Laguerre minimal surfaces and isotropic geometry

Helmut Pottmann, **Philipp Grohs**, Niloy J. Mitra

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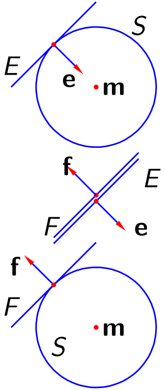

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surfaces are represented by their contact elements.

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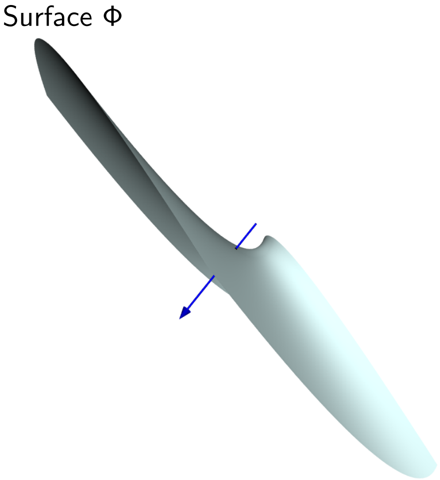
Middle spheres



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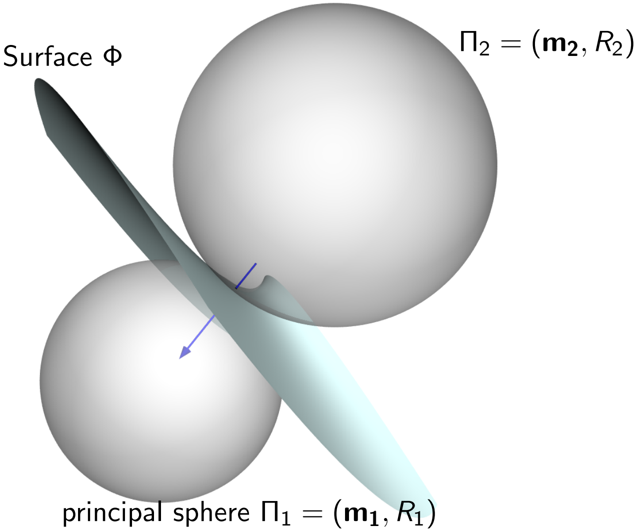

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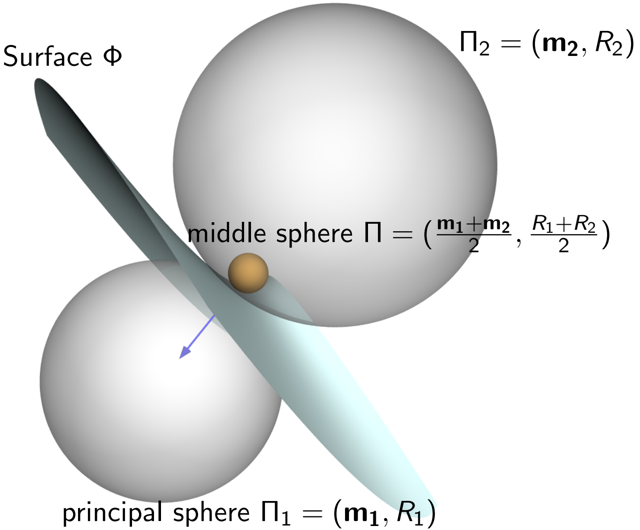

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Associate to every surface point p the corresponding middle sphere $\Pi_p = (\mathbf{m}_p, R_p) \in \mathbb{R}^4$ to get a 2-surface $\Phi_M \subseteq \mathbb{R}^4$.

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$$\mathbf{x} \cdot \mathbf{y} = x_1 y_1 + \cdots + x_3 y_3 - x_4 y_4.$$

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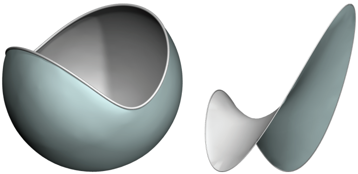
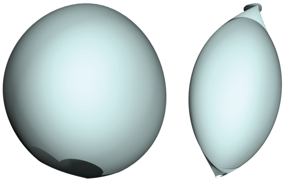
$$\left(= \text{ Minimizer of } \int_\Phi \frac{(H^2 - K)}{K} dA. \right)$$

