MULTIDIMENSIONAL CONTINUED FRACTIONS

JUNE 22 – 26, TU GRAZ

Schedule of the workshop

	Saturday 22 June	Sunday 23 June	Tuesday 25 June	Wednesday 26 June
9:00–9:45	**T. Garrity-I**	**R. Nair-II**	**M. Skopenkov-II**	**C. Elsholtz**
9:45–10:30	**R. Nair-I**	**V. Goryunov**	**T. Garrity-II**	**T. Garrity-III**
10:30–11:00	Coffee break	Coffee break	Coffee break	Coffee break
11:00–11:45	**A. Ustinov-I**	**A. Ustinov-II**	**O. German**	**J. Thuswaldner**
11:45–12:30	**M. Skopenkov-I**	**T. Pejković**	**R. Nair-III**	**O. Karpenkov**
12:30–14:30	Lunch break	Lunch break	Lunch break	Lunch break
14:30–15:15	**V. Petričević**	**A.A. Illarionov**	**M. Avdeeva**	
15:15–15:50			**M. Monina**	

This conference is supported by the Austrian Science Fund (FWF), grant M 1273-N18.

Some extra announcements

Saturday, 22 June, 8:45. Registration of the participants.

Monday, 24 June, 8:45. For those of adventurous mind who are in a good physical shape, the climb through this narrow gorge is a memorable experience. A train takes us to Mixnitz, where we walk for one hour and a half and then climb a total 164 ladders. In the afternoon we end up at the Teichalm plateau at 1100m above sea level to have dinner and return to Graz by train. Total time of the hike is approximately four hours. Any special boots are not required, although it could be easier to climb the wooden ladders with firm shoes.

A word of caution: You must not be afraid of heights. The organizers strongly urge you not to attempt the climb if you feel unsure.

We are meeting at Graz Hauptbahnhof train station in front of the train ticket office. Our train (S1) will leave at 9:05, direction Bruck/Mur.

Tuesday, 25 June, 17:00. The conference dinner would be at 18:00 in one of the nice restaurants on a hill around Graz ("Stoffbauer"). To reach the restaurant we should make a small walk, so we meet at 17:00 at the University.

Abstracts

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On Statistical Properties of Quotients of Irreducible Fractions with Fixed Denominator

Let $C : \mathbb{N} \to \mathbb{R}$ be a sequence of real number (cost) such that $C \neq 0$ and $|C(n)| \leq \log_2(n+1)$ for each $n \in \mathbb{N}$.

For rational $r \in (0,1)$ consider the continued fraction expansion

$$r = [0; q_1, q_2, \dots, q_s] \qquad (q_i = q_i(r) \quad \text{are positive integers})$$

of length $s = s(r)$. We define

$$s_C(r) = \sum_{i=1}^{s(r)} C(q_i(r)) \qquad \text{and} \qquad M(C) = \sum_{n=1}^{\infty} C(k) \log\left(1 + \frac{1}{n(n+2)}\right).$$

Let $\mathcal{F}_N$ be a set of all irreducible $r = m/n \in [0,1]$ with $n \leq N$ ($N \in \mathbb{N}$) and $\Phi(n)$ be a number of elements of the set $\mathcal{F}_N$.

It is know that exists (see [1])

$$\lim_{N\to\infty} \frac{1}{\Phi(n)} \sum_{r \in \mathcal{F}_N} \left| \frac{s_C(r) - \frac{12}{\pi^2} M(C) \log N}{\sqrt{\log N}} \right|^2 = D(C) > 0$$

and uniformly for $-\infty \leq \alpha < \beta \leq \infty$

$$\frac{1}{\Phi(n)} \cdot \# \left\{ r \in \mathcal{F}_N \;\middle|\; \alpha \leq \frac{s_C(r) - \frac{12}{\pi^2} M(C) \log N}{\sqrt{D(C) \log N}} \leq \beta \right\} =$$

$$= \frac{1}{\sqrt{2\pi}} \int_{\alpha}^{\beta} e^{-\frac{x^2}{2}} dx + O\left(\frac{1}{\sqrt{\log N}}\right).$$

We use these results and ideas from [2] as base to prove the following estimation.

