Study's kinematic mapping – a tool for motion design

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Abstract: Via Study's kinematic mapping $\mathfrak{S}$ the 6-parametric Lie group $SE(3)$ of direct Euclidean displacements can be identified with a certain hyperquadric $\mathcal{M}_6$ in 7-dimensional real projective space. The mapping has nice geometric properties; for instance one parametric rotation groups are represented by straight lines on $\mathcal{M}_6$, coordinate transformations in Euclidean 3-space are represented by special automorphisms of $\mathcal{M}_6$. With the help of $\mathfrak{S}$ Euclidean kinematics can be considered as a point-geometry in the sense of Felix Klein's Erlangen program.

We give an application in the field of motion design: The problem of constructing a motion interpolating a sequence of given positions can be solved by constructing an appropriate curve interpolating the corresponding points on Study's quadric.

1 Introduction

In the following we consider the Euclidean 3-space $\mathbb{E}_3$ as affine part of the real projective 3-space $\mathbb{P}_3$. A point X in $\mathbb{P}_3$ is described by its homogeneous coordinates $(x_0, x_1, x_2, x_3)^t$. If X is a proper point ($x_0 \neq 0$) then the inhomogeneous coordinates of X are $(\frac{x_1}{x_0}, \frac{x_2}{x_0}, \frac{x_3}{x_0})^t$.

A direct displacement α in $\mathbb{E}_3$ can be described by

$$\alpha : \mathbf{x}^* = \begin{pmatrix} m_{00} & 0 & 0 & 0 \\ m_{10} & m_{11} & m_{12} & m_{13} \\ m_{20} & m_{21} & m_{22} & m_{23} \\ m_{30} & m_{31} & m_{32} & m_{33} \end{pmatrix} \cdot \mathbf{x} \tag{1}$$

where for the matrix $\mathbf{M} := (m_{ij})_{i,j\in\{1,2,3\}}$ the conditions

$$m_{00} \neq 0, \quad \mathbf{M}\cdot\mathbf{M}^t = m_{00}^2 \cdot \mathbf{E}, \quad \det \mathbf{M} = m_{00}^3 \tag{2}$$

have to be fulfilled. Here $\mathbf{x}$ and $\mathbf{x}^*$ denote the homogeneous coordinate vectors of a point X and its image $X^* = \alpha(X)$, respectively and $\mathbf{E}$ denotes the 3×3-unit matrix. The set of all Euclidean displacements is a 6-parametric Lie group which is usually denoted by $SE(3)$. This group can be embedded into a 7-dimensional projective space $\mathbb{P}_7$ via Study's kinematic mapping:

$$\mathfrak{S} : \left\{ \begin{array}{ccl} SE(3) & \longrightarrow & \mathbb{P}_7 \\ \alpha & \longrightarrow & A \ldots \mathbf{a} = (a_0, \ldots, a_7)^t \end{array} \right\} \tag{3}$$

where the components of the homogeneous coordinate vector $\mathbf{a}$ representing the point A in $\mathbb{P}_7$ are determined by the relations [1]:

$$\begin{array}{ll} a_0 : a_1 : a_2 : a_3 & = \\ m_{00}+m_{11}+m_{22}+m_{33} : m_{23}-m_{32} : m_{31}-m_{13} : m_{12}-m_{21} & = \\ m_{23}-m_{32} : m_{00}+m_{11}-m_{22}-m_{33} : m_{12}+m_{21} : m_{31}+m_{13} & = \\ m_{31}-m_{13} : m_{12}+m_{21} : m_{00}-m_{11}+m_{22}-m_{33} : m_{23}+m_{32} & = \\ m_{12}-m_{21} : m_{31}+m_{13} : m_{23}+m_{32} : m_{00}-m_{11}-m_{22}+m_{33}, & \end{array} \tag{4}$$

$$\begin{array}{lcl} 2\cdot m_{00}\cdot a_4 & = & +a_1\cdot m_{10}+a_2\cdot m_{20}+a_3\cdot m_{30}, \\ 2\cdot m_{00}\cdot a_5 & = & -a_0\cdot m_{10}+a_3\cdot m_{20}-a_2\cdot m_{30}, \\ 2\cdot m_{00}\cdot a_6 & = & -a_3\cdot m_{10}-a_0\cdot m_{20}+a_1\cdot m_{30}, \\ 2\cdot m_{00}\cdot a_7 & = & +a_2\cdot m_{10}-a_1\cdot m_{20}-a_0\cdot m_{30}. \end{array} \tag{5}$$

The homogeneous coordinates of a point A, computed via eqs. (4), (5), satisfy

$$x_0\cdot x_4+x_1\cdot x_5+x_2\cdot x_6+x_3\cdot x_7 \;=\; 0 \tag{6}$$

which shows us that the points A lie on a hyperquadric $\mathcal{M}_6$ (*Study's quadric*), given by this equation.

