A Saturated Packing of 37 Regular Tetrahedra

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Abstract. The paper is devoted to special saturated packings of regular tetrahedra. The 'smallest' saturated packing in 3-space which is presently known, consists of 16 tetrahedra and was first presented by H. Harborth and M. Möller in [1]. In this paper the authors ask for examples of such packings with an odd number of congruent regular tetrahedra. Such arrangements are of high interest as the planar case does not provide this possibility: The number of regular triangles used for a planar saturated packing of triangles has to be even. In this paper we are able to present a saturated packing of 37 congruent regular tetrahedra.

The presented satureated packing consists of three parts: Two are formed by 'sliced copies with unsaturated vertices' of the model by H. Harborth and M. Möller. A third part consisting of 5 regular tetrahedra is used to close the gaps and yields the saturated packing of 37 congruent regular tetrahedra.

1. Introduction.

A set of congruent regular triangles in the plane (congruent regular tetrahedra in the space) is called a *saturated packing*, if every vertex is connected to exactly one vertex of a triangle (tetrahedron) of the set. In addition we neither not want the triangles (tetrahedra) to interfere with each other nor to form disconnected parts. In the Euclidean plane there exists an example involving $n = 42$ regular triangles [3, 4]. But it is still unknown if there do exist examples for $n \leq 42$.

If a planar saturated packing consists of n triagles, there are $3n$ vertices which are pairwise coincident. $3n/2$ is the number of the corresponding linkages. As this number has to be a positive integer, n has to be even in the case of a planar saturated packing (see [2]).

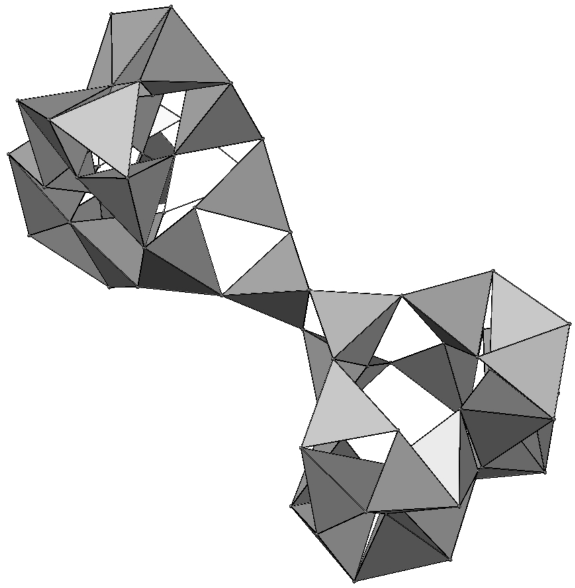

Figure 1: Saturated packing of 37 tetrahedra

The saturated packing of regular tetrahedra in 3-space with the smalles number of parts which is presently known, consists of 16 tetrahedra. It was first presented by H. Harborth and M. Möller in [1] (see left side of Figure 3). It consists of 16 congruent regular tetrahedra which are interlinked at the 32 vertices. Different from the planar case for saturated packings in space there is no restriction of parity for the number n of tetrahedra. In [1] H. Harborth and M. Möller ask for examples with an odd number of tetrahedra.

We will use this model to generate a saturated packing of 37 congruent regular tetrahedra (see Figure 1). We don't know if there are saturated packings of this type consisting of a smaller odd number of tetrahedra.

For the sequel we assume that all these tetrahedra have normed edge length $a := 1$.

2. The Saturated Packing of 37 Congruent Regular Tetrahedra. The idea of the following is to take two saturated packings described by H. Harborth and M. Möller, modify them and connect them with a suitable part with 5 regular tetrahedra.

We are going to describe this model in several steps:

Step 1: The 'middle part' consisting of five tetrahedra. We take a regular triangle (side length a) and arrange three regular tetrahedra with one of their edges in the sides of this triangle. This is done in a way that the three tetrahedra lie symmetric with respect to the plane of the triangle (see left hand side of Figure 2). Now we take one of these three tetrahedra as 'central tetrahedron' with center C. A rotary reflection (axis r, rotation about the angle $\pi/2$, center C) transforms the other two tetrahedra into two further copies fitting to the other vertices of the central tetrahedron. In total we have got a 'middle part' consisting of five regular tetrahedra. It has eight unsaturated vertices (see right side fo Figure 2). The unsaturated vertices of two neighbouring tetrahedra of this part form two congruent rectangles. Their side lengths are given by

$$\begin{aligned} a &= 1 \qquad\qquad \text{and} \\ b &:= (1+\sqrt{6})/2. \end{aligned} \tag{1}$$

