Meshes with offset properties

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Overview

- Nodes without torsion
- Parallel meshes and geometric support structure
- offsets and discrete Gauss image
- circular meshes, conical meshes, EO meshes
- mesh optimization
- References: [Liu et al. SIGGRAPH 06], [Pottmann et al. SIGGRAPH 07], [Bobenko and Suris 2007]

Beams and Nodes in meshes

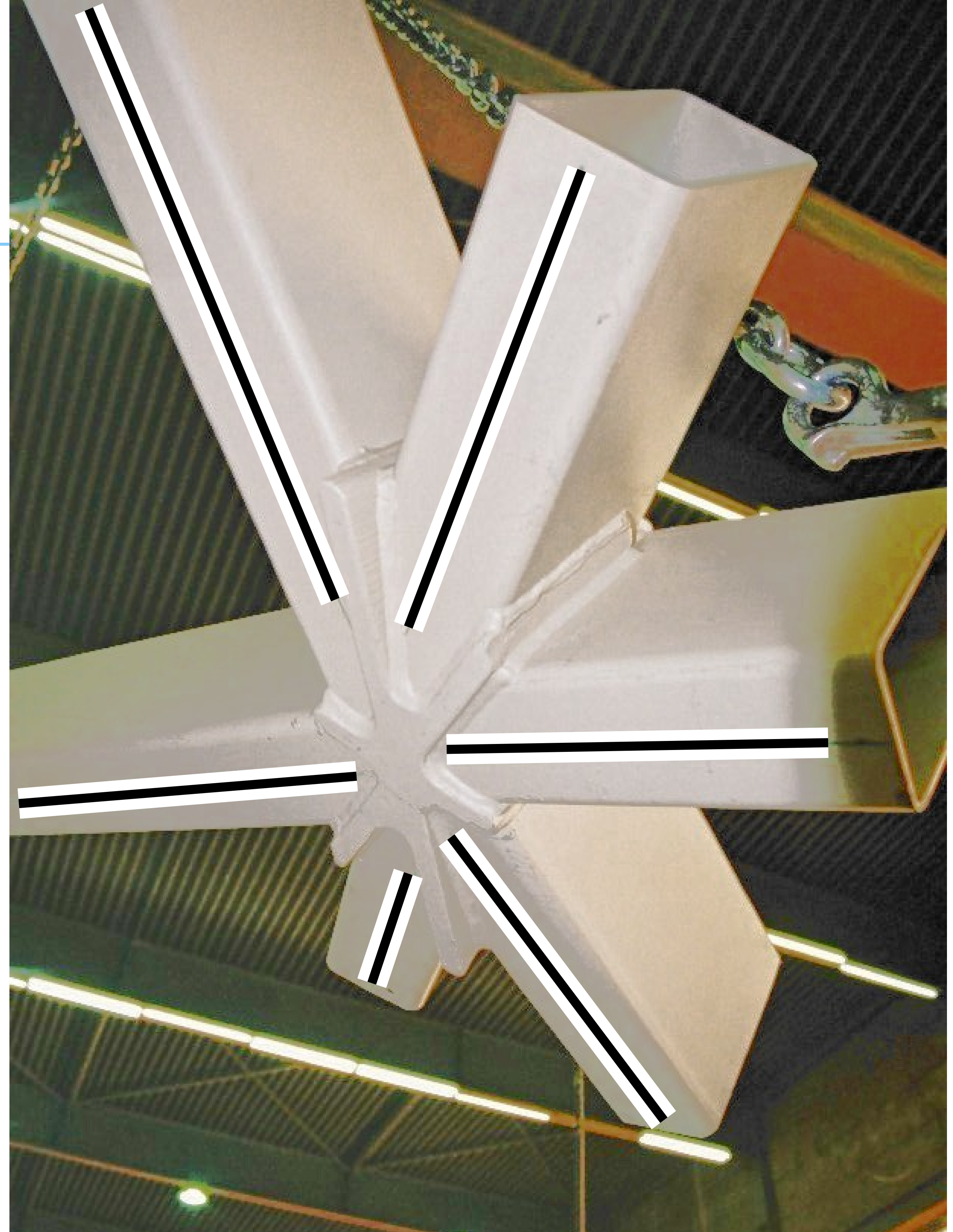
(Image courtesy Waagner Biro, Vienna)

Example of node

(Image courtesy Waagner Biro, Vienna)