Conversely, for any point $A\ldots\mathbf{a}=(a_0,\ldots,a_7)^t$ on $\mathcal{M}_6$ with $(a_0,a_1,a_2,a_3)^t\neq(0,0,0,0)^t$ there is exactly one preimage $\alpha\in SE(3)$: The corresponding matrix-entries m_{ij} have to be computed by

$$\begin{array}{lcl} m_{00} & = & a_0^2+a_1^2+a_2^2+a_3^2, \\ m_{10} & = & 2\cdot(a_2\cdot a_7-a_3\cdot a_6-a_0\cdot a_5+a_1\cdot a_4), \\ m_{11} & = & a_0^2+a_1^2-a_2^2-a_3^2, \\ m_{12} & = & 2\cdot(a_1\cdot a_2+a_0\cdot a_3), \\ m_{13} & = & 2\cdot(a_3\cdot a_1-a_0\cdot a_2), \\ m_{20} & = & 2\cdot(a_3\cdot a_5-a_1\cdot a_7-a_0\cdot a_6+a_2\cdot a_4), \\ m_{21} & = & 2\cdot(a_1\cdot a_2-a_0\cdot a_3), \\ m_{22} & = & a_0^2-a_1^2+a_2^2-a_3^2, \\ m_{23} & = & 2\cdot(a_2\cdot a_3+a_0\cdot a_1), \\ m_{30} & = & 2\cdot(a_1\cdot a_6-a_2\cdot a_5-a_0\cdot a_7+a_3\cdot a_4), \\ m_{31} & = & 2\cdot(a_3\cdot a_1+a_0\cdot a_2), \\ m_{32} & = & 2\cdot(a_2\cdot a_3-a_0\cdot a_1), \\ m_{33} & = & a_0^2-a_1^2-a_2^2+a_3^2. \end{array} \tag{7}$$

Hence, after denoting the 3-dimensional projective subspace of $\mathbb{P}_7$ defined by $x_0=x_1=x_2=x_3=0$ with $\mathcal{G}^u$, we can say that $\mathfrak{S}$, regarded as a mapping from $SE(3)$ to $\mathcal{M}_6\setminus\mathcal{G}^u$, is bijective. Moreover we see from the eqs. (4), (5) that $\mathfrak{S}$ is quadratic in the matrix-entries m_{ij}. Due to the eqs. (7) the inverse mapping $\mathfrak{S}^{-1}$ too is quadratic in the homogeneous coordinates of the points A.

Interpreting the vector $\mathbf{a}$ as biquaternion $\mathbf{a}=\mathbf{a}_1+\mathbf{a}_2\cdot\mathbf{e}$ with $\mathbf{e}^2=0$ and the Hamilton quaternions $\mathbf{a}_1=(a_0,a_1,a_2,a_3)^t$ and $\mathbf{a}_2=(a_4,a_5,a_6,a_7)^t$ we also get a simple formula[2], describing the displacement α:

$$\alpha:\;\; \mathbf{y}^* \;=\; \bar{\mathbf{a}}\cdot\mathbf{y}\cdot\mathbf{a}_e. \tag{8}$$

[1] Compare with Study [11], pp. 174–177. The parameters a_0, a_1, a_2, a_3 are the Euler parameters of the rotational part of the displacement.

[2] Compare with Blaschke, [1], chapter 4. See also Study's original papers: Study [7], [8], [9] and [10].