Theorem. Put $\mathbb{Z}'_d = \left\{ a \in \mathbb{Z} \;\middle|\; 1 \leq a \leq d, \gcd(a,d) = 1 \right\}$ and $\varphi(d) = \#\mathbb{Z}'_d$. Then $\forall \varepsilon > 0$ uniformly for $\gamma \geq 1$

$$\frac{1}{\varphi(d)} \# \left\{ a \in \mathbb{Z}'_d \;\middle|\; \frac{\left| s_C(\frac{a}{d}) - \frac{12}{\pi^2} M(C) \log\, d \right|}{\sqrt{D(C) \log d}} \geq \gamma \right\} \quad \underset{C,\varepsilon}{\ll} \quad \frac{1}{\gamma} e^{-\frac{1}{4}\gamma^2} \quad + \quad \log^{-\frac{1}{2}+\varepsilon} d.$$

This theorem strengthen the result obtained in [3].

References

[1] V. Baladi, B. Vallée, *Euclidean algorithms are Gaussian. J. Number Theory* ***110*** *(2005), no. 2, 331–386.*

[2] V. A. Bykovskii, *An estimate for the dispersion of lengths of finite continued fractions. Fundam. Prikl. Mat.* ***11*** *(2005), No. 6, 15–26; translation in J. Math. Sci. (N.Y.), 2007, V. 146, No. 2, P. 5643-5643.*

[3] M.O. Avdeeva, V. A. Bykovskii, *Statistical Prorerties of Finite Continued Fractions with Fixed Denominator. Doklady Akademii Nauk* ***449*** *(2013), No. 3, 255–258; translation in Doklady Mathematics* ***87*** *(2013) No. 2, P. 160–163.*

Christian Elsholtz

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Sums of fractions

In this talk we give a survey on results and methods on solutions of the equation

$$\frac{m}{n} = \frac{1}{x_1} + \cdots + \frac{1}{x_k}$$

in positive integers.

The questions we study include the following:

(1) For fixed m, n and k, what can we say about the number $f_k(m, n)$ of solutions of

$$\frac{m}{n} = \frac{1}{x_1} + \cdots + \frac{1}{x_k}\,?$$

Even the case $m = n = 1$ is for large k widely open.

(2) Fix m and k. What can we say about those n for which there is no solution of

$$\frac{m}{n} = \frac{1}{x_1} + \cdots + \frac{1}{x_k}\,?$$

For the special case $m = 4, k = 3$ the famous Erdős-Straus conjecture states that for $n > 1$ there is always a solution, but this conjecture is open.

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I. On the Hermite Problem and Multidimensional Continued Fractions

We will discuss the advantages of motivating the study of multidimensional continued fractions via the rhetoric of the Hermite problem, namely by the problem of finding ways of representing real number by a sequence of integers so that periodicity of the sequence corresponds to the initial real being a cubic irrational. This first part will be emphasizing how to explain and motivate multidimensional continued fractions to non-experts.

We will then look at TRIP maps, a family of multidimensional continued fraction algorithms.

II. On TRIP Maps and the Hermite Problem

In the first part we will provide more details about TRIP maps, and in particular how a certain family of TRIP maps links periodicity with units in any cubic number field.

We will then shift gears and talk about how to link some ideas motivated from thermodynamics to multidimensional continued fractions.

III. Some Functional Analysis behind Multidimensional Continued Fractions: Transfer Operators

Most multidimensional continued fractions can be naturally interpreted as dynamical systems. These dynamical systems often are iterations of triangles. This leads to natural links between ergodic theory and multidimensional continued fractions. The study of the statistics of the possible sequences of integers arising from a given multidimensional continued fraction leads to the natural question of generalizing Gauss-Kuzmin-Wirsing results from traditional continued fractions. This in turns leads naturally to the study of the spectrum of certain operators (called transfer operators) mapping spaces of functions on the initial triangle to spaces of function on the triangle. Thus we will be discussing how to generalize the work of Mayer and others on transfer operators of the Gauss map to multidimensional continued fractions.

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Klein polyhedra and Oppenheim conjecture for linear forms

We present a survey of results on Klein polyhedra which connect properties of a lattice to be algebraic and to have positive norm minimum with some properties of Klein polyhedra's boundaries. These results allow reformulating the famous Oppenheim conjecture for linear forms in terms of Klein polyhedra.