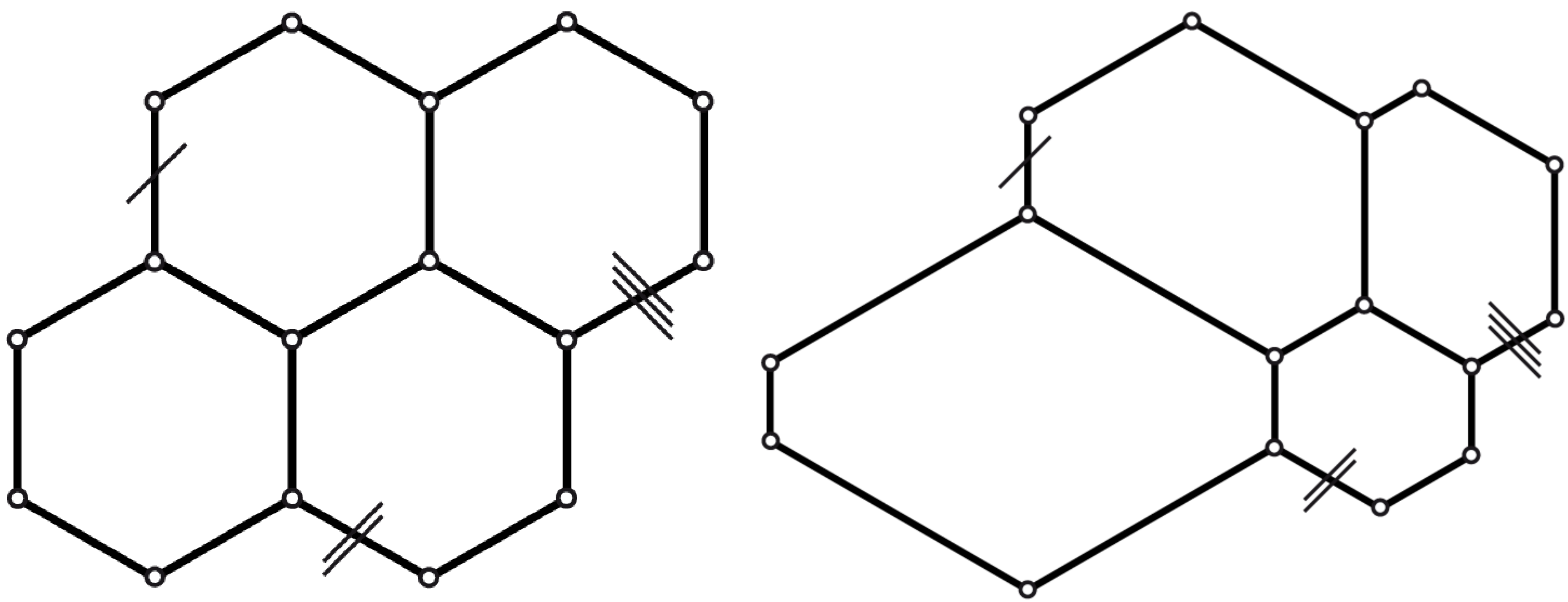
Node Torsion

- Problems with node geometry:
- Symmetry planes of beams do not intersect



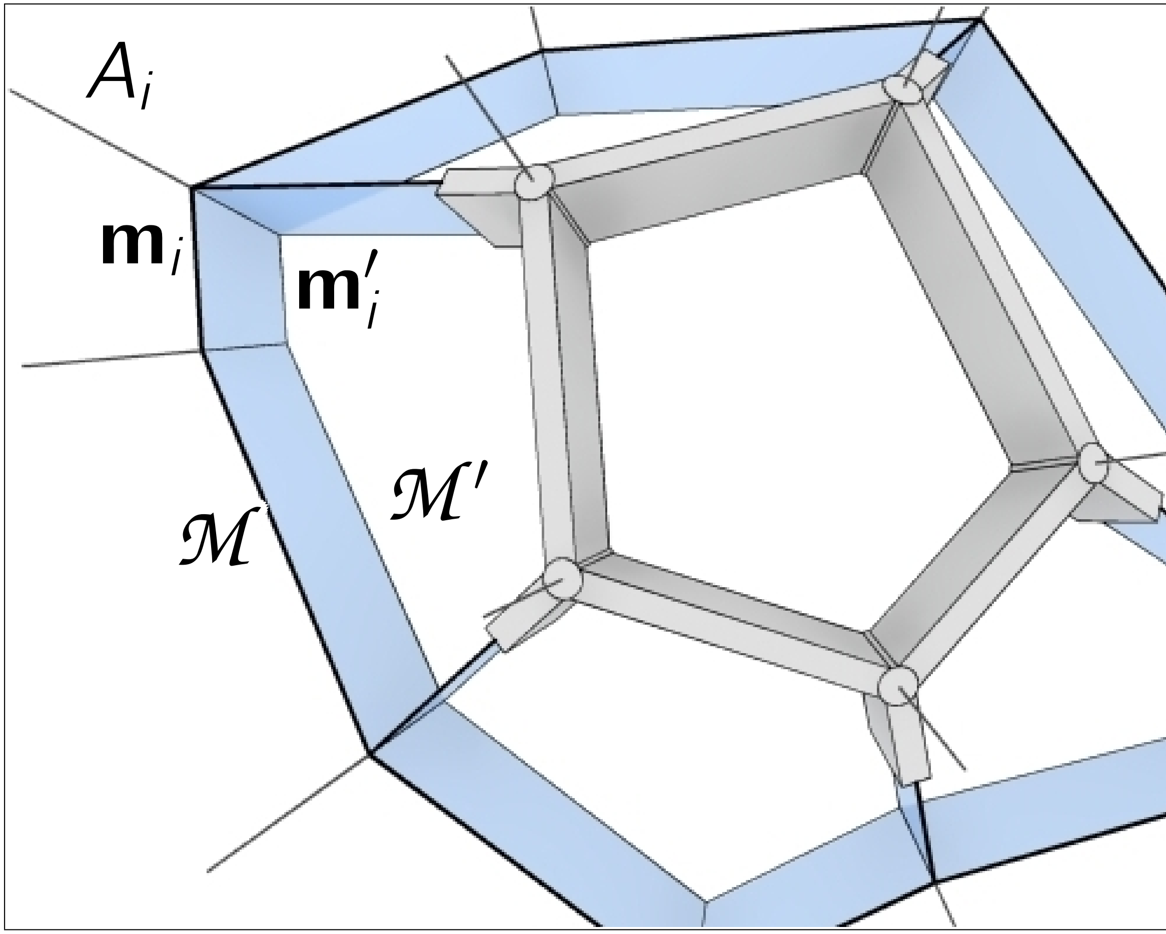
Parallel meshes

- Meshes are parallel, if they are combinarially equivalent, and corresponding edges are parallel.



Mesh parallelity / Node torsion

- Suppose symmetry planes of beams co-incide with planes spanned by corr. parallel edges
- $\Longrightarrow$ node axes exist, no torsion

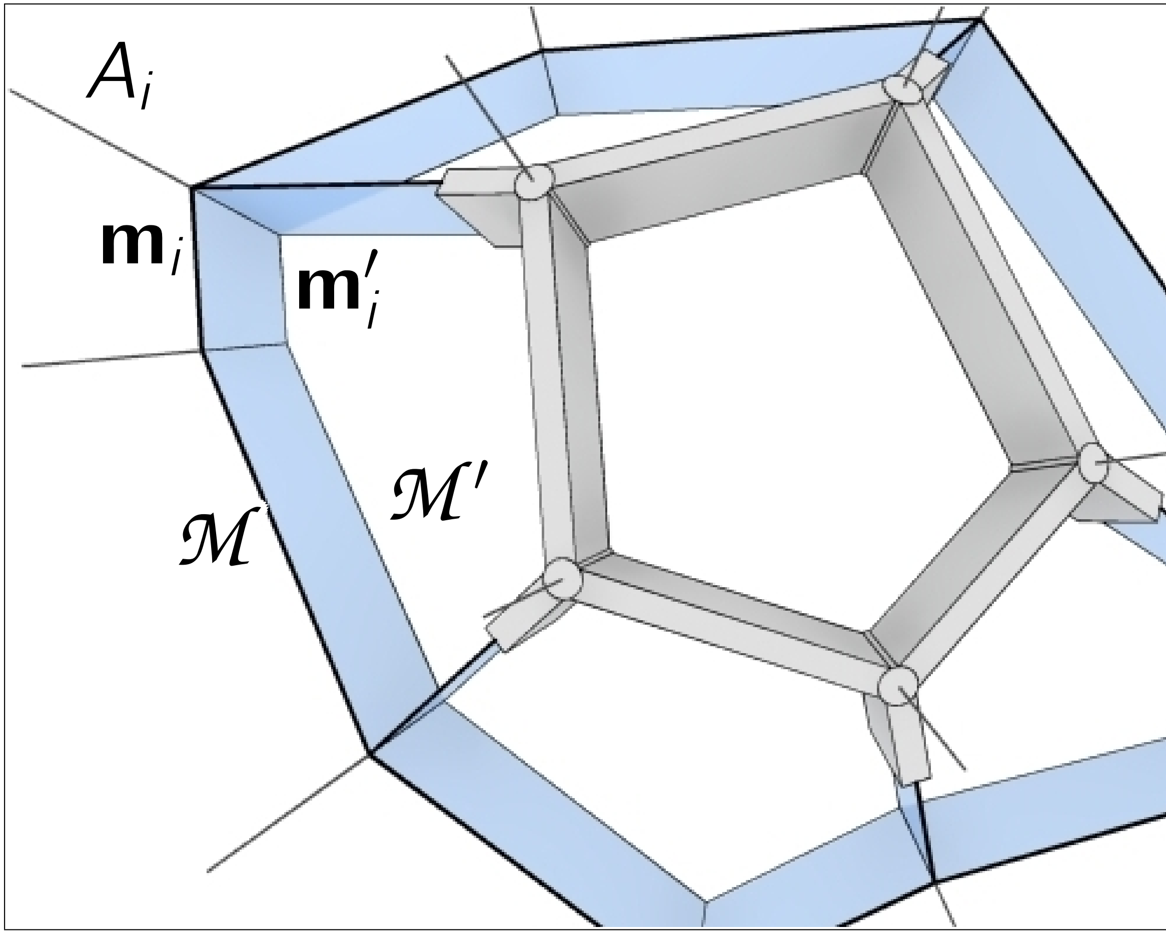


Parallel triangle meshes

- For 2 triangles Δ, Δ' with parallel edges there exists a central similarity transformation with $\Delta \mapsto \Delta'$
- Factor of similarity determined by edge length ratio
- $\implies$ If connected triangle meshes are parallel then they are homothetic.

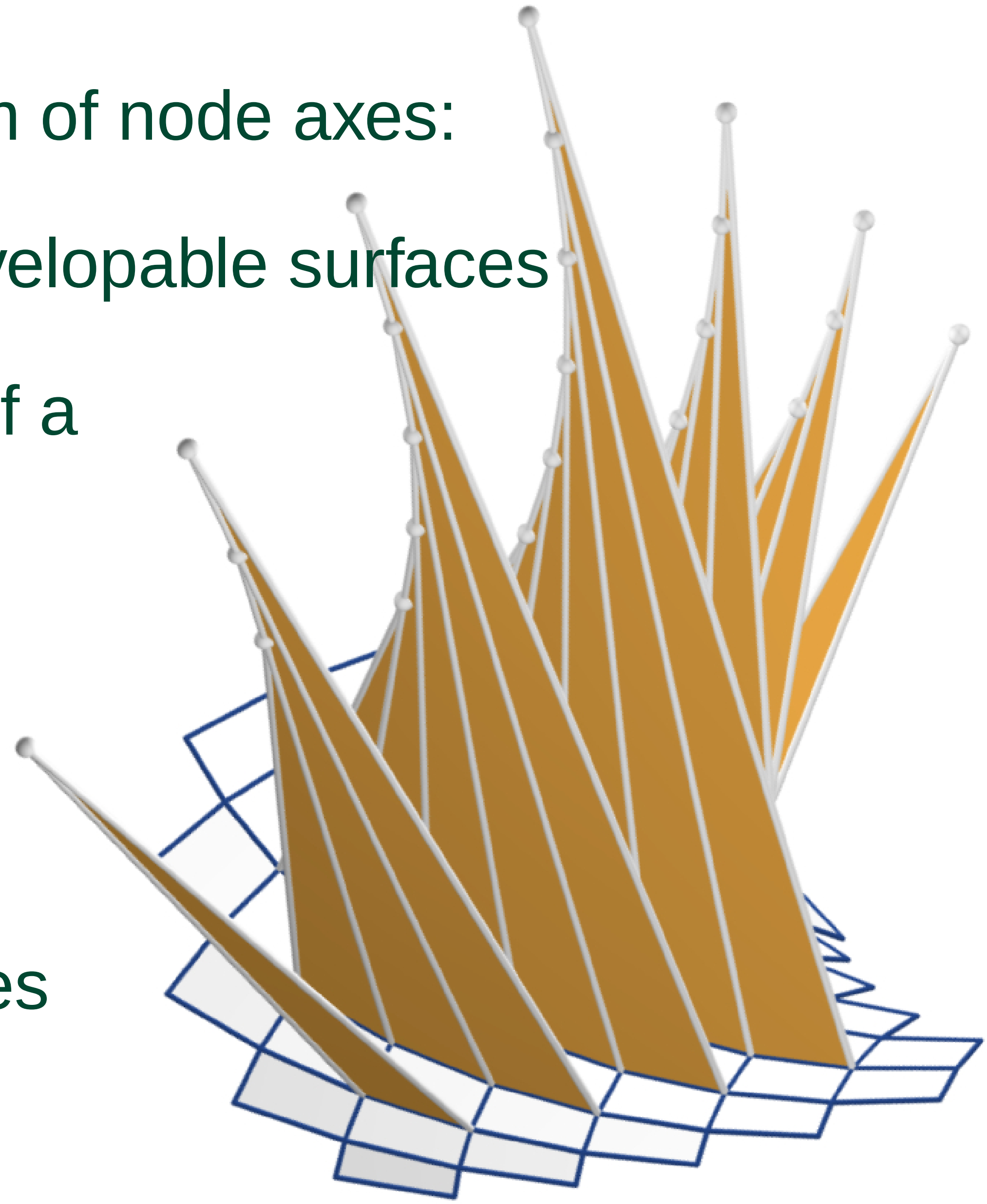
Geometric support structure

- consists of 2 parallel meshes plus connecting planes
- plus node axes
- neighbouring node axes **intersect each other**