Here $\mathbf{y}$ and $\mathbf{y}^*$ are the biquaternions $(x_0, 0, 0, 0, 0, x_1, x_2, x_3)^t$ and $(x_0^*, 0, 0, 0, 0, x_1^*, x_2^*, x_3^*)^t$, when $\mathbf{x} = (x_0, x_1, x_2, x_3)^t$ and $\mathbf{x}^* = (x_0^*, x_1^*, x_2^*, x_3^*)^t$ are the homogeneous coordinate vectors of a point X and its image $X^* = \alpha(X)$. Moreover $\bar{\mathbf{a}} = \bar{\mathbf{a}}_1 + \bar{\mathbf{a}}_2 \cdot \mathbf{e} = (a_0, -a_1, -a_2, -a_3, a_4, -a_5, -a_6, -a_7)^t$ denotes the *conjugate* biquaternion and $\mathbf{a}_e = \mathbf{a}_1 - \mathbf{a}_2 \cdot \mathbf{e}$ the **e**-*conjugate* biquaternion to **a**.

2 Coordinate transformations

Let α be a displacement in $SE(3)$, described by eq. (8) and let

$$\mathbf{z} \;=\; \bar{\mathbf{b}} \cdot \mathbf{y} \cdot \mathbf{b}_e, \;\; \mathbf{z}^* \;=\; \bar{\mathbf{c}} \cdot \mathbf{y}^* \cdot \mathbf{c}_e \tag{9}$$

be coordinate-transformations in the original and the displaced space, respectively. Then we get the following biquaternion-representation of the displacement α with respect to the new coordinate-frames:

$$\mathbf{z}^* \;=\; \bar{\mathbf{c}} \cdot \bar{\mathbf{a}} \cdot \mathbf{b} \cdot \mathbf{z} \cdot \bar{\mathbf{b}}_e \cdot \mathbf{a}_e \cdot \mathbf{c}_e. \tag{10}$$

What is the effect of a pair of coordinate-transformations β and γ on the points of Study's quadric? Therefore we have to investigate the transformation $\tau : X \longrightarrow X^*$ in $\mathbb{P}_7$ defined by

$$\tau : \;\; \mathbf{x}^* \;=\; \bar{\mathbf{b}} \cdot \mathbf{x} \cdot \mathbf{c}. \tag{11}$$

The answer is given by the following[3]

Theorem 1. *The transformation τ given by eq. (11) induced by a pair β, γ of Euclidean coordinate-transformations in $\mathbb{E}_3$ is an automorphic collineation of Study's quadric $\mathcal{M}_6$ with the additional properties:*
(a) τ fixes $\mathcal{G}^u$ and
(b) The restriction of τ to $\mathcal{G}^u$ is a direct elliptic transformation.

Conversely any automorphism τ of $\mathcal{M}_6$ with the properties (a), (b) is induced by an appropriate pair of Euclidean coordinate transformations β and γ.

3 Linear subspaces on $\mathcal{M}_6$

Study's quadric $\mathcal{M}_6$ carries two 6-parametric sets of generators, each of them being a 3-dimensional projective subspace in $\mathbb{P}_7$. Moreover we have[4]

Theorem 2. *(a) Two points $A_0, A_1 \in \mathcal{M}_6/\mathcal{G}^u$ are conjugate[5] with respect to $\mathcal{M}_6$ iff $\alpha_1^{-1} \circ \alpha_0$ is a rotation.[6]*
(b) A generator $\mathcal{G} \neq \mathcal{G}^u$ on $\mathcal{M}_6$ represents a set of the form $\alpha \cdot \Gamma \cdot \beta$, where α, β are arbitrary displacements and Γ is the set of rotations which axes either run through a common point O or lie in a common plane $\mathcal{P}$.

[3] Although the proof can be given straight forward, the author has not found it in the classical literature.
[4] For the proof see Weiss [12, pp. 118–119].
[5] Two points on a hyperquadric are conjugate with respect to this hyperquadric if the line connecting them is part of the hyperquadric.
[6] Here translations are also included by interpreting a translation as a rotation with an improper axis.

Remark 1. (a) If Γ is a set of rotations with axes running through a common point O, then we of course can identify Γ either with the subgroup $SO(3)$ (in case that O is a proper point) or with the subgroup $SE(2)$ (in case that O is an improper point) of $SE(3)$.
(b) If Γ is a set of rotations with axes lying in a common plane $\mathcal{P}$ then Γ is no subgroup of $SE(3)$ but in the exceptional case that $\mathcal{P}$ is the plane at infinity – here Γ is the 3-parametric subgroup of all translations in $\mathbb{E}_3$.