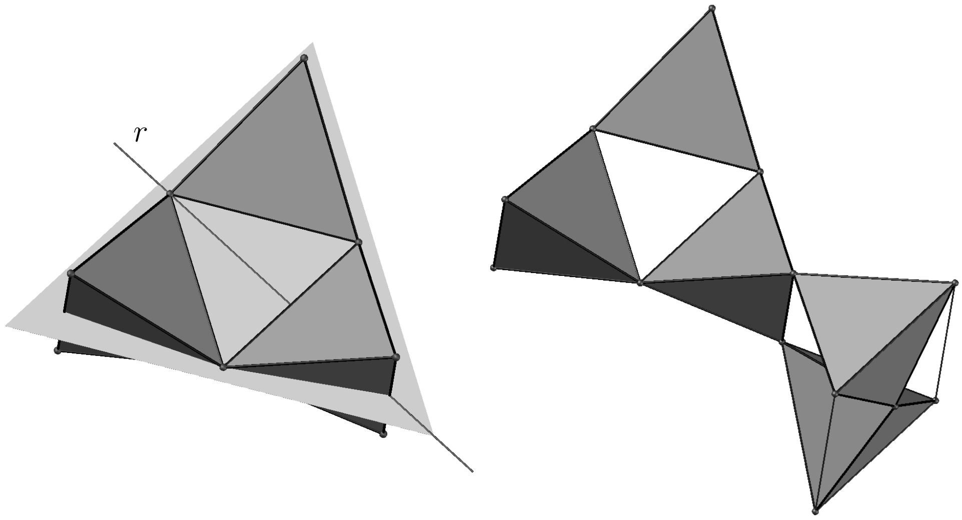

Figure 2: Middle part

Step 2: Opening the saturated model. In a second step we open the model by H. Harborth and H. Möller with it's 16 tetrahedra at two vertices such that the unsaturated vertices of a rectangle of the middle part fit to this opening. We know that this model admits at least a two-parametric self-motion (see [5]). Due to its generation it has two orthogonal planes of symmetry in each position. One of these planes contains the four vertices $1, 2, 3, 4$ - see Figure 3. We disconnect two of them (e.g. 1 and 2) and open the model by rotating one part about the line $[3, 4]$. The points $1, 2$ and its corresponding vertices $1', 2'$ form a rectangle. Its shape depends on the motion-parameters and the angle φ of the rotation used for the opening process (see Figure 3).

Now we want to saturate the four vertices $1, 2, 1', 2'$ of this opening by adding the unsaturated rectangular part of the middle part (see Figure 4). According to (1) this is possible if $\overline{11'} = \overline{22'} = 1$ and $\overline{12} = \overline{1'2'} = b$. The parameters of the self-motions of the model can be used to achieve $\overline{12} = \overline{1'2'} = b$. Then the angle φ is chosen such that $\overline{11'} = \overline{22'} = 1$. There remains one free parameter which can be used to gain arrangements without self-intersections.

Step 3: Closing the chain. Once such a configuration is given the middle part can be used to fill the gap $1, 2, 1', 2'$. The last four unsatured vertices of the middle part are closed with an anologue procedure with a second modified model by H. Harborth and M. Möller. This yields a saturated packing of

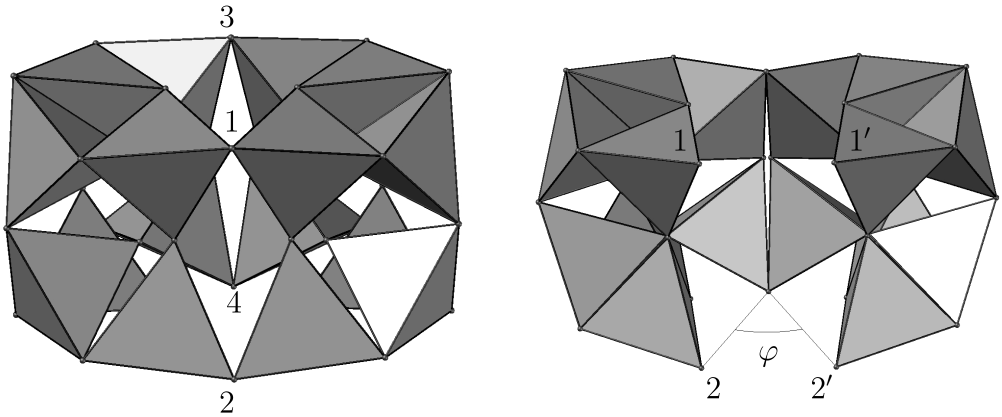

Figure 3: Model by Harborth and Möller in it's original form and in the open position

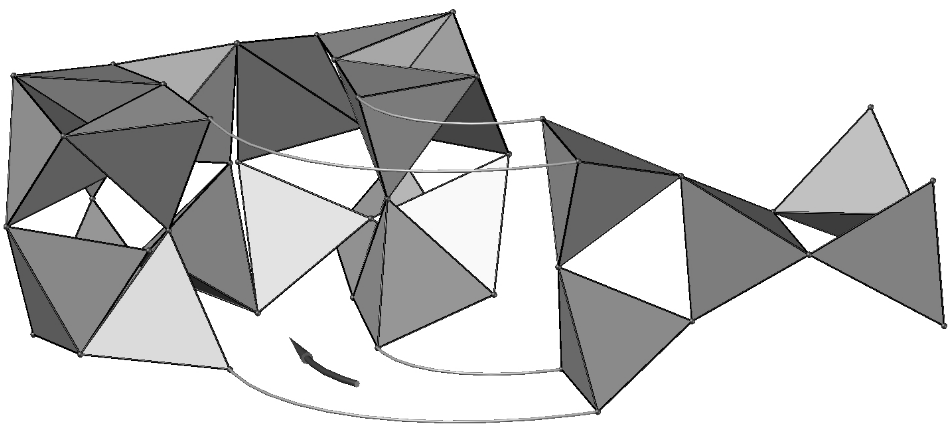

Figure 4: Combining two parts

regular tetrahedra consisting of 37 parts. Figure 1 displays this saturated packing.

3. Conclusion.

We presented a saturated packing of 37 congrunet regular tetrahedra. This packing is the first example for such a saturated packing wiht an odd number of tetrahedra. This particular example consists of three parts: Two of them are generated by an opening procedure of the model by H. Harborth and M. Möller, the third is some 'middle part' consisting of 5 regular tetrahedra. We suppose that this packing is the one with the smallest odd number of regular tetrahedral parts.

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