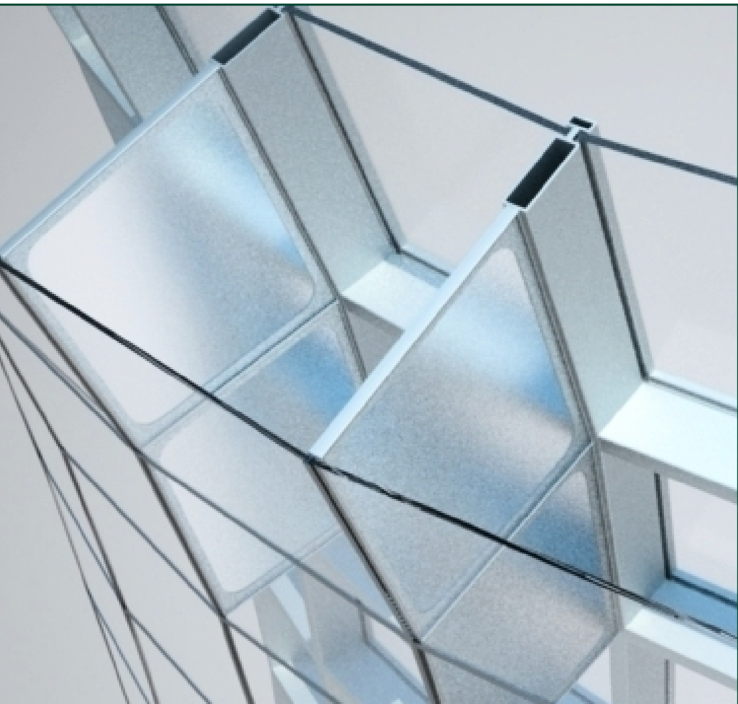
Geometric Support Structure

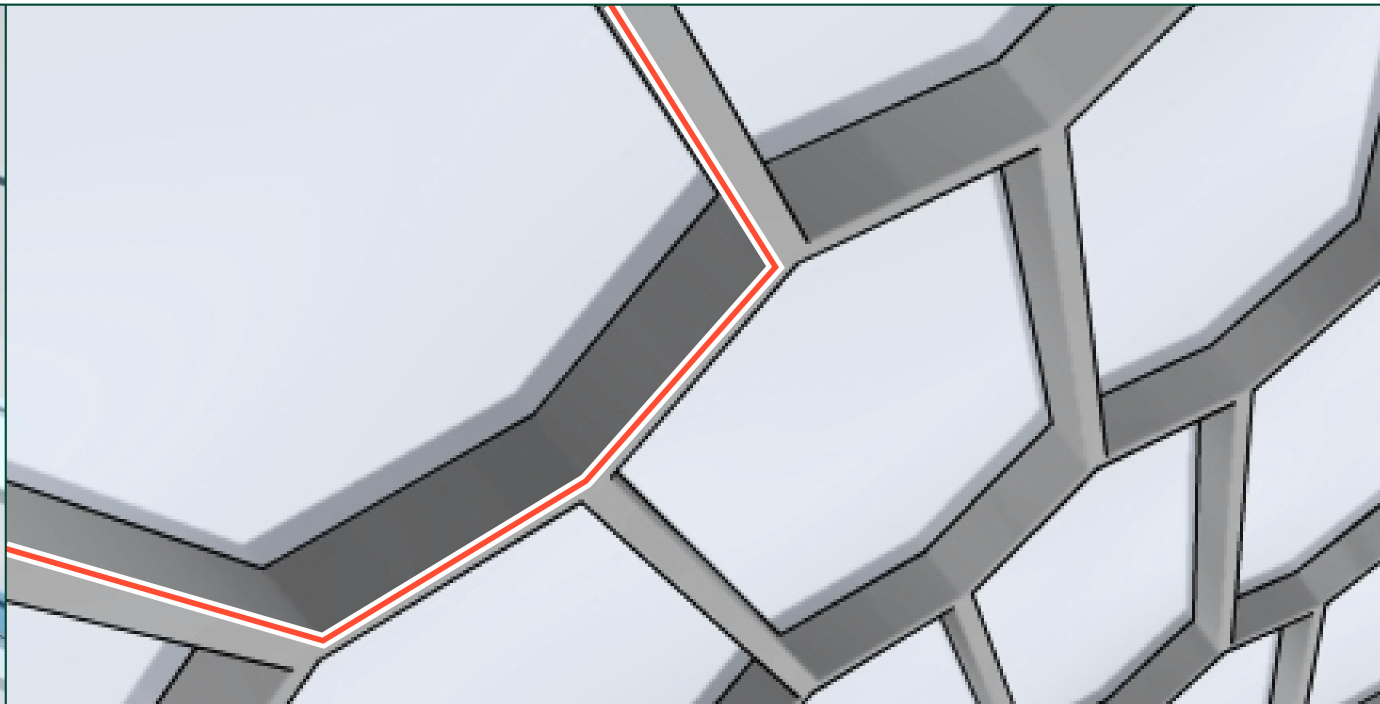
- Structure of the system of node axes:
- They form discrete developable surfaces
- (just like the normals of a principal curvature line parametrization)
- Parallel mesh can be reconstructed from axes



Meshes at constant distance

- Offsets meshes are used for multilayer constructions
- Beams also count as multilayer constructions (faces are not physically realized)





Meshes at constant distance

- Parallel meshes $\mathcal{M}$, $\mathcal{M}'$ are:
- **vertex offsets** $\iff$ the distance of corresponding vertices is constant throughout the mesh
- **edge offsets** $\iff$ corr. edges have constant distance
- **face offsets** $\iff$ corr. faces have constant distance
- **approximate offsets** $\iff$ corresponding vertices/ edges/faces are at approximately constant distance.

Discrete Gauss image: v-offsets

- If meshes $\mathcal{M}$, $\mathcal{M}'$ are at constant **vertex-vertex** distance d, then the **vertices** of $\boxed{\mathcal{S} = \dfrac{\mathcal{M}' - \mathcal{M}}{d}}$ have distance 1 from the origin.
- $\implies$ There is a ‘spherical’ mesh $\mathcal{S}$ with $\mathcal{M}' = \mathcal{M} + d\mathcal{S}$.
- View vertices of $\mathcal{S}$ as unit normal vectors

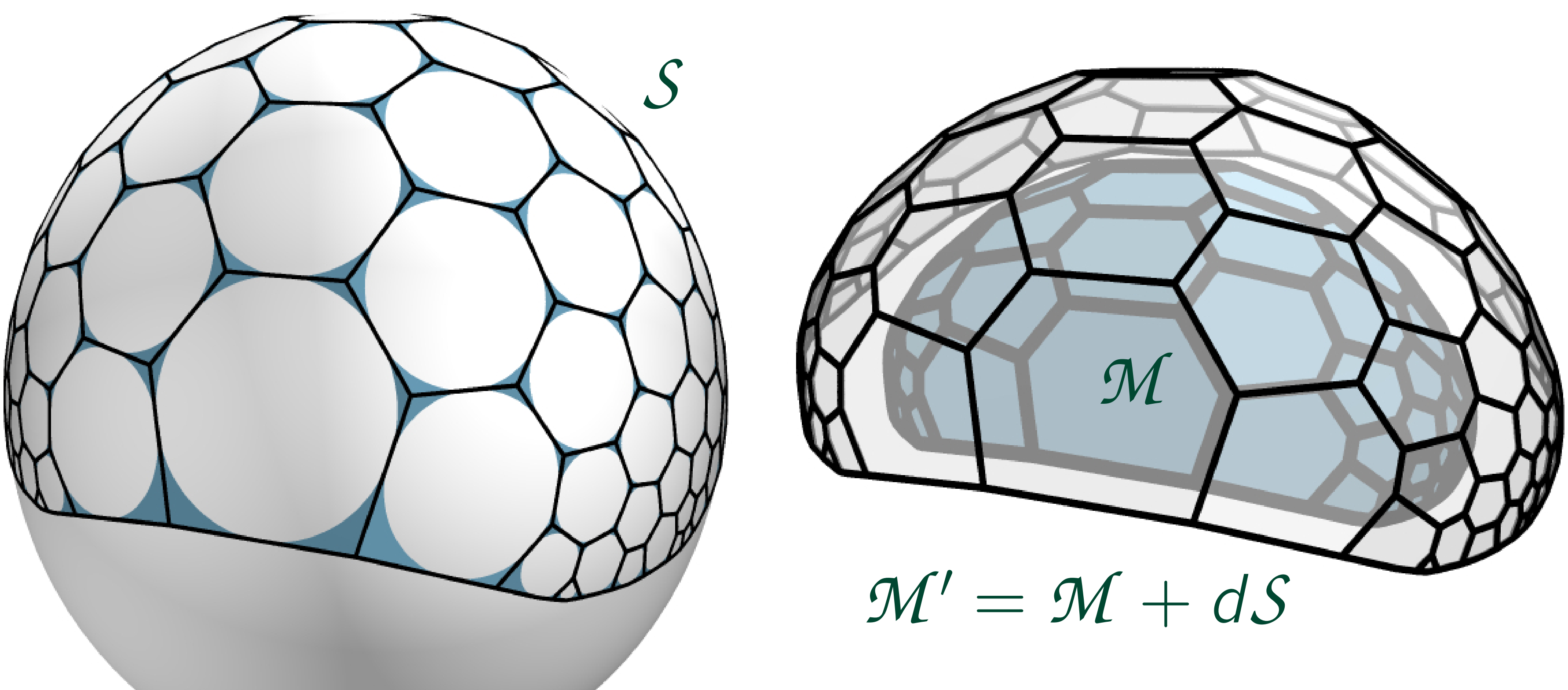
  ($\mathcal{S}$ is Gauss image)
- $\mathcal{M} \parallel \mathcal{M}' \implies \mathcal{M} \parallel \mathcal{S}$

Discrete Gauss image: f-offsets

- If meshes $\mathcal{M}$, $\mathcal{M}'$ are at constant **face-face** distance d, then the **faces** of $\boxed{\mathcal{S} = \dfrac{\mathcal{M}' - \mathcal{M}}{d}}$ have distance 1 from the origin.
- $\implies$ There is a ‘spherical’ mesh $\mathcal{S}$ with $\mathcal{M}' = \mathcal{M} + d\mathcal{S}$.
- View vertices of $\mathcal{S}$ as unit normal vectors ($\mathcal{S}$ is Gauss image)
- $\mathcal{M} \parallel \mathcal{M}' \implies \mathcal{M} \parallel \mathcal{S}$

Discrete Gauss image: e-offsets

- If meshes $\mathcal{M}$, $\mathcal{M}'$ are at constant **edge-edge** distance d, then the **edges** of $\boxed{\mathcal{S} = \dfrac{\mathcal{M}' - \mathcal{M}}{d}}$ have distance 1 from the origin.