Example 1. (a) The subgroup $SO(3)$ (this is the set of rotations with axes through the point $O_0 \ldots (1,0,0,0)^t$) is represented by the generator $x_4 = x_5 = x_6 = x_7 = 0$ on $\mathcal{M}_6$.
(b) The subgroup $SE(2)$ (this is the set of rotations with axes through the improper point $O_3 \ldots (0,0,0,1)^t$) is represented by the generator $x_1 = x_2 = x_4 = x_7 = 0$ on $\mathcal{M}_6$.

4 Motion design by Study's kinematic mapping – an example

The $\mathfrak{S}$-image of a k-parametric Euclidean motion is a k-dimensional submanifold on Study's quadric $\mathcal{M}_6$; especially if $k = 1$ we have a 1-parametric motion which is represented by a curve on $\mathcal{M}_6$. Due to the birationality of $\mathfrak{S}$ rational motions are mapped into rational submanifolds and vice versa.

The interpolation of given positions by an appropriate (1-parametric) motion is an important problem in the field of robotics. This problem can be formulated as follows:
Problem (MIP)**:** Find an interpolating 1-parametric motion $\gamma(t)$ for $n+1$ given displacements α_i and corresponding parameter values t_i.

Using Study's kinematic mapping $\mathfrak{S}$ we can try to solve this problem in the following way:
1.) Map the given displacements onto $\mathcal{M}_6$ via $\mathfrak{S}$. This yields the points $A_i := \mathfrak{S}(\alpha_i)$ on $\mathcal{M}_6$.
2.) Try to find an interpolation curve $\mathcal{C} \ldots \mathbf{c}(t)$ which satisfies the data set $\{A_i, t_i\}$ and is moreover contained[7] in $\mathcal{M}_6$.
3.) The one-parametric motion described by the quaternion equation[8]

$$\mathbf{y}^* \;=\; \overline{\mathbf{c}(t)} \cdot \mathbf{y} \cdot \mathbf{c}(t)_e \qquad (12)$$

solves our problem.

To be compatible with CAGD-standards[9] we additionally postulate that the solution should be a *rational* motion or in case of a spline interpolant consist of *rational* subspline-motions. This postulate will be abbreviated by (RM). This means that the curve $\mathcal{C}$ on $\mathcal{M}_6$ will have to be *rational*.

Additionally the construction of the motion should be invariant under Euclidean coordinate transformations (CTI), which means that applying coordinate transformations in the fixed and moving frames should not change the resulting motion. Thus the construction of the corresponding curve $\mathcal{C}$ on $\mathcal{M}_6$ has to be invariant under the special automorphisms of $\mathcal{M}_6$ (compare with section 2). This can be guaranteed by constructing the curve $\mathcal{C}$ only out of its *control points* A_i and by making use only of projectively invariant properties of $\mathcal{M}_6$. We shall give an example for such a construction in the following.

[7] Of course there is no problem in constucting a curve $\mathcal{C}$ interpolating the points A_i – this can be done by standard spline-techniques. The real problem is to force the curve $\mathcal{C}$ to lie on $\mathcal{M}_6$. But in the last years much research work has been done on the construction of free-form curves on (hyper)quadrics (see for instance Dietz, Hoschek and Jüttler [2], Wenping and Barry [13], Gfrerrer [3], [4]).
[8] One can also compute a matrix representation of this motion using $\mathfrak{S}^{-1}$, eqs. (7).
[9] Compare with Röschel [6].

Definition 1. Let $\mathcal{Q}_{d-1}$ be a hyperquadric in d-dimensional projective space $\mathbb{P}_d$, let $A_{i_0} \dots \mathbf{a}_{i_0}$, $A_{i_1} \dots \mathbf{a}_{i_1}$, $A_{i_2} \dots \mathbf{a}_{i_2}$ be three points on $\mathcal{Q}_{d-1}$ and let moreover t_{i_0}, t_{i_1}, t_{i_2} be three pairwise distinct real numbers. Then we can define the parametrization

