HARMONIC COHOMOLOGY OF SYMPLECTIC FIBER BUNDLES

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Abstract. We show that every de Rham cohomology class on the total space of a symplectic fiber bundle with closed Lefschetz fibers, admits a Poisson harmonic representative in the sense of Brylinski. The proof is based on a new characterization of closed Lefschetz manifolds.

1. Introduction and main result

Suppose P is a Poisson manifold [10] with Poisson tensor π. Let d denote the de Rham differential on $\Omega(P)$ and write i_π for the contraction with the Poisson tensor. Recall that Koszul's [5] codifferential $\delta := [i_\pi, d] = i_\pi d - d i_\pi$ satisfies $\delta^2 = 0$ and $[d, \delta] = d\delta + \delta d = 0$. Differential forms $\alpha \in \Omega(P)$ with $d\alpha = 0 = \delta\alpha$ are called *(Poisson) harmonic.* Brylinski [2] asked for conditions on a Poisson manifold which imply that every de Rham cohomology class admits a harmonic representative.

In the symplectic case, this question has been settled by Mathieu. Recall that a symplectic manifold (M, ω) of dimension $2n$ is called *Lefschetz* iff, for all k,

$$[\omega]^k \wedge H^{n-k}(M; \mathbb{R}) = H^{n+k}(M; \mathbb{R}).$$

According to Mathieu [6], see [11] for an alternative proof, a symplectic manifold is Lefschetz iff it satisfies the Brylinski conjecture, i.e. every de Rham cohomology class of M admits a harmonic representative.

In this paper we study the Brylinski problem for smooth symplectic fiber bundles [7]. Recall that the total space of a symplectic fiber bundle $P \to B$ is canonically equipped with the structure of a Poisson manifold obtained from the symplectic form on each fiber. Locally, the Poisson structure on P is product like, that is, every point in B admits an open neighborhood U such that there exists a fiber preserving Poisson diffeomorphism $P|_U \cong M \times U$. Here M denotes the typical symplectic fiber, equipped with the corresponding Poisson structure, and U is considered as a trivial Poisson manifold. This renders the symplectic foliation of P particularly nice, for its leaves coincide with the connected components of the fibers of the bundle $P \to B$.

The aim of this note is to establish the following result, providing a class of Poisson manifolds which satisfy the Brylinski conjecture.

Theorem 1. *Let M be a closed symplectic Lefschetz manifold, and suppose $P \to B$ is a smooth symplectic fiber bundle with typical symplectic fiber M. Then every de Rham cohomology class of P admits a Poisson harmonic representative. Moreover, the analogous statement for compactly supported cohomology holds true.*

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This result, as well as a characterization of closed Lefschetz manifolds similar to Theorem 2 below, has been established in the first author's diploma thesis, employing sightly different methods than those of the present work, see [3].

The proof presented in Section 3 below is based on a handle body decomposition $\emptyset = B_0 \subseteq B_1 \subseteq B_2 \subseteq \cdots$ of B. Given a cohomology class of P, we will inductively produce representatives which are harmonic on $P|_{B_k}$, for increasing k. The crucial problem, of course, is to extend harmonic forms across the handle, from $P|_{B_k}$ to $P|_{B_{k+1}}$. This issue is addressed in Theorem 2, see also Lemma 6.

2. Extension of harmonic forms

Let M be a closed symplectic manifold and consider the trivial symplectic fiber bundle $P := M \times \mathbb{R}^p \times D^q$ where D^q denotes the q-dimensional closed unit ball. In other words, the Poisson structure on P is the product structure obtained from the symplectic form on M and the trivial Poisson structure on $\mathbb{R}^p \times D^q$. Note that the boundary $\partial P = M \times \mathbb{R}^p \times \partial D^q$ is a Poisson submanifold. It turns out that the Lefschetz property of M is equivalent to harmonic extendability of forms, from ∂P to P.

To formulate this precisely, we need to introduce some notation which will be used throughout the rest of the paper. For every Poisson manifold P we let $Z(P) := \{\alpha \in \Omega(P) \mid d\alpha = 0\}$ and $Z_0(P) := \{\alpha \in \Omega(P) \mid d\alpha = 0 = \delta\alpha\}$ denote the spaces of closed and harmonic differential forms, respectively. Moreover, we write $H_0(P) := \ker(d) \cap \ker(\delta) / \operatorname{img}(d) \cap \ker(\delta)$ for the space of de Rham cohomology classes which admit a harmonic representative, $H_0(P) \subseteq H(P)$. If $\iota : S \hookrightarrow P$ is a Poisson submanifold, then the relative complex $\Omega(P, S) := \{\alpha \in \Omega(P) \mid \iota^*\alpha = 0\}$ is invariant under δ, and we define the relative harmonic cohomology $H_0(P, S) \subseteq H(P, S)$ in an analogous manner. Finally, if Q is a Poisson manifold and B is a smooth manifold we let $\Omega_{\mathrm{vc}}(Q \times B)$ denote the space of forms with vertically compact support (with respect to the projection $Q \times B \to Q$), and define the harmonic cohomology with vertically compact supports $H_{\mathrm{vc},0}(Q \times B) \subseteq H_{\mathrm{vc}}(Q \times B)$ in the obvious way.

