Discrete Conformal Structures

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THE PROBLEM

Find “nice” texture maps

- simplicial surface
  \[ K = \{V, E, T\} \]
- metric data
  \[ L = \{l_{ij} | e_{ij} \in E, l_{ij} > 0\} \]

\[ V = \{v_i | 1 \leq i \leq n\} \]
\[ E = \{e_{ij} | i, j \in V\} \]
\[ T = \{t_{ijk} | e_{ij}, e_{jk}, e_{ki} \in E\} \]
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- new (flat) metric \( \tilde{L} \)
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**Ansatz**

Seek conformally equivalent metric

- **data:** simplified complex & lengths
- **output:** new metric (i.e., lengths)

\[ \tilde{g} = e^{2u} g \quad K \neq 0 \quad \rightarrow \quad \tilde{K} = 0 \]

- ignore boundary for the moment

new metric

conformal factor

original metric

ignore boundary for the moment
Ansatz

Seek conformally equivalent metric

- data: simpl. complex & lengths
- output: new metric (i.e., lengths)
- $\tilde{g} = e^{2u}g$ \quad $K \neq 0 \rightarrow \tilde{K} = 0$
- variables at vertices
  
  $u(v_i) = u_i$  \quad $\tilde{l}_{ij} = l_{ij}e^{u_i+u_j}$
Ansatz

Seek conformally equivalent metric

- data: simpl. complex & lengths
- output: new metric (i.e., lengths)
  \[ \tilde{g} = e^{2u} g \quad K \neq 0 \longrightarrow \tilde{K} = 0 \]
- variables at vertices
  \[ u(v_i) = u_i \quad \tilde{l}_{ij} = l_{ij} e^{u_i+u_j} \]
- goal: desired angle sums
  \[ \Theta(v_i) = \sum_{t,ijk} \alpha_{jk}^i(u_i, u_j, u_k) = \tilde{\Theta}_i \]
**Non-Linear Problem**

Find $u(V)$ to satisfy angle sum targets from lengths to angles

$\forall i \in V : \Theta_i = \sum_{t_{ijk} \ni v_i} \alpha^{i}_{jk}(u_i, u_j, u_k)$

$2 \tan^{-1} \sqrt{\frac{(-a+b+c)(a+b-c)}{(a-b+c)(a+b+c)}}$

Watch out for triangle inequality!
Non-Linear Problem

Find $u(V)$ to satisfy angle sum targets

- from lengths to angles

$$\forall i \in V : \tilde{\Theta}_i = \sum_{t_{ijk} \ni v_i} \alpha_{jk}^i (u_i, u_j, u_k)$$

- ... a miracle occurs ...

This system of equations can be integrated!
The Energy

Find minimum of a convex energy

- Milnor’s Lobachevsky function

\[ \Pi(x) = - \int_0^x \log 2|\sin t|dt \]

\[ \frac{l_{jk}}{R} = 2 \sin \alpha_{jk} \]

\[ \downarrow \]

\[ \log l_{jk} - \log R = \log 2 \sin \alpha_{jk} \]
The Energy

Find minimum of a convex energy

- Milnor’s Lobachevsky function
- for each triangle

\[ f(x_{12}, x_{23}, x_{31}) = \alpha_1 x_{23} + \alpha_2 x_{31} + \alpha_3 x_{12} + \Pi(\alpha_1) + \Pi(\alpha_2) + \Pi(\alpha_3) \]

\[ x_{ij} = \lambda_{ij} + u_i + u_j \]

[Diagram: Triangle with logarithmic input lengths]
Find minimum of a convex energy

- Milnor’s Lobachevsky function
- for each triangle

\[
\frac{d}{du_i} f(x_{ij}, x_{jk}, x_{ki}) = \pi - \alpha_i
\]

\[
E(u) = \sum_{t_{ijk} \in T} (f(u_i, u_j, u_k) - \pi (u_i + u_j + u_k)) + \sum_{\nu_i \in V} \tilde{\Theta}_i u_i
\]
The Energy

Properties

- convex: Hessian is pos. semi-def.

\[ u^T H u = \sum_{e_{ij} \in E} (\cot \alpha_{jk}^i + \cot \alpha_{kj}^l)(u_k - u_j)^2 \]

only one term for boundary edges
The Energy

Properties

- convex: Hessian is pos. semi-def. 
  \[ u^T H u = \sum_{e_{ij} \in E} \left( \cot \alpha_{jk}^i + \cot \alpha_{kj}^l \right) (u_k - u_j)^2 \]
- solution exists \( \Rightarrow \) is unique \( \min E(u) \)
- gradient flow is curvature flow 
  \[ \frac{d}{dt} u(t) = -\nabla E(u(t)) = \tilde{K} - K(t) \]
The Energy

Properties

- convex: Hessian is pos. semi-def.
  \[ u^T H u = \sum_{e_{ij} \in E} \left( \cot \alpha_{jk}^i + \cot \alpha_{kj}^l \right) (u_k - u_j)^2 \]

- solution exists \( \Rightarrow \) is unique \( \min E(u) \)

- gradient flow is curvature flow
  \[ \frac{d}{dt} u(t) = -\nabla E(u(t)) = \tilde{K} - K(t) \]

- what about triangle inequality?!
Domain of Definition

Not just any $u$ value is cool...