Discrete Gauss image: v-offsets

- A *quad* mesh $\mathcal{M}$ has vertex offsets

  $\implies$ $\exists$ a quad mesh $\mathcal{S}$ with $\mathcal{S} \parallel \mathcal{M}$ and vertices in S^2

  $\implies$ Every quad of $\mathcal{M}$ has a circumcircle

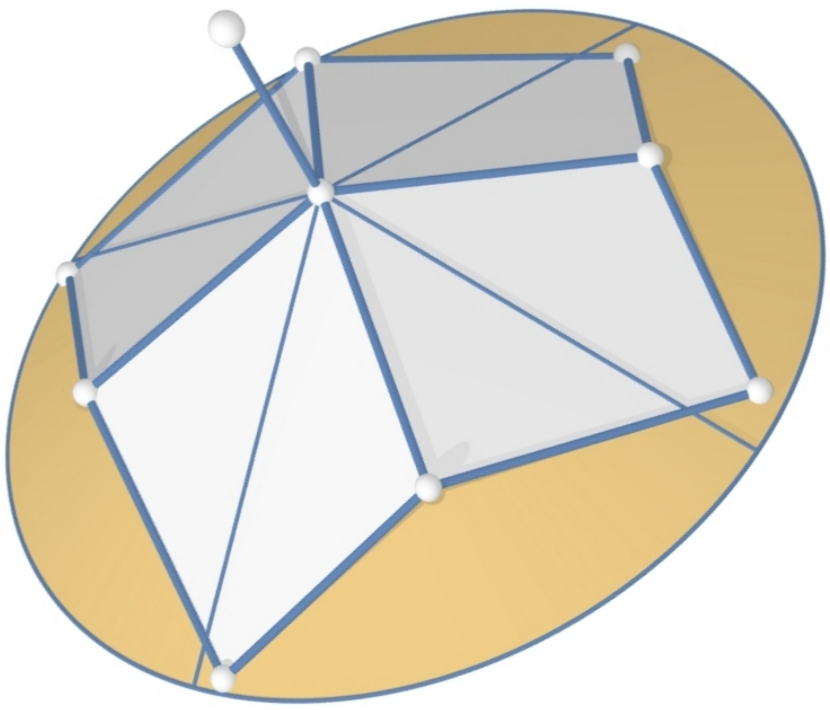
  $\implies$ $\mathcal{M}$ is a *circular quad mesh*

- The converse is true for simply connected meshes.

  [Konopelchenko and Schief 1998], [Pottmann et al 2007]

Discrete Gauss image: f-offsets

- A mesh $\mathcal{M}$ has face offsets
- $\iff$ $\exists$ a mesh $\mathcal{S}$ with $\mathcal{S} \parallel \mathcal{M}$ and faces tangent to S^2
- $\iff$ for $\mathcal{S}$, the faces adjacent to a vertex lie in a cone of revolution
- $\iff$ for $\mathcal{M}$ the same is true
- $\iff$ $\mathcal{M}$ is a conical mesh
- [Liu et al SIGGRAPH 2006]



Discrete Gauss image: e-offsets

- A mesh $\mathcal{M}$ has edge offsets $\Longleftrightarrow$
- $\exists$ a mesh $\mathcal{S}$ with $\mathcal{S} \parallel \mathcal{M}$ and edges tangent to S^2
- $\mathcal{S}$ is then a Koebe polyhedron

  cf. [Bobenko, Springborn 2005ff]
- Edges emanating from a vertex

  are tangent to a cone

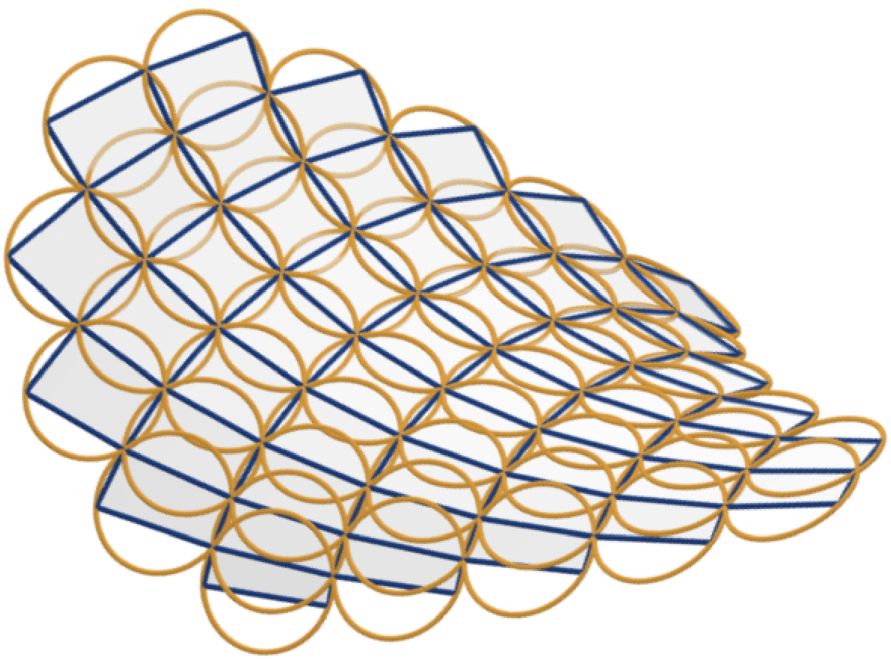
  (for both $\mathcal{S}$ and $\mathcal{M}$)

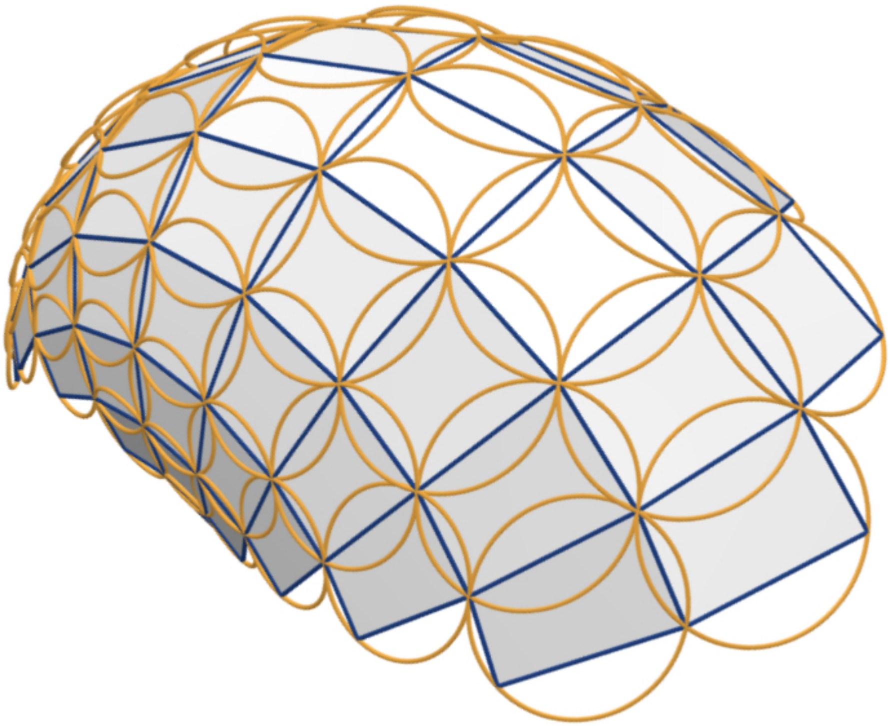
Example

- EO meshes are realizable as beams of constant height
- Here: 3D printout

Geometric Transformations: Möbius

- Apply a Möbius transformation to vertices of quad mesh which has v-offsets: result has v-offsets.
- Passage to parallel mesh keeps v-offset property