$$\mathbf{c}_{i_0,i_1,i_2}(t) = a_{i_1,i_2} \cdot f_{i_0}(t) \cdot \mathbf{a}_{i_0} + a_{i_0,i_2} \cdot f_{i_1}(t) \cdot \mathbf{a}_{i_1} + a_{i_0,i_1} \cdot f_{i_2}(t) \cdot \mathbf{a}_{i_2} \tag{13}$$

with $a_{i,j} := \langle \mathbf{a}_i, \mathbf{a}_j \rangle$, where $\langle .,. \rangle$ denotes the bilinear form belonging to the quadric $\mathcal{Q}_{d-1}$ and for $i, j, k \in \{i_0, i_1, i_2\}$ pairwise distinct

$$f_i(t) \quad := \quad (t_i - t_j) \cdot (t_i - t_k) \cdot (t - t_j) \cdot (t - t_k). \tag{14}$$

If $a_{i,j} \neq 0$ for $i \neq j \in \{i_0, i_1, i_2\}$ (this is the condition that the span of the 3 points A_{i_0}, A_{i_1}, A_{i_2} is a plane intersecting $\mathcal{Q}_{d-1}$ in a regular second-order curve) the parametrization (13) represents a curve $\mathcal{C}_{i_0,i_1,i_2}$ on $\mathcal{Q}_{d-1}$ which is the conic section on $\mathcal{Q}_{d-1}$ determined by the three points.[10] Moreover eq. (13) yields a homogeneous coordinate vector of A_i if substituting t_i, $i \in \{i_0, i_1, i_2\}$. Thus $\mathcal{C}_{i_0,i_1,i_2}$ is a rational solution curve for the interpolation problem of the three points A_{i_0}, A_{i_1}, A_{i_2} on $\mathcal{Q}_{d-1}$.

Definition 2. Let again $\mathcal{Q}_{d-1}$ be a hyperquadric in d-dimensional projective space $\mathbb{P}_d$ and let now I denote a sequence of five indices: $I := (i_0, i_1, i_2, i_3, i_4)$; let furthermore $A_i \dots \mathbf{a}_i$, $i \in I$ be five points on $\mathcal{Q}_{d-1}$ and t_{i_0}, t_{i_2}, t_{i_4} be three pairwise distinct real numbers. Then we can define the parametrization

$$\mathbf{c}_I(t) \quad := \quad \sum_{j=0}^{4} q_{I,j}(t) \cdot \mathbf{a}_{i_j} \tag{15}$$

where the functions $q_{I,j}(t)$ are 4-th order polynomials given by

$$\begin{aligned}
q_{I,0}(t) &:= f_{i_0}(t) \left(a_{i_1,i_4} a_{i_2,i_3} f_{i_0}(t) + a_{i_1,i_3} a_{i_2,i_4} f_{i_2}(t) + a_{i_1,i_2} a_{i_3,i_4} f_{i_4}(t) \right), \\
q_{I,1}(t) &:= a_{i_0,i_4} a_{i_2,i_3} f_{i_0}(t) f_{i_2}(t), \\
q_{I,2}(t) &:= a_{i_0,i_4} a_{i_1,i_3} f_{i_2}^2(t), \\
q_{I,3}(t) &:= a_{i_0,i_4} a_{i_1,i_2} f_{i_2}(t) f_{i_4}(t), \\
q_{I,4}(t) &:= f_{i_4}(t) \left(a_{i_0,i_1} a_{i_2,i_3} f_{i_0}(t) + a_{i_0,i_2} a_{i_1,i_3} f_{i_2}(t) + a_{i_0,i_3} a_{i_1,i_2} f_{i_4}(t) \right).
\end{aligned} \tag{16}$$

Theorem 3. *The rational curve $\mathcal{C}_I$ represented by eq. (15) has the following properties*[11] *(see fig. 1):*
(a) $\mathcal{C}_I$ is part of the hyperquadric $\mathcal{Q}^{d-1}$; this means: $\langle \mathbf{c}_I(t), \mathbf{c}_I(t) \rangle \equiv 0$.
(b) If $a_{i,j} \neq 0$ for $i \neq j \in I$ then $\mathcal{C}_I$ interpolates of A_{i_0}, A_{i_2}, A_{i_4} for the parameter values t_{i_0}, t_{i_2}, t_{i_4}, respectively (starting-, mid-, end-point interpolation). Moreover the conic section $\mathcal{C}_{i_0,i_1,i_4}$ on $\mathcal{Q}_{d-1}$ defined by the points A_{i_0}, A_{i_1}, A_{i_4} (see definition 1) has a common tangent with $\mathcal{C}_I$ in A_{i_0}. Analogously the conic section $\mathcal{C}_{i_0,i_3,i_4}$ defined by the points A_{i_0}, A_{i_3}, A_{i_4} is tangent to $\mathcal{C}_I$ in A_{i_4}.