Here is the main result that will be established in this section.

Theorem 2. *Let M be a closed symplectic manifold, suppose $p, q \in \mathbb{N}_0$, and consider the Poisson manifold $P := M \times \mathbb{R}^p \times D^q$. Then the following are equivalent:*

- *(i) M is Lefschetz, i.e. $H_0(M) = H(M)$ according to* [6].
- *(ii) $H_0(P, \partial P) = H(P, \partial P)$.*
- *(iii) $H_{\mathrm{vc},0}(P \setminus \partial P) = H_{\mathrm{vc}}(P \setminus \partial P)$ with respect to the projection along $D^q \setminus \partial D^q$.*
- *(iv) If $\alpha \in Z(P)$ is harmonic on a neighborhood of ∂P, then there exists $\beta \in \Omega(P)$, supported on $P \setminus \partial P$, so that $\alpha + d\beta$ is harmonic on P.*
- *(v) If $\alpha \in Z(P)$ and $\delta\iota^*\alpha = 0$, then there exists $\beta \in \Omega(P)$ with $\iota^*\beta = 0$, so that $\alpha + d\beta$ is harmonic on P. Here $\iota : \partial P \hookrightarrow P$ denotes the canonical inclusion.*

An essential ingredient for the proof of Theorem 2 is the following $d\delta$-Lemma.

Lemma 1 ($d\delta$-Lemma, [4, 8])**.** *A closed symplectic manifold is Lefschetz if and only if* $\ker(\delta) \cap \operatorname{img}(d) = \operatorname{img}(d\delta)$.

We will also make use of the following averaging argument.

Lemma 2. *Suppose G is a connected compact Lie group acting on a Poisson manifold P via Poisson diffeomorphisms, and let $r : \Omega(P \times I) \to \Omega(P \times I)^G$,*

$r(\alpha) := \int_G g^*\alpha\, dg$, denote the standard projection onto the space of G-invariant forms, $I := [0,1]$. Then there exists an operator $A : \Omega(P \times I) \to \Omega(P \times I)$, commuting with d, i_π and δ, so that $A(\alpha) = \alpha$ in a neighborhood of $P \times \{1\}$ and $A(\alpha) = r(\alpha)$ in a neighborhood of $P \times \{0\}$, for all $\alpha \in \Omega(P \times I)$.

Proof. Choose finitely many smoothly embedded closed balls $D_i \subseteq G$ such that $\bigcup_i \mathring{D}_i = G$. Let λ_i denote a partition of unity on G so that $\operatorname{supp}(\lambda_i) \subseteq D_i$. Choose smooth contractions $h_i : D_i \times I \to G$ so that $h_i(g,t) = g$ for $t \leq 1/3$ and $h_i(g,t) = e$ for $t \geq 2/3$, $g \in D_i$. Here e denotes the neutral element of G. Using the maps

$$\phi_{i,g} : P \times I \to P \times I, \quad \phi_{i,g}(x,t) := (h_i(g,t) \cdot x, t), \qquad g \in D_i,$$

we define the operator $A : \Omega(P \times I) \to \Omega(P \times I)$ by

$$A(\alpha) := \sum_i \int_{D_i} \lambda_i(g) \phi_{i,g}^* \alpha \; dg$$

where integration is with respect to the invariant Haar measure of G. It is straightforward to verify that A has the desired properties, the relations $[A, i_\pi] = 0 = [A, \delta]$ follow from the fact that each $\phi_{i,g}$ is a Poisson map. □

The following application of Lemma 2 will be used in the proof of Theorem 1.

Lemma 3. *Let M be a symplectic manifold and consider the Poisson manifold $P := M \times \mathbb{R}^p \times A^q$ where $A^q := \{\xi \in \mathbb{R}^q \mid \frac{1}{2} \leq \xi \leq 1\}$ denotes the q-dimensional annulus. Moreover, suppose $\alpha \in \Omega(P)$ is harmonic on a neighborhood of $\partial_+ P := M \times \mathbb{R}^p \times \partial D^q$. Then there exist $\beta \in \Omega(P)$, supported on $P \setminus \partial_+ P$, and $\beta_1, \beta_2 \in Z_0(M)$, so that $\tilde{\alpha} := \alpha + d\beta$ is harmonic on P, and $\tilde{\alpha} = \sigma^*\beta_1 + \sigma^*\beta_2 \wedge \rho^*\theta$ in a neighborhood of $\partial_- P := M \times \mathbb{R}^p \times \frac{1}{2}\partial D^q$. Here $\sigma : P \to M$ and $\rho : P \to \partial D^q$ denote the canonical projections, and θ denotes the standard volume form on ∂D^q.*[1]