Examples of L-minimal surfaces





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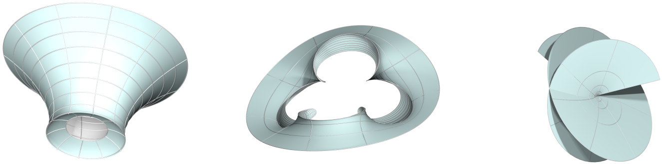

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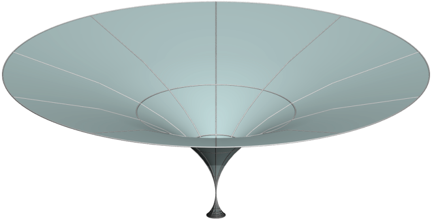

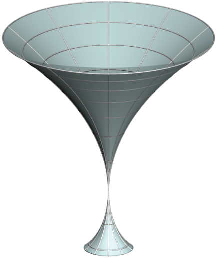

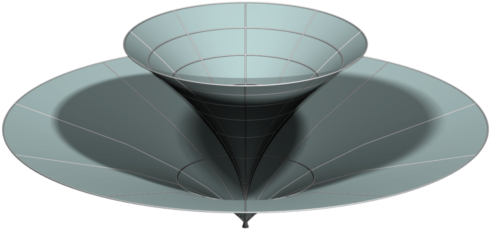

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The vector $(\mathbf{n}, h) = (n_1, n_2, n_3, h)$ gives a coordinate representation of the set of planes with values in $S^2 \times \mathbb{R} \subseteq \mathbb{R}^4$ (Blaschke cylinder).

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Spheres are represented by the image of their oriented tangent planes on the Blaschke cylinder (ellipsoids).

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planes $\mapsto$ points on $S^2 \times \mathbb{R}$

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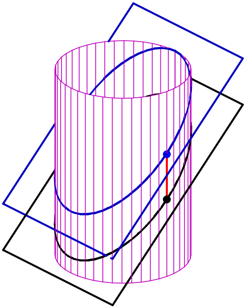
Laguerre trafos are projective transformations of the projective 4-space that leave the Blaschke cylinder invariant.



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apply stereographic projection from the point $(0, 0, -1, 0) \in S^2 \times \mathbb{R}$ to map planes to points in $\mathbb{R}^3$.

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add a line to $\mathbb{R}^3$ to make the mapping bijective and well defined.

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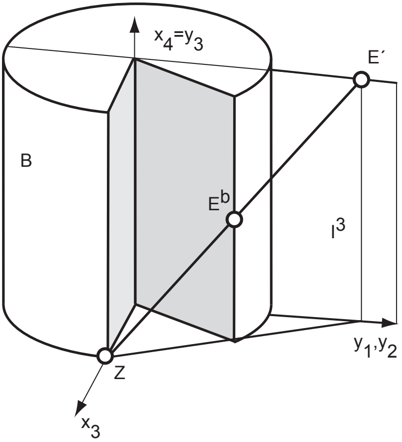

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$$\begin{aligned} \text{planes} &\mapsto \text{points} \\ \text{spheres} &\mapsto \text{paraboloids of revolution with } z\text{-parallel axis.} \end{aligned}$$

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$$\begin{aligned} \text{planes} &\mapsto \text{points} \\ \text{spheres} &\mapsto \text{paraboloids of revolution with } z\text{-parallel axis.} \\ \text{Tangent planes of surfaces } \Phi &\mapsto \Phi_I = \text{Graph of a function } f(x,y). \end{aligned}$$

Note that points are just spheres with radius 0!

Minimal surfaces

Theorem (Pottmann - G - Mitra 07)

Let Φ be an L-surface and $\Phi_I : f(x, y)$ its isotropic counterpart. Φ is L-minimal if and only if f is a biharmonic function, i.e.

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$\rightsquigarrow$ Problem of finding L-minimal surface $\cong$ Problem of finding graphs of biharmonic functions (well-studied i.e. in linear elasticity!).

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Harmonic functions

The bisecting planes of the principal curvature centers envelope the **middle evolute**.

Theorem (Kommerell)

A surface is L-minimal if and only if its middle evolute is a Euclidean minimal surface.

