hence $d_\infty(Sx, Tx) \leq C \cdot D_\Omega(x)$, so the statement follows from Theorem 1. □

Thus, Corollary 7 leads to the identification of the following convergent schemes:

Corollary 11. *Suppose the barycentric scheme S possesses a mask whose support is an integer quad with edges of length at least 2, or $\operatorname{supp}(a) = Z(X) \cap \mathbb{Z}^s$ with $Z(X)$ an unimodular centered zonotope. Then S converges.*

Corollary 12. *On Hadamard spaces, the barycentric analogues of B-spline schemes of arbitrary degree converge.*

We conclude our reasoning with a statement highlighting the effect of the contractivity constant on the quality of convergence.

Proposition 13. *Suppose Ω is a balanced convex set with nonempty interior such that $\operatorname{supp}(a) \subseteq 4\Omega^\circ$ and $D_\Omega(Sx) \leq \gamma D_\Omega(x)$, $\gamma < 1$. Then S converges and*

$$d(S^\infty x(j/2^n), S^n x_j) \leq \gamma^n \left(\frac{D_\Omega(x)}{1-\gamma} \right).$$

Proof. By inequality (7), we have

$$d(Sx_{2j}, x_j) \leq \sum_{\rho_\Omega(2(j-k))<4} a_{2(j-k)} d(x_k, x_j) \leq D_\Omega(x),$$

and consequently

$$d((S^m x)_{2^{m-n}j}, S^n x_j) \leq \gamma^n \left(\frac{D_\Omega(x)}{1-\gamma} \right). \tag{12}$$

According to Theorem 10, S converges, hence the first statement follows by taking the limit in m on the left hand side of equation (12). □

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