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Rational Tangles and Rational Knots

The talk is devoted to the classical objects of knot theory, and to their classifications obtained by Schubert and Conway.

Take two parallel threads in R^3, and make a sequence of twists on pairs of their ends. The result of the sequence is called a *rational tangle.* The term reflects Conway's observation that there is a natural way to relate a rational number or ∞ to such an object. He has shown that this relation enumerates the isotopy classes of rational tangles, and each such tangle is isotopy-equivalent to the rational tangle read from the continued fraction expression of the related number.

We close a rational tangle by gluing up pairs of neighbouring ends. The result is called a *rational knot* (to be precise, it is actually either a 1- or 2-component link). A theorem by Schubert gives exact conditions on the rational numbers associated to tangles for the rational links to be isotopic.

In the talk, I will explain some details of a combinatorial approach to the proofs of these results due to Kauffman and Lambropoulou.

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On statistical properties of Klein polyhedra and local minima

Two multidimensional generalizations of the classical continued fraction algorithm were suggested at the end of the nineteenth century. One is due to Klein, and the other was developed by Voronoi and independently by Minkowski.

Klein's construction deals with the Klein polyhedra $K_\theta(\Gamma)$, where $\theta = (\theta_1, \dots, \theta_s)$, $\theta_i = \pm 1$, and

$$\Gamma = \{n_1 a^{(1)} + \ldots + n_s a^{(s)} : \ \ n_i \in \mathbb{Z}, i = \overline{1, s}\}$$

is a s-dimensional lattice with basis $a^{(1)}, \ldots a^{(s)} \in \mathbb{R}^s$. Recall that $K_\theta(\Gamma)$ is defined as the convex hull of nonzero points of Γ lying in the s-hedral angle

$$\{x \in \mathbb{R}^s : \ \ \theta_i x_i \geq 0, \ \ i = \overline{1, s}\}.$$

The Voronoi-Minkowski construction deals with the set $\mathfrak{M}(\Gamma)$ of all nonzero points $\gamma \in \Gamma$ such that there exists no nonzero point $\eta \in \Gamma$ satisfying

$$|\eta_i| \leq |\gamma_i|, \ \ i = \overline{1, s}, \qquad \sum_{i=1}^{s} |\eta_i| < \sum_{i=1}^{s} |\gamma_i|.$$

Voronoi refers to elements of $\mathfrak{M}(\Gamma)$ as relative minima. We use the term "local minimum", well established in English literature

We present some results on statistical properties of local minima and Klein polyhedra of integer lattices with given determinant.

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Gauss-Kuzmin statistics for faces of Klein polyhedra

In this talk we consider multidimensional geometric continued fractions in the sense of Klein, which is an alternative approach to Jacobi-Perron continued fraction algorithms. Klein continued fractions are certain surfaces equipped with polyhedral structure. In the algebraic case the polyhedral structure has a periodic nature. We show several examples of multidimensional continued fractions and explain how to use Möbius geometry to generalize Gauss-Kuzmin ergodic statistics from the case of ordinary continued fractions to the multidimensional case.

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About the Arithmetic Nature of Some Identities of the Elliptic Functions Theory

In the series of papers published in *Journal de mathématiques pures et appliquées* in 1858-1865 under the common title "Sur quelques formules générales qui peuventétre utiles dans la théorie des nombres", the French mathematician Liouville gave, without proof, many arithmetic identities (they are listed in [4]). Using these identities, he calculated the number of of representations of a positive integer by quadratic forms of special form and published these calculations, also without proof, in numerous notes. Liouville's methods were reconstructed by Baskakov, Nazimov, Uspenskii, and other authors (see [4], [5], [6] and [7]); new applications have also been developed.

A new arithmetic method for proving certain classical identities from the theory of theta-functions is suggested. This method is applied, in particular, to obtain identities for triple, quintuple and octuple product.

Let L be a nonzero linear form in s independent variables $x_1, \dots, x_s \quad (s = 2, 3, \dots)$ with integer coefficients, and let

$$J, U_-, U_+, R : \mathbb{R}^s \to \mathbb{R}^s$$

be four linear transformations determined by $s \times s$ matrices with integer coefficients and determinants ± 1.