Harmonic functions

This leads to a construction of Euclidean minimal surface from L-minimal surfaces of the spherical type:

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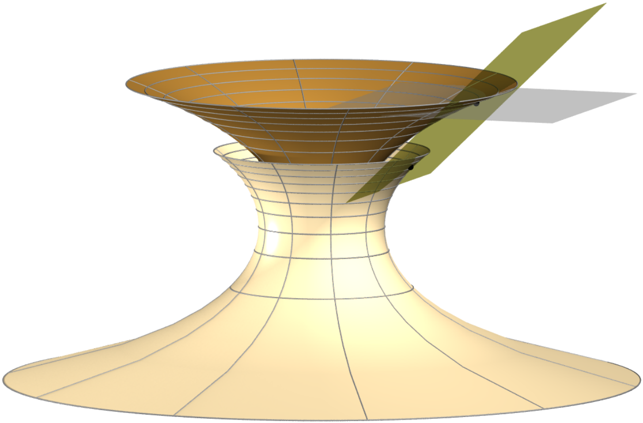
Idea of proof: Show that the resulting surface is the middle evolute of another L-minimal surface.

Middle evolute



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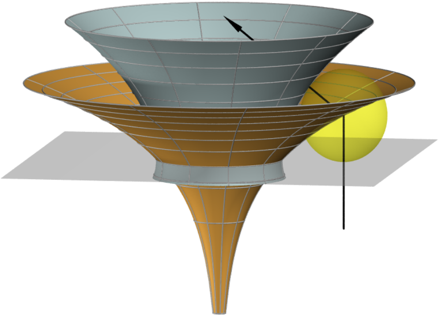
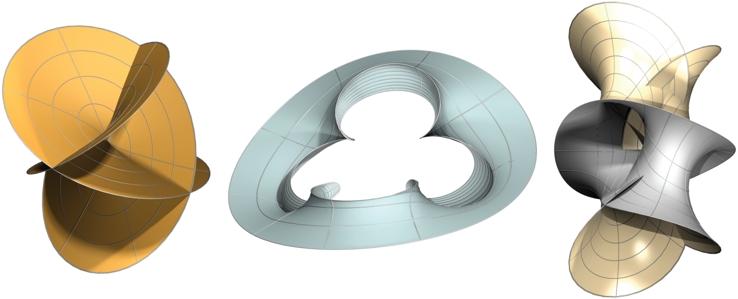
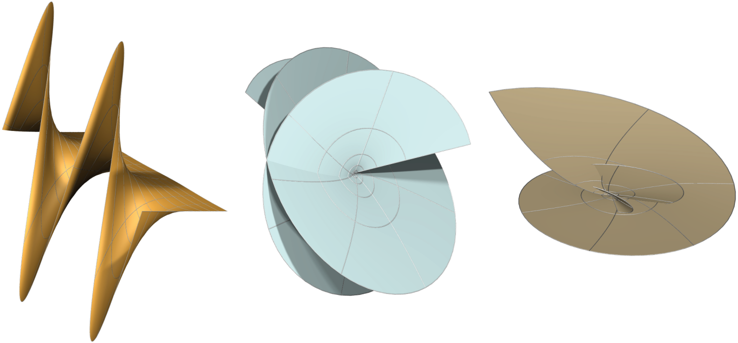


Enneper minimal surface



Scherk minimal surface



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Given a set of planes find an L-minimal surface that touches all of these planes.

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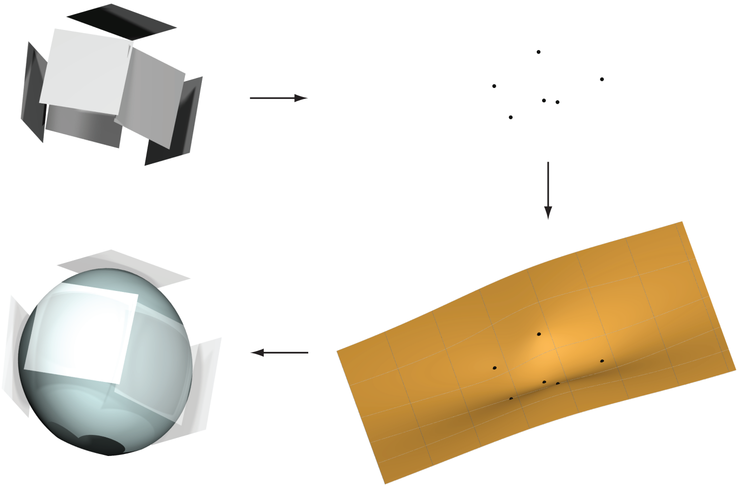

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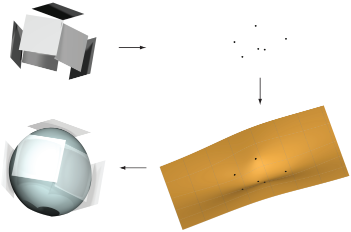

Can solve the interpolation problem explicitely.