Geometric Transformations: Laguerre

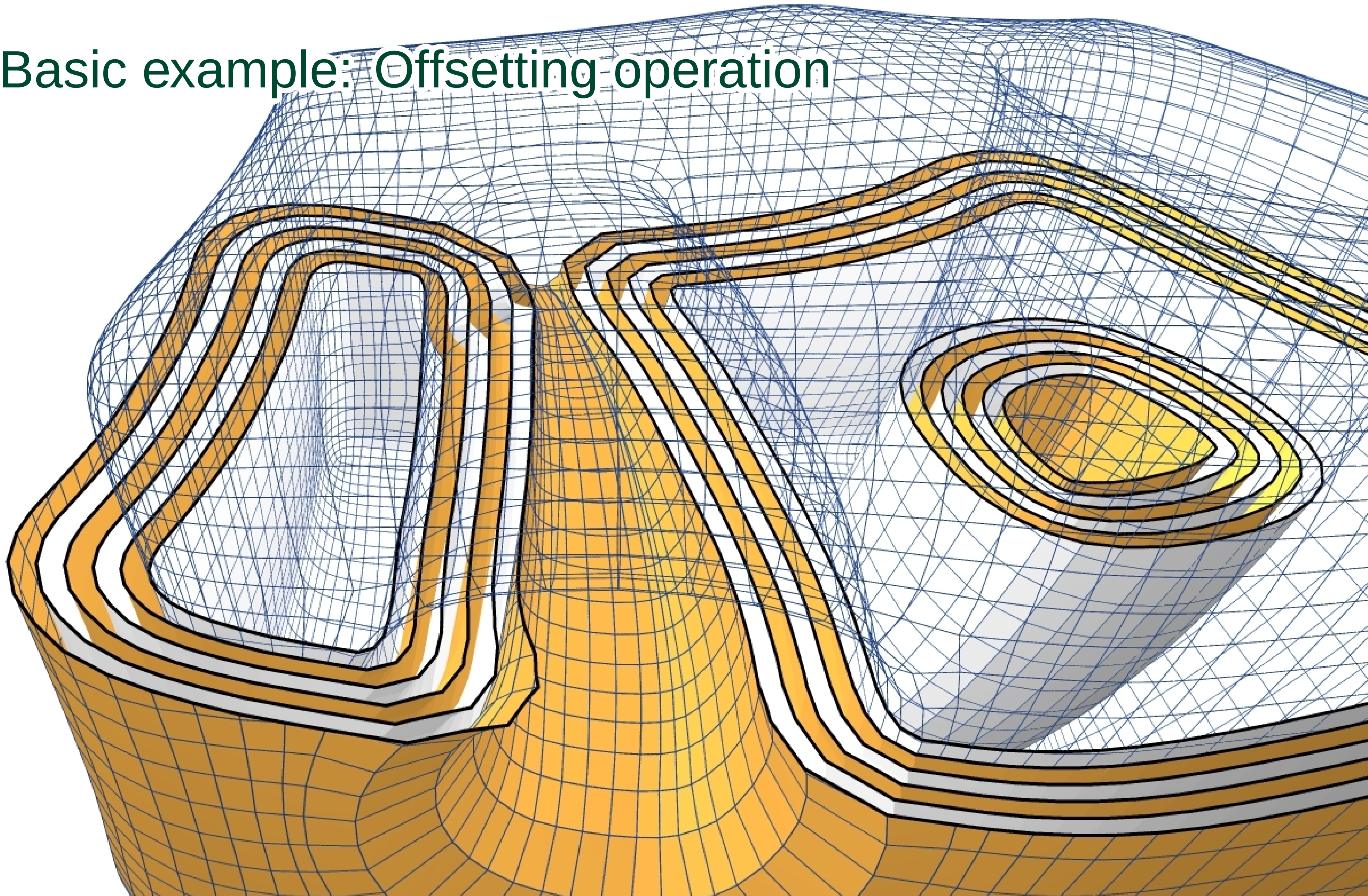
- A sphere with center (m_1, m_2, m_3) and radius $r \in \mathbb{R}$ is identified with a point of $\mathbb{R}^4$.
- For $J =$diag$(1, 1, 1, -1)$, an affine mapping $x \mapsto Ax + a$ in $\mathbb{R}^4$ is an L-transformation $\iff A^T JA = J$.
- spheres tangent to a plane/cone retain this property $\implies$ L-transformation applies to planes/cones.
- L-trafos keep contact sphere/plane/cone.

Geometric Transformations: Laguerre

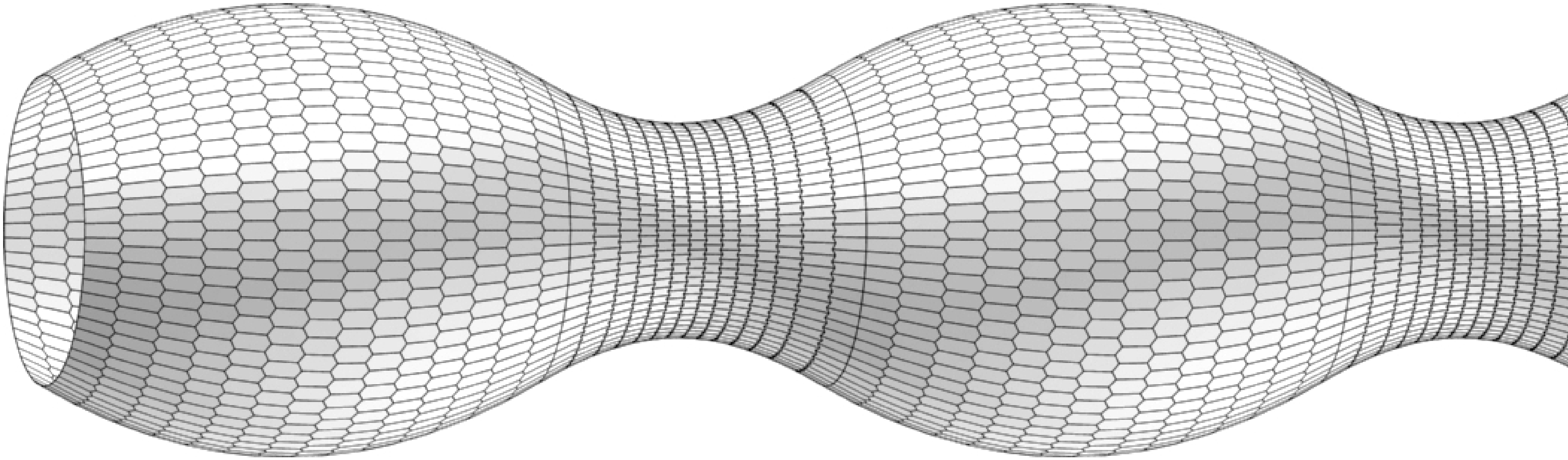
- Apply a Laguerre transformation to faces of mesh which has f-offsets: Result has f-offsets
- Apply a Laguerre transformation to edge cones associated with the vertices of an EO mesh: Result consists of edge cones of an e-offset mesh. [Pottmann 06]

L-trafo example for conical mesh

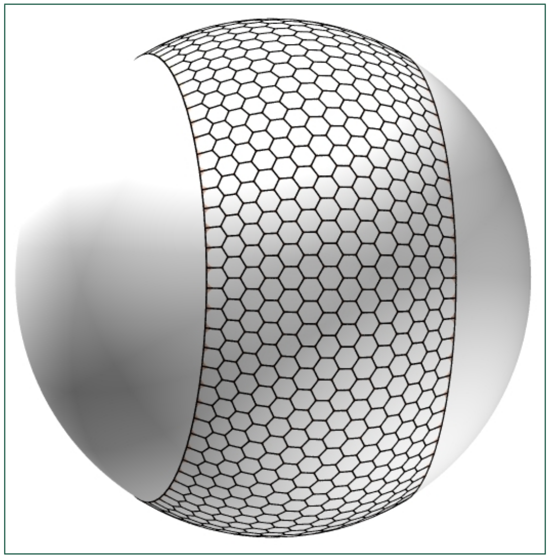
- Basic example: Offsetting operation



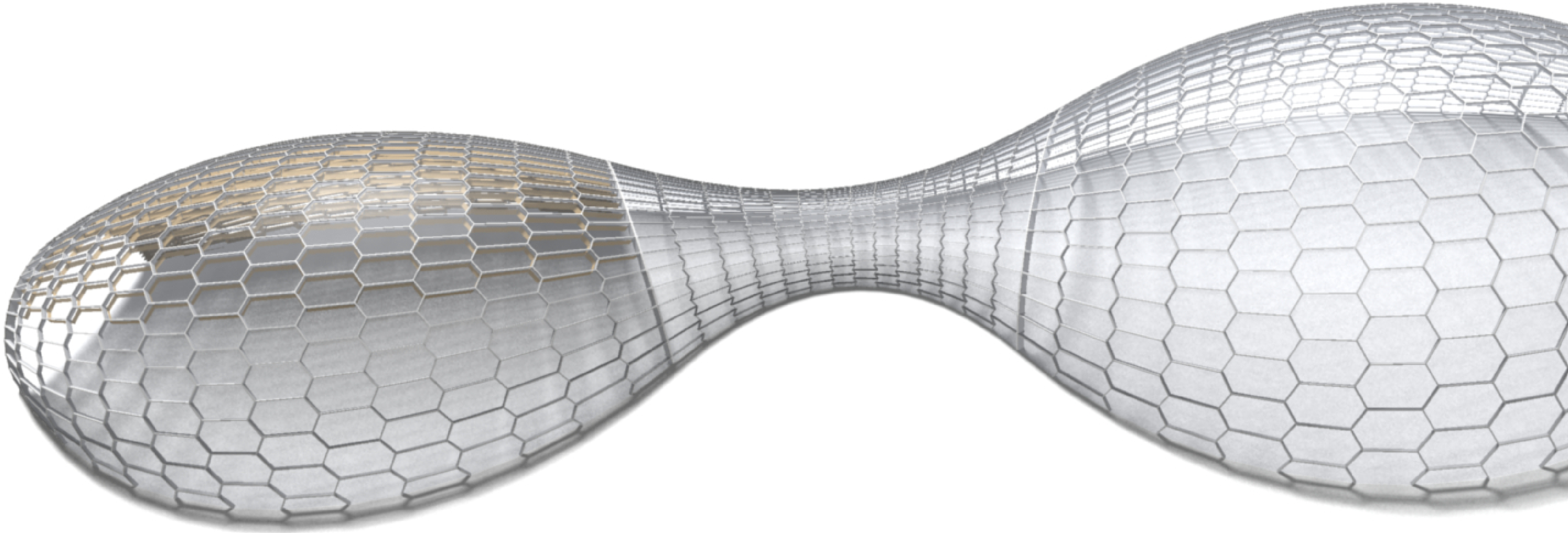
L-trafo example for EO mesh



- Start with a Koebe polyhedron $\mathcal{S}$
- Find an EO mesh $\mathcal{M}$ parallel to $\mathcal{S}$