Proof. W.l.o.g. we may assume $\alpha \in Z_0(P)$ and $\alpha = \tau^*\gamma$ in a neighborhood of $\partial_- P$ where $\gamma \in Z_0(M \times \partial D^q)$ and $\tau = (\sigma, \rho) : P \to M \times \partial D^q$ denotes the canonical projection. Applying the operator A from Lemma 2 to α, we obtain $\tilde{\alpha} \in Z_0(P)$ so that $\tilde{\alpha} = \alpha$ in a neighborhood of $\partial_+ P$, and $\tilde{\alpha} = \tau^*\tilde{\gamma}$ in a neighborhood of $\partial_- P$, where $\tilde{\gamma} \in Z_0(M \times \partial D^q)$ is $SO(q)$-invariant. We conclude that $\tilde{\gamma}$ is of the form $\tilde{\gamma} = \beta_1 + \beta_2 \wedge \theta$ with $\beta_1, \beta_2 \in Z_0(M)$, whence $\tilde{\alpha} = \sigma^*\beta_1 + \sigma^*\beta_2 \wedge \rho^*\theta$ in a neighborhood of $\partial_- P$. Clearly, there exists $\beta \in \Omega(P)$, supported on $P \setminus \partial_+ P$, such that $\tilde{\alpha} - \alpha = d\beta$. □

Lemma 4. *Let P be a Poisson manifold, and suppose B is an oriented smooth manifold with boundary. Then integration along the fibers $\int_B : \Omega_{\mathrm{vc}}(P \times B) \to \Omega(P)$ commutes with i_π and δ.*

Proof. The relation $i_\pi \int_B \alpha = \int_B i_\pi \alpha$ is obvious. Combining this with Stokes' theorem, that is $[\int_B, d] = \int_{\partial B} \iota^*$, we obtain

$$[\textstyle\int_B, \delta] = [\int_B, [i_\pi, d]] = [[\int_B, i_\pi], d] + [i_\pi, [\int_B, d]] = [i_\pi, \int_{\partial B} \iota^*] = 0.$$

Here $\iota : P \times \partial B \hookrightarrow P \times B$ denotes the canonical inclusion. □

[1]To be specific, in the case $q = 1$ we assume $\theta(-1) = -1/2$ and $\theta(1) = 1/2$, so that $\int_{\partial D^q} \theta = 1$ with respect to orientation on ∂D^q induced from the standard orientation of D^q.

Lemma 5. *Suppose Q is a Poisson manifold, and consider the Poisson manifold $P := Q \times D^q$. Then the Thom (Künneth) isomorphism restricts to an isomorphism of harmonic cohomology, i.e. $H_0^{*-q}(Q) = H^*_{\mathrm{vc},0}(P \setminus \partial P) = H_0^*(P, \partial P)$.*

Proof. Choose $\eta \in \Omega^q(D^q)$, supported on $D^q \setminus \partial D^q$, such that $\int_{D^q} \eta = 1$. Clearly, the chain map $\Omega(Q) \to \Omega_{\mathrm{vc}}(P \setminus \partial P) \subseteq \Omega(P, \partial P)$, $\alpha \mapsto \alpha \wedge \eta$, commute with δ. This map induces the Thom isomorphism which therefore preserve harmonicity. Its inverse is induced by integration along the fibers $\int_{D^q} : \Omega(P, \partial P) \to \Omega(Q)$, and this commutes with δ too, see Lemma 4. □

Now the table is served and we proceed to the

Proof of Theorem 2. Set $Q := M \times \mathbb{R}^p$ and note that the isomorphism $H(Q) = H(M)$ induced by the canonical projection restricts to an isomorphism of harmonic cohomology $H_0(Q) = H_0(M)$. The equivalence of the first three statements thus follows from Lemma 5. Let us continue by showing that (iii) implies (iv). Assume $\alpha \in Z(P)$ is harmonic on a neighborhood of ∂P. Let $\rho : P \setminus (M \times \mathbb{R}^p \times \{0\}) \to \partial D^q$ and $\sigma : P \to M$ denote the canonical projections. In view of Lemma 3, we may w.l.o.g. assume $\alpha = \sigma^*\beta_1 + \sigma^*\beta_2 \wedge \rho^*\theta$ in a neighborhood of ∂P where $\beta_1, \beta_2 \in Z_0(M)$ and θ denotes the standard volume form on ∂D^q. Using Stokes' theorem for integration along the fiber of $M \times D^q \to M$, we obtain