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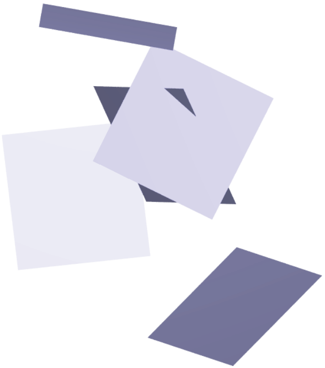

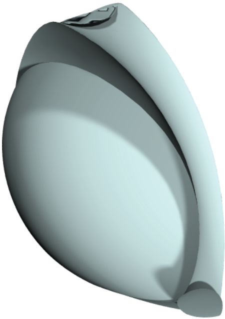

Solving the Boundary value problem (Plateau problem)

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Solution:

- Transform the initial data into the isotropic model
- Solve biharmonic BVP
- Transform back

Solving the Boundary value problem

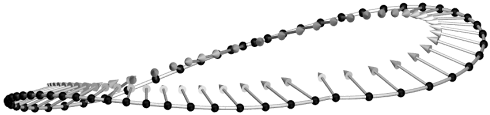
Theorem

A biharmonic function $f(x, y)$ is uniquely determined by a boundary strip. This means that for a Domain D, and two functions g, h defined on ∂D, there exists a unique biharmonic function f with

$$\begin{aligned} f(x,y) &= g \quad \text{for } (x,y) \in \partial D \\ \frac{\partial}{\partial n} f(x,y) &= h \quad \text{for } (x,y) \in \partial D \end{aligned}$$

where n is the unit normal of ∂D pointing towards the inside of D.

Solving the Boundary value problem (isotropic model)



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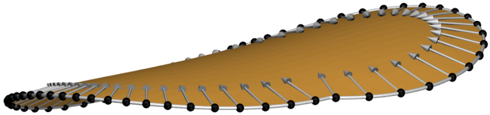
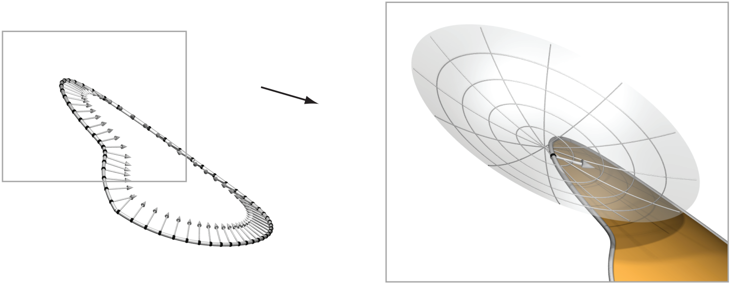


Transferring the L-boundary data to isotropic boundary data



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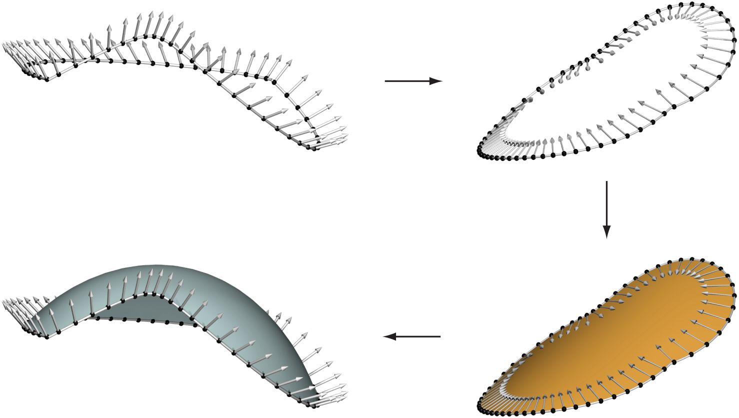
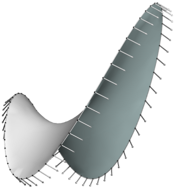
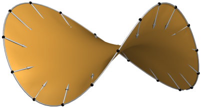


other examples





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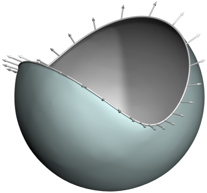

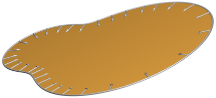

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- Interpretation of Problems in linear elasticity in terms of L-geometry
- Interpretation in terms of geometrical optics is possible ($\rightarrow$ paper)

Future tasks

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- construct and classify discrete (isothermic) L-minimal surfaces

The end

Thank you for your attention!