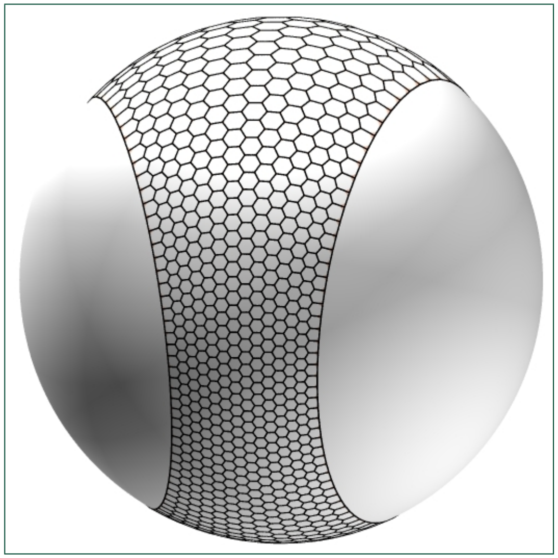


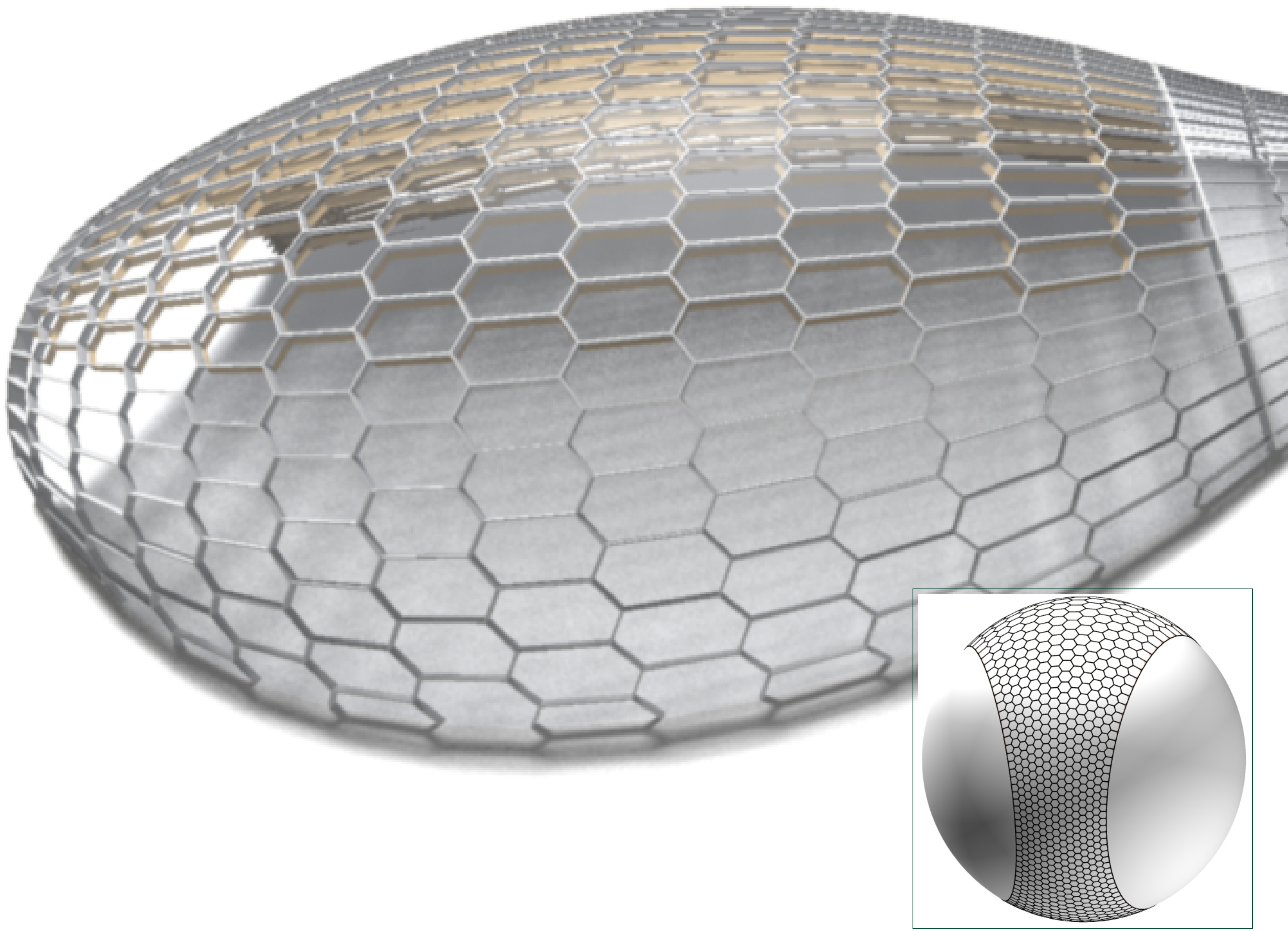
L-trafo example for EO mesh

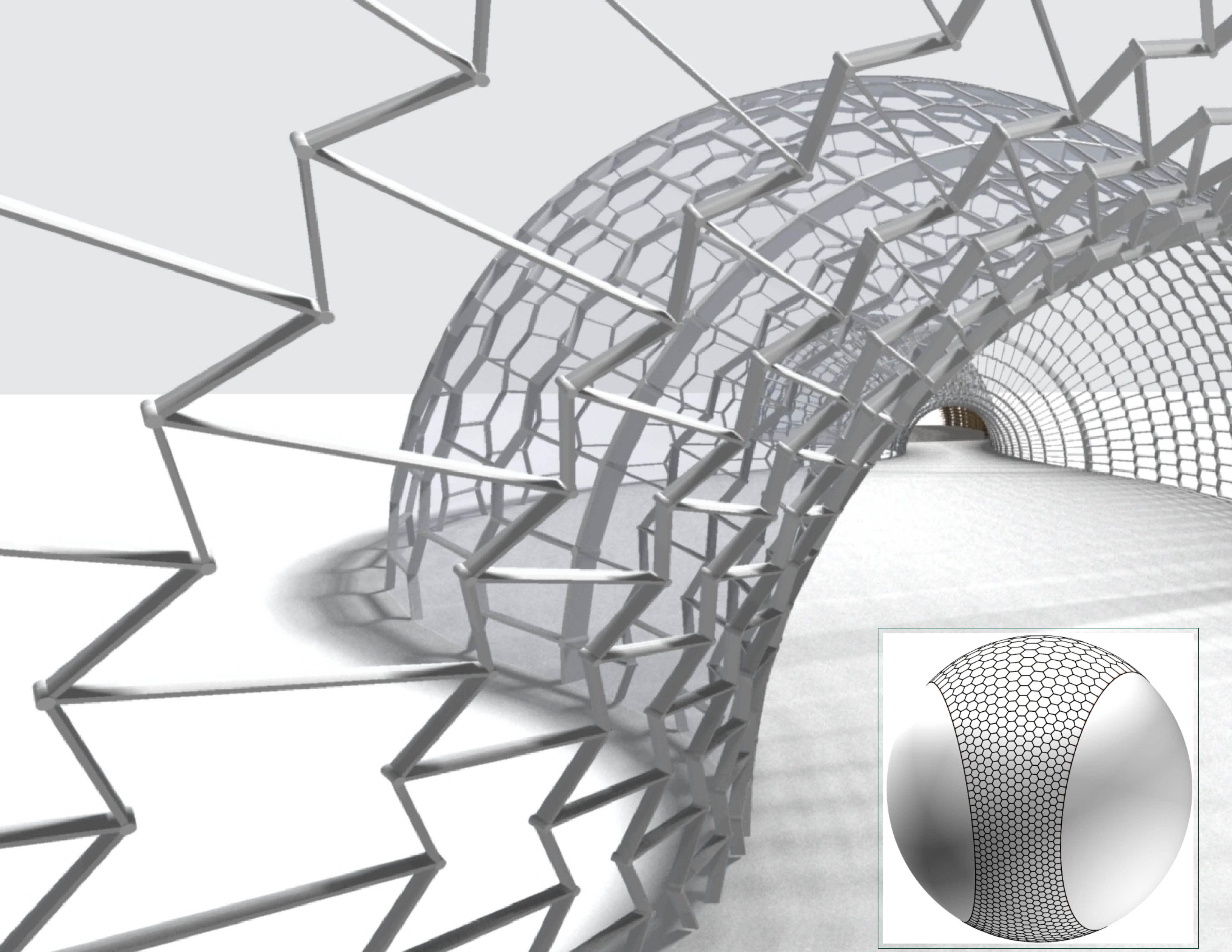


- Apply Laguerre transformation

$\Longrightarrow$ EO mesh again.