[10] see Gfrerrer [4].
[11] For the proofs see Gfrerrer [4].

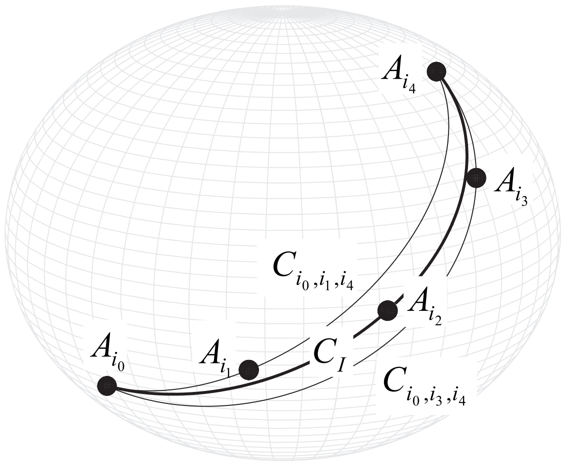

Figure 1: Quartic QB-curve $\mathcal{C}_I$ and the conic sections $\mathcal{C}_{i_0,i_1,i_4}$, $\mathcal{C}_{i_0,i_3,i_4}$ defined by the control-point triples A_{i_0}, A_{i_1}, A_{i_4} and A_{i_0}, A_{i_3}, A_{i_4} on a hyperquadric.

The curve defined by the parametrization eq. (15) is quartic and we will call it a *quartic QB-curve.*[12]

Let now a series of points $A_0 \dots \mathbf{a}_0, \dots, A_n \dots \mathbf{a}_n$ on $\mathcal{M}_6$ and corresponding parameter values t_i be given, $n \geq 4$ being an even number. Then we can construct a GC^1-spline curve[13] on $\mathcal{M}_6$ with quartic QB-curves as subsplines in the following way (see fig. 2):

Algorithm 1. (Construction of a GC^1-spline on $\mathcal{M}_6$ with quartic subsplines)
1.) For $i \in \{0, \dots, \frac{n}{2} - 2\}$: Determine the conic sections $\mathcal{C}_{2\cdot i, 2\cdot i+2, 2\cdot i+4}$ on $\mathcal{M}_6$ belonging to the point triples $A_{2\cdot i}$, $A_{2\cdot i+2}$, $A_{2\cdot i+4}$ and corresponding parameter triples $t_{2\cdot i}$, $t_{2\cdot i+2}$, $t_{2\cdot i+4}$ (see definition 1).
2.) For $i \in \{0, \dots, \frac{n}{2} - 2\}$: Choose two values $s_{i,3}$ and $s_{i+1,1}$ (**design parameters**) with $t_{2\cdot i+1} < s_{i,3} < t_{2\cdot i+2} < s_{i+1,1} < t_{2\cdot i+3}$ and compute the corresponding points $A_{i+1,1}$, $A_{i,3}$ on the conic section $\mathcal{C}_{2\cdot i, 2\cdot i+2, 2\cdot i+4}$.
3.) Choose points $A_{0,1}$, $A_{\frac{n}{2}-1,3}$ on $\mathcal{M}_6$ arbitrarily.
4.) For $i \in \{0, \dots, \frac{n}{2} - 1\}$: Set $A_{i,0} := A_{2\cdot i}$, $A_{i,2} := A_{2\cdot i+1}$, $A_{i,4} := A_{2\cdot i+2}$ and $t_{i,0} := t_{2\cdot i}$, $t_{i,2} := t_{2\cdot i+1}$, $t_{i,4} := t_{2\cdot i+2}$ and construct the quartic QB-curve $\mathcal{C}_i$ belonging to the control points $A_{i,j}$, $j \in \{0, 1, 2, 3, 4\}$ and the parameter values $t_{i,0}$, $t_{i,2}$, $t_{i,4}$.