Suppose that Ω is finite subset in

$$\mathbb{Z}^s = \{m = (m_1, \ldots, m_s) \mid m_i \in \mathbb{Z}\},$$

which is partitioned into three disjoint subset Ω_0, Ω_- and Ω_+ by the constraints

$$L(m) = 0, \quad L(m) < 0 \ \text{ and } \ L(m) > 0.$$

Theorem. *Let $\Phi : \mathbb{Z}^s \to A$ be an arbitrary function with values in additive abelian group A such that*

$$\Phi(R(m)) = -\Phi(m) \qquad \forall m \in \mathbb{Z}^s.$$

Then

$$\sum_{m\in\Omega} \Phi(m) = \sum_{m\in\Omega_0} \Phi(m).$$

Specializing Φ, Ω, L, J, U_-, U_+ for the form $Q(x_1, x_2, x_3, x_4) = x_2^2 + x_3^2 + 2x_1x_2$ we obtain identity

$$-\sum_{b_1^2+b_2^2+8ac=d} \chi_{-4}(b_1)\chi_{-4}(b_2)h(2a + b_1, b_2 + 2c) =$$

$$= \sum_{n_1^2+n_2^2=d} \chi_{-4}(n_1)\chi_{-4}(n_2)(n_2h(n_1, n_1) - 2 \sum\nolimits''_{\substack{-n_1\le t\le n_1\\ t \text{ is odd}}} h(t, n_2)),$$

where $h(b_1, b_2) = u^{b_1}v^{b_2} - u^{-b_1}v^{-b_2} + v^{b_1}u^{b_2} - v^{-b_1}u^{-b_2}$ and $\chi_{-4} : \mathbb{Z} \to \{-1, 0, 1\}$ is quadratic character modulo 4.

Performing fairly simple calculations, we obtain the logarithmic derivative of classical identity

$$\frac{u^2+1}{u^2-1} + \frac{v^2+1}{v^2-1} + 2\sum_{n_1,n_2=1}^{\infty} (u^{-2n_1}v^{-2n_2} - u^{2n_1}v^{2n_2})q^{8n_1n_2} =$$

$$= \frac{G(uv; q)G'(1; q)}{G(u; q)G(v; q)},$$

which goes back to Jacobi, with

$$G(w; q) = \sum_{b=-\infty}^{\infty} \chi_{-4}(b)w^bq^{b^2}.$$

A detailed proof can be found in [3].

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[3]. V.A. Bykovskii and M.D. Monina, *About the arithmetic nature of some identities of the elliptic functions theory.* Far Eastern Mathematical Journal, V. 13 (1), 15–34 (2013).
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Ergodic Methods for Continued Fractions

The study of the statistical properties of continued fractions was initiated by C.F. Gauss. Building on subsequent of R.O. Kuzmin, A. Khinchin and W. Doeblin, C. Ryll-Nardzewski showed that this theory could be based on Birkhoff's pointwise ergodic theorem. This has evolved into a substantial field called the metric theory of algorithms, covered by 11K in Mathematics Reviews. In these talks I describe this subject and its modern development. Topics covered should include, The Euclidean Algorithm, The Gauss Map, The invariant measure, ergodicity and mixing, Birkhoff's pointwise ergodic theory, subsequence ergodic theory, the natural extension map, Hurwitz constants, entropy, the Bernoulli shift, Markov maps of the unit interval, invariant measures for Markov maps, the set of badly approximable points, relatives of the continued fraction maps – the nearest integer continued fraction expansion. Non-archemedian continued fractions, the continued fraction expansion in positive characteristic and the Schneider p-adic continued fraction expansion.

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Quadratic approximation in $\mathbb{Q}_p$

Let p be a prime number. Let w_2 and w_2^* denote the exponents of approximation defined by Mahler and Koksma, respectively, in their classifications of p-adic numbers. It is well-known that every p-adic number ξ satisfies $w_2^*(\xi) \leq w_2(\xi) \leq w_2^*(\xi) + 1$, with $w_2^*(\xi) = w_2(\xi) = 2$ for almost all ξ. By means of Schneider's continued fractions, we give explicit examples of p-adic numbers ξ for which the function $w_2 - w_2^*$ takes any prescribed value in the interval $(0, 1]$.