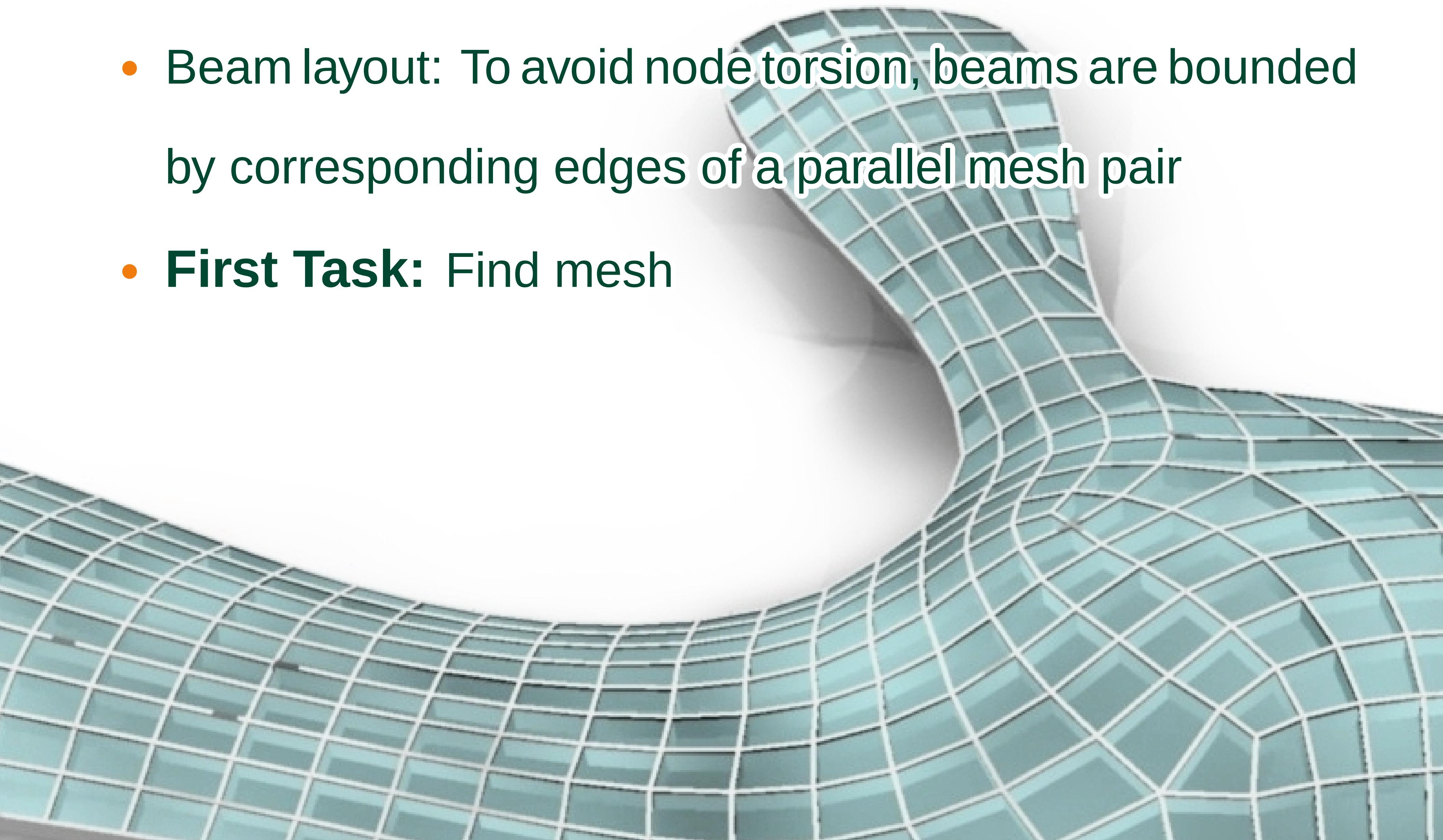
Approximation problems

Given is a surface “Φ”. We ask:

- Does some quad mesh (triangle mesh, hex mesh) with planar faces approximate Φ? YES
- Does some circular/conical mesh approximate Φ? YES
- are exactly constant beam heights possible? NO
- are approximately constant beam heights possible? YES *(generic answers)*

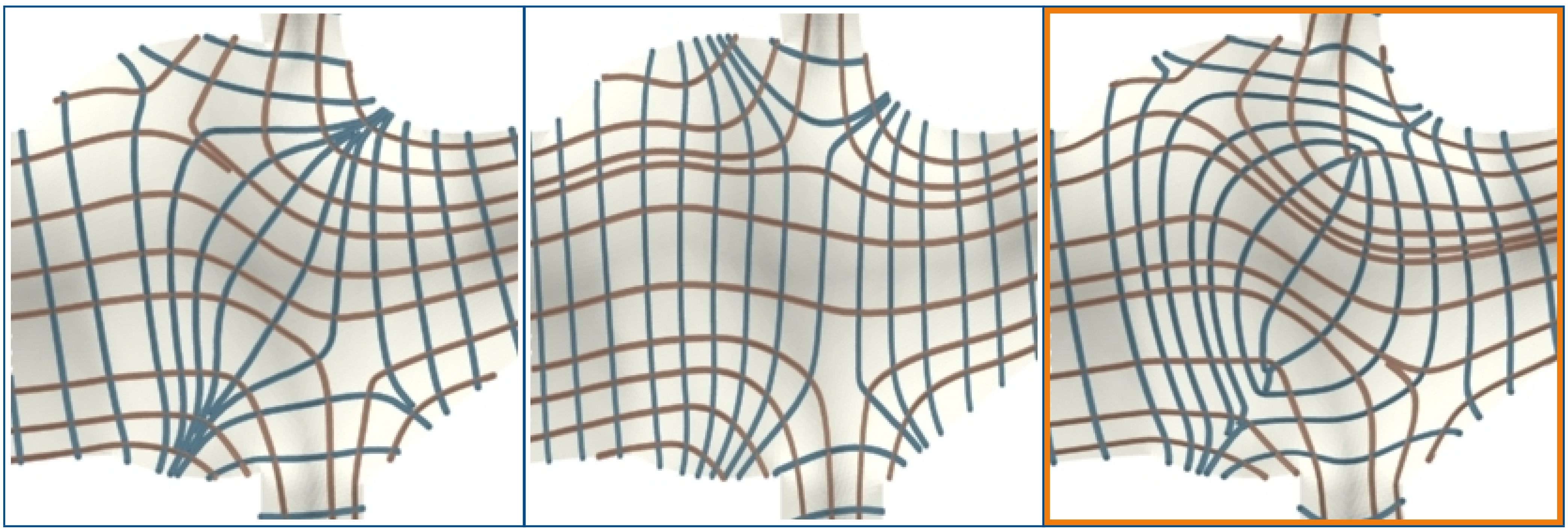
Processing pipeline

- Beam layout: To avoid node torsion, beams are bounded by corresponding edges of a parallel mesh pair
- **First Task:** Find mesh



Processing pipeline

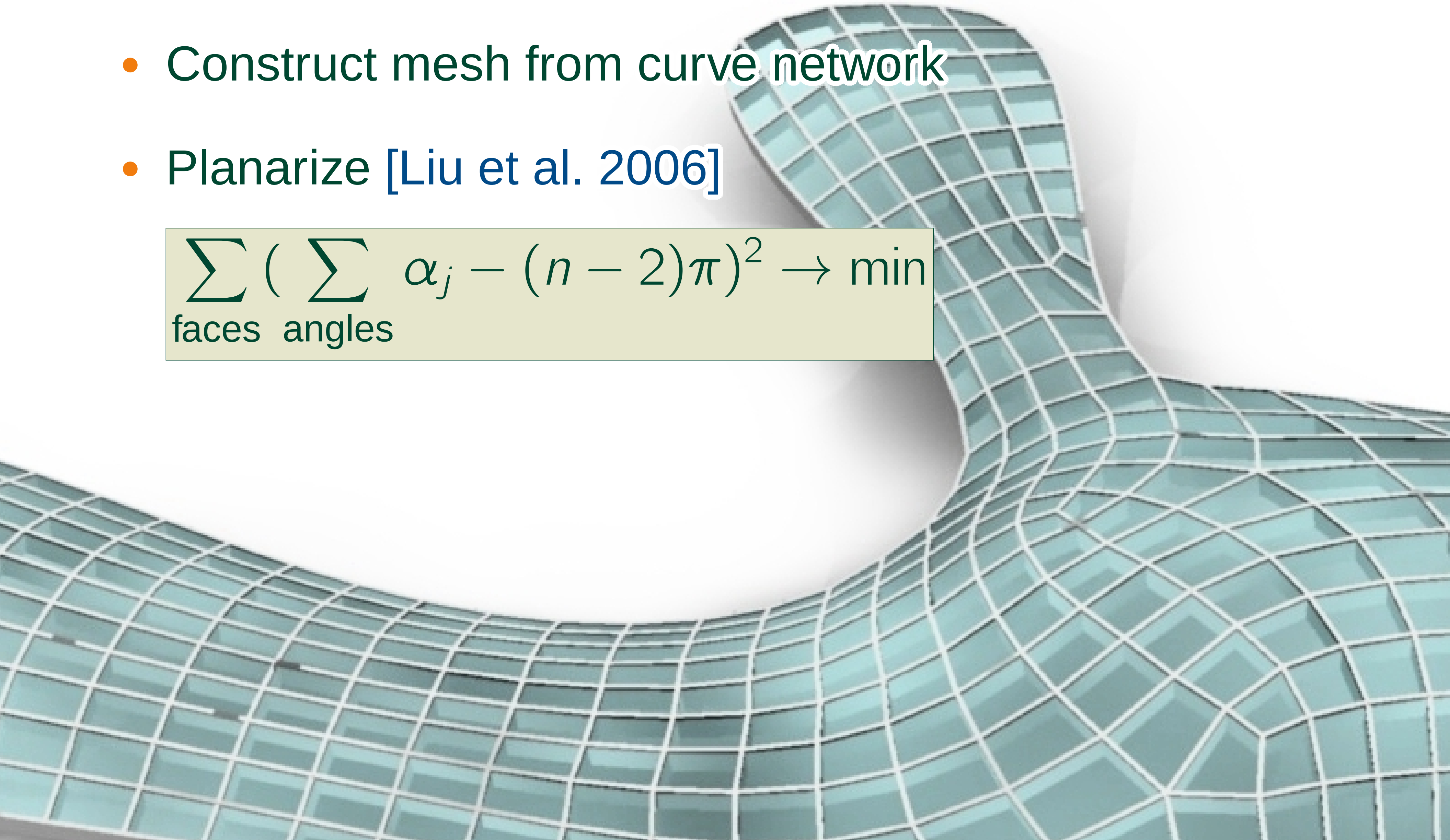
- A quad-dominant mesh with planar faces is a discrete network of conjugate curves
- Choose from several conjugate networks