If we moreover assume[14] that

$$\begin{aligned} \langle \mathbf{a}_i, \mathbf{a}_{i+2} \rangle, \langle \mathbf{a}_i, \mathbf{a}_{i+4} \rangle, \langle \mathbf{a}_{i+2}, \mathbf{a}_{i+4} \rangle, \langle \mathbf{a}_{i,1}, \mathbf{a}_{i,4} \rangle, &\neq 0, \\ \langle \mathbf{a}_{i,2}, \mathbf{a}_{i,3} \rangle, \langle \mathbf{a}_{i,0}, \mathbf{a}_{i,4} \rangle, \langle \mathbf{a}_{i,1}, \mathbf{a}_{i,3} \rangle, \langle \mathbf{a}_{i,0}, \mathbf{a}_{i,3} \rangle, \langle \mathbf{a}_{i,1}, \mathbf{a}_{i,2} \rangle &\neq 0 \end{aligned} \tag{17}$$

then the spline-curve $\mathcal{C}$ with the subsplines $\mathcal{C}_i$ constructed in algorithm 1 has the following properties:

[12] Here the "Q" stands for quadric and the "B" for Bézier. The "B" is justified as QB-curves have similar properties as ordinary Bézier curves. QB-curves can be defined for a control structure consisting of an arbitrary odd number of control points (see Gfrerrer [4]).
[13] "GC^1" means that adjacent subsplines have the same tangent in their common point.
[14] Compare with theorem 3.

Theorem 4. *(a) $\mathcal{C}$ interpolates A_i for the parameter value t_i.*
(b) Adjacent subsplines $\mathcal{C}_i$ and $\mathcal{C}_{i+1}$ meet with GC^1-continuity at their common point $A_{2\cdot i+2}$.
(c) In the points $A_{2\cdot i+1}$ we have C^∞-continuity.

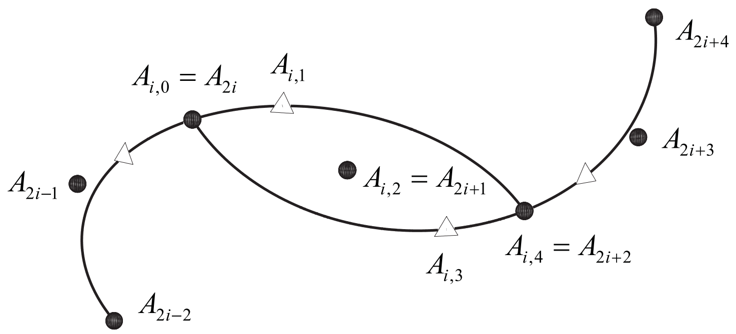

Figure 2: An algorithm for constructing a GC1-spline on a hyperquadric.

The curve defined by the parametrization eq. (15) is quartic and we will call it a *quartic QB-curve*. [15] So we suggest the following procedure to solve our motion-interpolation problem (MIP):
Input: $n+1$ direct Euclidean displacements α_i (positions of the moving space) by their transformation matrices $(m_{i,\,jk})_{j,k\in\{0,1,2,3\}}$ (see eqs. (1), (2)) and corresponding parameter values t_i, $n \geq 4$ being an even number.
Procedure: 1.) Map the given displacements onto $\mathcal{M}_6$ via $\mathfrak{S}$ (see eqs. (4), (5)). This yields the points $A_i := \mathfrak{S}(\alpha_i)$ on $\mathcal{M}_6$.
2.) Use algorithm 1 to construct the GC^1-spline $\mathcal{C} \ldots \mathbf{c}(t)$ belonging to the data set $\{A_i, t_i\}$.
Output: GC^1-spline-motion $\gamma(t)$, given by

$$\gamma(t): \ \mathbf{y}^* \ = \ \overline{\mathbf{c}(t)} \cdot \mathbf{y} \cdot \mathbf{c}(t)_e \tag{18}$$

interpolating the given positions α_i for the given parameter values t_i.