- triangle inequality

legal range

$u_1 + u_2 + u_3 = 0$

$\lambda_{12} = \lambda_{23} = \lambda_{31} = 0$
Domain of Definition

Not just any $u$ value is cool...

- triangle inequality
- extend definition

$$\alpha = \Re \left( 2 \tan^{-1} \sqrt{\frac{(-a+b+c)(a+b-c)}{(a-b+c)(a+b+c)}} \right)$$

Real part only

Functional remains $C^1$ in $u$

$u_1 + u_2 + u_3 = 0$

$\lambda_{12} = \lambda_{23} = \lambda_{31} = 0$
Domain of Definition

Not just any $u$ value is cool...

- triangle inequality
- extend definition
  \[ \alpha = \text{Re} \left( 2 \tan^{-1} \sqrt{\frac{-(a+b+c)(a+b-c)}{(a-b+c)(a+b+c)}} \right) \]
- minimum may occur at illegal values...
- conditions for existence guarantee?
Boundary Conditions

Fixing variables

- fix $u_i$ let $\Theta_i$ vary
  - $u_i = 0$: isometric bndry.
  - nice for cut-boundaries!
  - unknown cone angles
  - arbitrary topology
- fix $\Theta_i$ let $u_i$ vary
  - rectangle, disk...
Convex optimization

- Newton-Steihaug trust region
Convex optimization
- Newton-Steinhaug trust region
- Petsc/TAO library
- SSOR precon for cotan system
- layout: dual spanning tree
  - achieves $10^{-9}$ to $10^{-13}$ acc.
  - alternatively: Dirichlet problem
Conformal Equivalence

From continuous to pair of meshes

\[ \tilde{g} = e^{2u} g \]
\[ \tilde{l}_{ij} = e^{ui} l_{ij} e^{uj} \]

equivalently:

\[ \tilde{c}_{ij} = \tilde{l}_{il} \tilde{l}_{lj}^{-1} \tilde{l}_{jk} \tilde{l}_{ki}^{-1} \]

\[ e^{-uj} \quad e^{uj} \]
**Conformal Equivalence**

From continuous to pair of meshes

\[ \tilde{g} = e^{2u} g \]

\[ \tilde{l}_{ij} = e^{ui} l_{ij} e^{uj} \]

equivalently:

\[ \tilde{c}r_{ij} = \tilde{l}_{il} \tilde{l}_{lj}^{-1} \tilde{l}_{jk} \tilde{l}_{ki}^{-1} = cr_{ij} \]

Length cross ratios are preserved
WHY $u = 0$ ON BOUNDARY?

In the continuous setting

- conformally equivalent metric
  \[ \tilde{K} = e^{-2u}(K - \Delta u) \quad \text{flat} \quad \Rightarrow \Delta u = K \]
- choose “least varying”:
  \[
  \min_{\text{flat}} \int |du|^2
  \Rightarrow u|_{\partial M} = c
  \]
**WHY \( u = 0 \) ON BOUNDARY?**

In the continuous setting
- conformally equivalent metric
  \[
  \tilde{K} = e^{-2u}(K - \Delta u) \quad \text{flat} \implies \Delta u = K
  \]
- choose “least varying”:
  \[
  \min_{\text{flat}} u \int |du|^2
  \]
  \[
  \iff u|_{\partial M} = c
  \]
**Dual Functional**

Angles/lengths are dual variables

- functional in angles

\[
f(\alpha_i, \alpha_j, \alpha_k) = \lambda_{ij} \alpha_k + \lambda_{jk} \alpha_i + \lambda_{ki} \alpha_j + \Lambda(\alpha_i) + \Lambda(\alpha_j) + \Lambda(\alpha_k)
\]

\[
E(\alpha) = \sum_{t_{ijk} \in T} f(\alpha_i, \alpha_j, \alpha_k)
\]

Original logarithmic lengths

- \(\alpha_{jk} > 0\)
- \(\alpha_{jk}^i + \alpha_{ki}^j + \alpha_{ij}^k = \pi\)
- \(\sum_{t_{ijk} \ni i} \alpha_{jk}^i = \tilde{\theta}_i\)
**DUAL FUNCTIONAL**

Angles/lengths are dual variables

- functional in angles

\[
E(\alpha) = \sum_{t_{ijk} \in T} f(\alpha_i, \alpha_j, \alpha_k)
\]

- length cross ratios invariant

\[
\nabla E = 0 \iff \frac{l_{il}l_{jk}}{l_{il}l_{ki}} = \frac{\tilde{l}_{il}\tilde{l}_{jk}}{\tilde{l}_{il}\tilde{l}_{ki}}
\]

\[
\alpha^i_{jk} > 0 \\
\alpha^i_{jk} + \alpha^j_{ki} + \alpha^k_{ij} = \pi \\
\sum_{t_{ijk} \ni i} \alpha^i_{jk} = \tilde{\Theta}_i
\]
THE BIG PICTURE

Discrete conformal structure

- simplicial mesh
  \[ z_{il} z_{lj}^{-1} z_{jk} z_{ki}^{-1} \]
- preserve:
  - phase: circle patterns
    - can’t read off angles directly...
  - magnitude: new functional
    - CAN read off lengths directly!
TODO LIST

Future work
- conditions for existence
  - intrinsic Delaunay not required!
- automatic cone singularity placement
  - $u$ provides potential hook

\[ \Delta u^0 = K \]

- first Newton step
- sparse approximation problem