Processing pipeline

- Construct mesh from curve network
- Planarize [Liu et al. 2006]

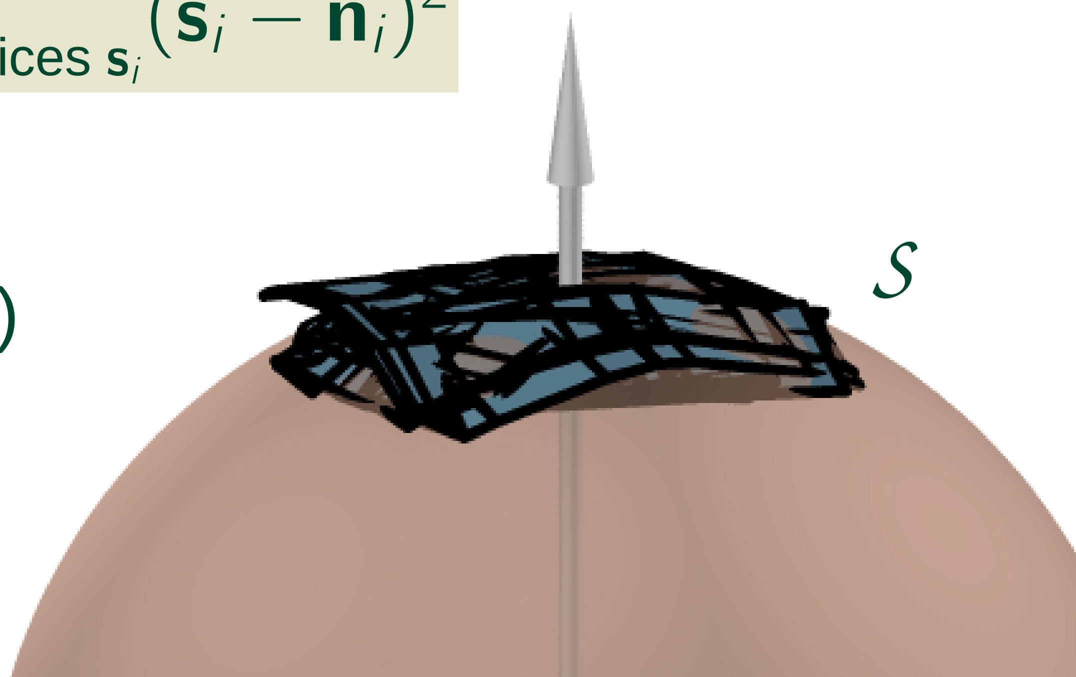
$$\sum_{\text{faces}} \Big(\sum_{\text{angles}} \alpha_j - (n-2)\pi \Big)^2 \to \min$$



Processing pipeline

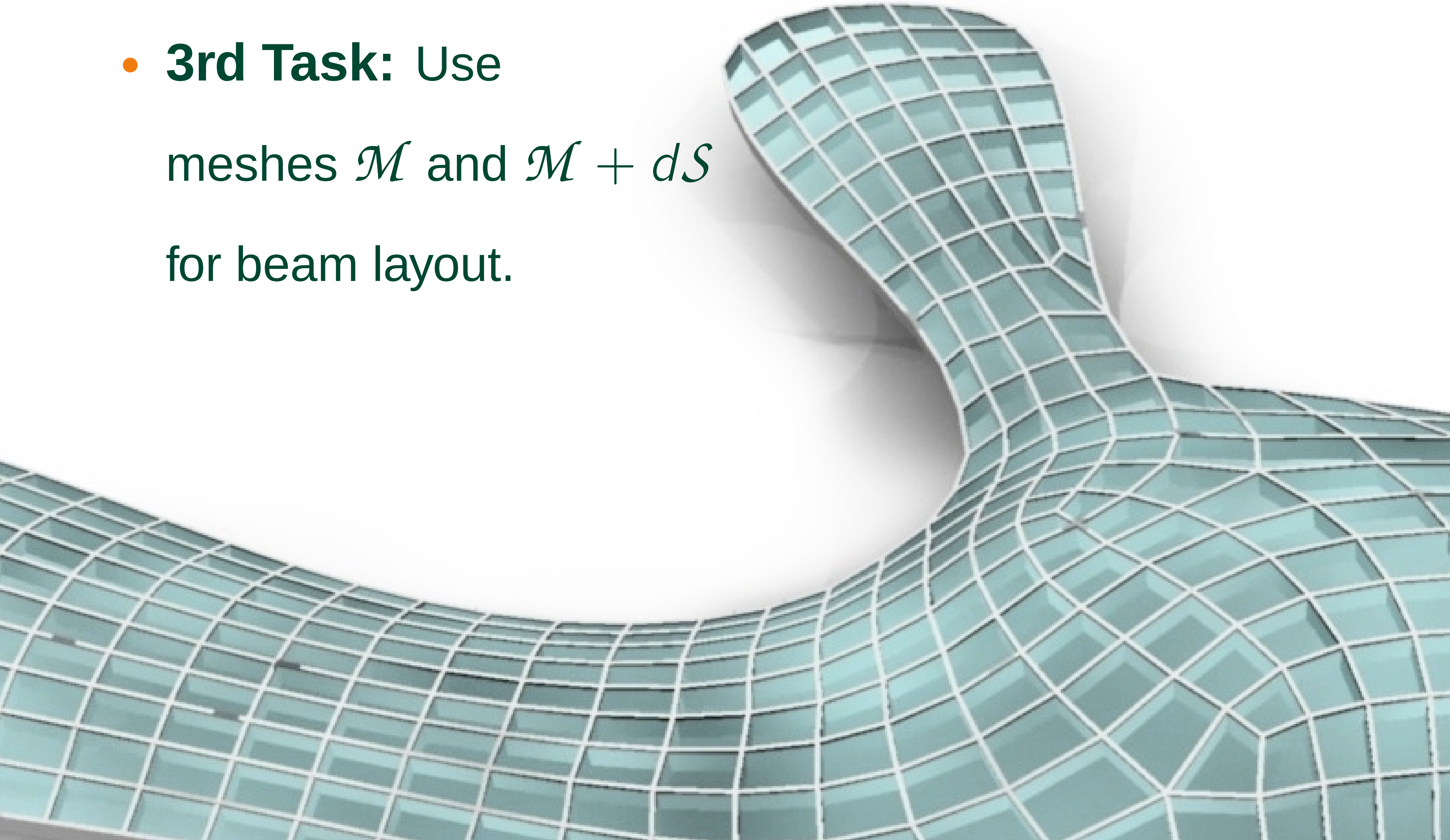
- **2nd Task:** Find spherical mesh $\mathcal{S}$ parallel to $\mathcal{M}$.
- fairness functional $\sum_{\text{vertices } \mathbf{s}_i} \left(\mathbf{s}_i - \frac{1}{\deg(\mathbf{s}_i)} \sum_{\mathbf{s}_j \in \text{link}(\mathbf{s}_i)} \mathbf{s}_j\right)^2$
- face functional $\sum_{\text{faces } F} (\mathbf{n}_F \cdot (\mathbf{s}_{i(F)} - \mathbf{n}_F))^2$
- vertex functional $\sum_{\text{vertices } \mathbf{s}_i} (\mathbf{s}_i - \tilde{\mathbf{n}}_i)^2$
- $\Longrightarrow$ optimize

(small weight for fairness)



Processing pipeline

- **3rd Task:** Use meshes $\mathcal{M}$ and $\mathcal{M} + d\mathcal{S}$ for beam layout.



Conclusion

- Parallel meshes: for nodes without torsion
- Offsets: for multilayer constructions, including beam layout
- Edge offset meshes (exactly constant beam heights)
- Geometric transformations
- Mesh optimization (approximately constant beam heights)