Remark 2. (Properties of the GC^1-spline-motion)
(a) The interpolant has the properties (RM) and (CTI).
(b) In the positions α_{2i+2}, $i \in \{0, \ldots, \frac{n}{2} - 2\}$ the spline motion displayes GC^1-continuity, which means that point-paths of adjacent subspline-motions meet with a common tangent. Anywhere else we have C^∞-continuity.
(c) The design parameters $s_{i,1}$, $s_{i,3}$ allow a subsequent modification of the $i-th$ subspline-motion $\gamma_i(t)$. This modification has local character as it does not change any other part of the motion. Figure 3 shows an example for $n = 6$: There we have three subspline-motions $\gamma_0(t)$, $\gamma_1(t)$, $\gamma_2(t)$. The two spline motions (a) and (b) differ only in the subspline $\gamma_1(t)$ due to a change of the design parameters $s_{1,1}$, $s_{1,3}$.

We have to emphasize that our algorithm will work correctly only if the properties (17) are fulfilled. For example we will get no result if we consider a set of displacements α_i which are all

[15] Here the "Q" stands for quadric and the "B" for Bézier. The "B" is justified as QB-curves have similar properties as ordinary Bézier curves. QB-curves can be defined for a control structure consisting of an arbitrary odd number of control points (see Gfrerrer [4]).

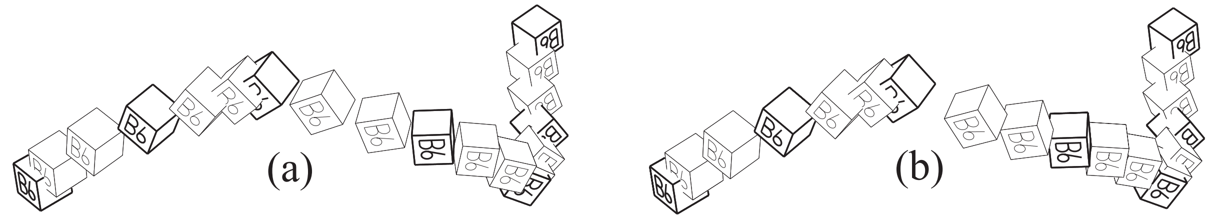

Figure 3: Spline motion consisting of 3 subsplines; the medium subspline in (a) differs from the one in (b). Control positions are drawn boldface.

pure rotations with axis through a common point (this means that the displacements are from $SO(3)$): In this case the corresponding points A_i on $\mathcal{M}_6$ lie all in a common generator (3-space) of $\mathcal{M}_6$ (compare with section 3) and thus $\langle \mathbf{a}_i, \mathbf{a}_j \rangle = 0$ for all $i, j \in \{0, \ldots, n\}$. But, if we work in $SO(3)$ it of course is better to use another kinematic mapping: The displacements in $SO(3)$ can be bijectively mapped into pairs of antipodal points on the unit hypersphere $\mathcal{S}_3$ in Euclidean 4-space $\mathbb{E}_4$ (The position vectors of the two antipodal points are given by the Euler parameters of the rotation-matrix representing the displacement). Thus we again have a hyperquadric as kinematic image space and hence can use a method like the one presented above for constructing an interpolating spline motion.

An analogous situation occurs if we deal with planar Euclidean displacements ($SE(2)$): The corresponding submanifold on $\mathcal{M}_6$ again is a generator (compare with section 3). We again can solve this problem by using another suitable kinematic mapping: The displacements in $SE(2)$ can be identified with pairs of antipodal points on a certain quadratic hypercylinder $\mathcal{Z}_3$ in $\mathbb{E}_4$.

5 Conclusion

In this paper we gave an application of Study's kinematic mapping $\mathfrak{S}$ in the field of motion design. We also wanted to show that Study's way to study kinematics has its justification among other methods, for instance Lie group techniques (see for example Karger and Nowak [5] or Zefran and Kumar [14], [15]). Study's quadric $\mathcal{M}_6$ is a *geometric* model for the group $SE(3)$ of direct Euclidean displacements, wheras Lie group techniques use the more analytic matrix-model of this group. Thus we think that Study's method gives us the possibility to gain a better insight into the geometry of Euclidean space kinematics.

We gave an example of how to construct a rational and coordinate-invariant GC^1-motion for the interpolation of a given sequence of positions by making use of $\mathfrak{S}$. Further research-efforts will be undertaken to investigate the possibility of constructing spline motions via $\mathfrak{S}$ which display a higher differentiation order in the control-positions.

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