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Householder's approximants and continued fraction expansion of quadratic irrationals

Let α be a quadratic irrational. It is well known that the continued fraction expansion of α is periodic. We observe Householder's approximant of order $m-1$ for the equation $(x-\alpha)(x-\alpha')=0$ and $x_0=p_n/q_n$: $R_n^{(m)}=\frac{\alpha(p_n/q_n-\alpha')^m-\alpha'(p_n/q_n-\alpha)^m}{(p_n/q_n-\alpha')^m-(p_n/q_n-\alpha)^m}$. We say that $R_n^{(m)}$ is good approximant if $R_n^{(m)}$ is a convergent of α. When period begins with a_1, there is a good approximant at the end of the period, and when period is palindromic and has even length ℓ, there is a good approximant in the half of the period. So when $\ell \leq 2$, then every approximant is good, and then it holds $R_n^{(m)}=\frac{p_{m(n+1)-1}}{q_{m(n+1)-1}}$ for all $n \geq 0$. We prove that to be a good approximant is the palindromic and the periodic property. Further, we define the numbers $j^{(m)}=j^{(m)}(\alpha,n)$ by $R_n^{(m)}=\frac{p_{m(n+1)-1+2j}}{q_{m(n+1)-1+2j}}$ if $R_n^{(m)}$ is a good approximant. We prove that $|j^{(m)}|$ is unbounded by constructing an explicit family of quadratic irrationals, which involves the Fibonacci numbers.

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Tiling of a rectangle, alternating current, and continued fractions

When a square can be tiled by rectangles similar to a given one? To be more precise, which can be side ratio of such rectangles? The answer to this question involves continued fractions, and the proof uses alternating-current networks.

The answer was obtained by C. Freiling, M. Laczkovich, D. Rinne and G. Szekeres [3,4]. We give a short physical proof reducing the result to an inverse problem for alternating-current circuits solved by R. Foster and W. Cauer in 1920s [2].

We are going to see how mathematical theory of electric networks appears naturally in studies of tilings of a rectangle [1]. Then we move to a more general question — *which polygons can be tiled by rectangles of given shapes* — and answer it in certain particular cases, each time going deeper into the theory of electric networks. For example, we prove the following new result.

Theorem [5] For a number $c > 0$ the following 3 conditions are equivalent:

- a rectangle of side ratio c can be tiled by rectangles of side ratio c (not all homothetic to each other);

- the number c^2 is algebraic and all its algebraic conjugates distinct from c^2 are negative real numbers.
- for certain positive rational numbers $d_1, \ldots, d_m$ we have $\dfrac{1}{d_1 c + \cdots + \dfrac{1}{d_m c}} = c.$

References

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S-adic words, Rauzy fractals, and torus rotations

In the late 1970s Rauzy observed that classical continued fraction expansions can be used to show that Sturmian words are natural codings of rotations on the one-dimensional torus $\mathbb{T}^1$ (this was originally proved by Morse and Hedlund using combinatorial methods). In 1991 Arnoux and Rauzy proposed a class of three letter words (now called Arnoux-Rauzy words) and conjectured that each Arnoux-Rauzy word is a natural coding of a rotation on $\mathbb{T}^2$. They set up a continued fraction algorithm that is suitable for this setting. However, up to now the conjecture could only be proved for examples that correspond to periodic continued fraction expansions. On the other hand, in 2000 Cassaigne, Ferenczi, and Zamboni exhibited examples of Arnoux-Rauzy words that cannot be codings of rotations on $\mathbb{T}^2$.

Setting up a general theory for the geometry of S-adic sequences, we are able to prove that the conjecture of Arnoux and Rauzy is true for almost all Arnoux-Rauzy words (w.r.t. a natural measure). We also exhibit concrete non-periodic Arnoux-Rauzy words that satisfy this conjecture. Moreover, we give examples for our new theory that correspond to S-adic words defined in terms of Brun's continued fraction algorithm.

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Kloosterman sums and continued fractions

Analytical approach based on the method of trigonometric sums and estimates of Kloosterman sums allows to solve different problems concerned with classical continued fractions. The first talk will contain a survey ofresults of this type. The second talk will be devoted to analogous 3-D tool. It is also based on the estimates of Kloosterman sums and uses Linnik-Skubenko ideas from their work "Asymptotic distribution of integral matrices of third order" (1964). In particular this tool allows to study statistical properties of Minkovski-Voronoi 3-D continued fractions.