$$\beta_2 = \int_{\partial D^q} j^*\alpha = -d \int_{D^q} j^*\alpha \in \mathrm{img}(d) \cap \ker(\delta)$$

where $j : M \times D^q \to M \times \{0\} \times D^q \subseteq P$ denotes the canonical inclusion. By the $d\delta$-Lemma 1, we thus have $\beta_2 = d\delta\gamma$ for some differential form γ on M. Let λ be a smooth function on P, identically 1 in a neighborhood of ∂P, identically 0 near $M \times \mathbb{R}^p \times \{0\}$, and constant in the M-direction. Then $\tilde\alpha := \sigma^*\beta_1 + d(\delta\sigma^*\gamma \wedge \lambda\rho^*\theta)$ is a harmonic on P, and $\alpha - \tilde\alpha = 0$ in a neighborhood of ∂P. Hence, using (iii), we find $\beta \in \Omega(P)$, supported on $P \setminus \partial P$, so that $\alpha - \tilde\alpha + d\beta$ is harmonic on P. Thus, β has the desired property. Let us next show that (iv) implies (v). Suppose $\alpha \in Z(P)$ and $\delta\iota^*\alpha = 0$. Clearly, there exists $\beta_1 \in \Omega(P)$, with $\iota^*\beta_1 = 0$, so that $\tilde\alpha := \alpha + d\beta_1$ satisfies $r^*\tilde\alpha = \tilde\alpha$ near ∂P, where $r : P \setminus (M \times \mathbb{R}^p \times \{0\}) \to \partial P$ denotes the canonical radial retraction. Particularly, $\tilde\alpha$ is harmonic on a neighborhood of ∂P. According to (iv) there exists $\beta_2 \in \Omega(P)$, supported on $P \setminus \partial P$, so that $\tilde\alpha + d\beta_2$ is harmonic on P. The form $\beta := \beta_1 + \beta_2$ thus has the desired property. Obviously, (v) implies (ii). □

3. Proof of Theorem 1

Choose a proper Morse function f on B, bounded from below, so that the preimage of each critical value consists of a single critical point [9]. We label the critical values in increasing order $c_0 < c_1 < \cdots$, and choose regular values r_k so that $c_{k-1} < r_k < c_k$. By construction, the sublevel sets $B_k := \{f(x) \leq r_k\}$ provide an increasing filtration of B by compact submanifolds with boundary, $\emptyset = B_0 \subseteq B_1 \subseteq B_2 \subseteq \cdots$. The statement in Theorem 1 is an immediate consequence of the following

Lemma 6. *Suppose $\alpha \in Z(P)$ is a closed form which is harmonic on a neighborhood of $P|_{B_k}$. Then there exists $\beta \in \Omega(P)$, supported on $P|_{B_{k+2} \setminus B_k}$, such that $\alpha + d\beta$ is harmonic on a neighborhood of $P|_{B_{k+1}}$.*

Proof. Let q denote the Morse index of the unique critical point in $B_{k+1} \backslash B_k$, and set $p := \dim B - q$. Recall [9] that there exists an embedding $j : \mathbb{R}^p \times D^q \to B_{k+1} \setminus \mathring{B}_k$ so that $j(\mathbb{R}^p \times \partial D^q) = j(\mathbb{R}^p \times D^q) \cap \partial B_k$. Moreover, there exists a vector field X on B, supported on $B_{k+2} \setminus B_k$, so that its flow φ_t maps B_{k+1} into any given neighborhood of $\partial B_k \cup j(\{0\} \times D^q)$, for sufficiently large t.

Trivializing the symplectic bundle P over the image of j, we obtain an isomorphism of Poisson manifolds $j^*P \cong M \times \mathbb{R}^p \times D^q$. Using Theorem 2(iv), we may thus assume that there exists an open neighborhood U of $\partial B_k \cup j(\{0\} \times D^q)$ so that α is harmonic on $P|_U$. Let $\tilde{X}$ denote the horizontal lift of X with respect to a symplectic connection [7] on P, and denote its flow at time t by $\tilde{\varphi}_t$. Clearly, each $\tilde{\varphi}_t$ is a Poisson map. Moreover, there exists t_0 so that $\tilde{\varphi}_{t_0}$ maps $P|_{B_{k+1}}$ into $P|_U$. Thus, $\tilde{\varphi}_{t_0}^*\alpha$ is harmonic on $P|_{B_{k+1}}$. Furthermore, $\tilde{\varphi}_{t_0}^*\alpha - \alpha = d\beta$ where $\beta := \int_0^{t_0} \tilde{\varphi}_t^* i_{\tilde{X}} \alpha \, dt$ is supported on $P|_{B_{k+2} \setminus B_k}$. □

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