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Axial Equiform Bricard Motions

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Abstract

Axial equiform Bricard motions are equiform motions where each point of the moving system sticks to a sphere and additionally a line (the axis) is fixed throughout the motion. In this paper we show that there are only two types of such motions. The first one is determined by requiring that the axis is translated along itself and another line, which is skew to the axis, always contains a point of the fixed system. The second type is a motion, which (besides the axis) fixes a point on this axis.

1 Introduction

In Euclidean kinematics there is only one non-trivial[1] type of motion with only spherical[2] orbits: This motion is 1-parametric and determined by the postulates that one line of the moving system Σ is translated along itself and additionally one point of Σ moves on a sphere.[3] So, this motion is an *axial motion* as one line (the axis) is fixed.

An equiform displacement only preserves angles but not necessarily distances. The group of equiform displacements is 7-parametric and it containes the 6-parametric group of Euclidean displacements as a subgroup. In the last years special

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[1]The trivial case occurs if one point is fixed; this yields a motion within the 3-parametic subgroup $SO(3)$ of all spherical displacements.

[2]Under a "sphere" we here always will understand a real sphere (ball) or a plane. A motion with only planar orbits will be called "Darboux motion", whereas we will speak of a "Bricard motion" if all points have spherical orbits and not all of these spheres degenerate into planes.

[3]See [3, chapter IV], [1], [8].

equiform Darboux motions have for instance been used to construct overconstrained (Euclidean) mechanisms (see [16], [13], [14], [15]), which shows that the study of equiform kinematics is not a pure theoretical task. Out of these considerations it seems to make sense to investigate equiform Darboux[4]- and Bricard motions. This paper deals with the 3-parametric group Γ_3 of axial equiform displacements, which is picked out of the full 7-parametric group of equiform displacements by the postulate that one line (the axis) is only moved along itself. We will show that beside Euclidean Bricard motions there exist exactly 2 types of Bricard motions within Γ_3. This will be done by making use of geometrical methods and of a certain *principle of transfer* (section 3). The latter goes back to Bricard (see [2]) who used it to classify motions where *some* points have spherical orbits. This principle of transfer was also adapted to treat analogous questions in other geometries (see [9], [10], [11], [5], [6]).

2 The group Γ_3

In the following $(\tilde{x}, \tilde{y}, \tilde{z})$ and (x, y, z) will denote the affine coordinates of a point in the fixed and moving system $\tilde{\Sigma}$ and Σ, respectively. Any axial equiform displacement can be analytically described by

$$\begin{pmatrix} \tilde{x} \\ \tilde{y} \\ \tilde{z} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ t \end{pmatrix} + s \cdot \begin{pmatrix} \cos u & -\sin u & 0 \\ \sin u & \cos u & 0 \\ 0 & 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} x \\ y \\ z \end{pmatrix} \tag{1}$$

where $s, t, u \in \mathbb{R}$ and $s \neq 0$. Here we have chosen the axis $\mathcal{L}$ which is only translated along itself as z-axis in the fixed as well as in the moving coordinate frame. The transformations (1) build up a 3-parametric group Γ_3 within the group of all equiform displacements. We will also use the following 2-parametric group

$$\Lambda_2 : \begin{pmatrix} \tilde{x} \\ \tilde{y} \\ \tilde{z} \end{pmatrix} = \begin{pmatrix} w \cdot \cos v & -w \cdot \sin v & 0 \\ w \cdot \sin v & w \cdot \cos v & 0 \\ 0 & 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} x \\ y \\ z \end{pmatrix} \tag{2}$$

of affine transformations ($v, w \in \mathbb{R}$ and $w \neq 0$). In every level plane[5] Λ_2 acts transitively: For any pair $X, Y \in \Sigma$ of points in the same level plane, none of them lying on the axis, there exists (exactly) one transformation $\lambda \in \Lambda_2$ with $\lambda(X) = Y$. The transformations in Λ_2 are compositions of axial dilatations with rotations around $\mathcal{L}$. Moreover the transformations of Λ_2 commute with the ones in Γ_3 which yields the following

[4] A complete classification of equiform Darboux motions is given in [7] or [12].

[5] These are the planes orthogonal to the axis.

Theorem 1. *If $P, Q \in \Sigma$ are two points in the same level plane, none of them lying on the axis and $\tilde{\Psi}_P$, $\tilde{\Psi}_Q$ denote their orbits under a motion γ in Γ_3, then there exists a displacement λ in Λ_2 with $\tilde{\Psi}_Q = \lambda(\tilde{\Psi}_P)$.*

Proof. There exists exactly one displacement $\lambda \in \Lambda_2$ with $\lambda(P) = Q$. Due to the commutativity of Γ_3 and Λ_2 we then have: $\tilde{\Psi}_Q = \gamma(Q) = \gamma(\lambda(P)) = \lambda(\gamma(P)) = \lambda(\tilde{\Psi}_P)$. □

This theorem shows us that orbits of points lying in the same level plane with respect to a motion in Γ_3 are affinely equivalent; the affine mapping is induced by the motion parameters.

As under the transformations of Λ_2 planes are transformed into planes, circles in level planes are transformed into such circles again and lines parallel to the axis $\mathcal{L}$ stay parallel to $\mathcal{L}$ we have the following conclusions:

Theorem 2. *If the orbit $\tilde{\Psi}_P$ of a point $P \notin \mathcal{L}$, lying in the level plane Π, is*

(a) in a plane, then all points $X \notin \mathcal{L}$ in Π have planar orbits.

(b) on a cylinder of revolution with axis parallel to $\mathcal{L}$, then this is true for all points of the moving space.

(c) on a ball, then all points $X \notin \mathcal{L}$ in Π have orbits on ellipsoids of revolution with axes parallel to $\mathcal{L}$ and centers in the level plane determined by the center of the orbit ball of P. Especially the orbits of the points on the circle through P with axis $\mathcal{L}$ also lie on balls.

Let us now assume that the orbit $\tilde{\Psi}_P$ of one point $P \notin \mathcal{L}$ is (in) a plane. This condition filters out a 2-parametric motion γ within Γ_3; we have to distinguish the following two cases:

Case 1. The orbit plane $\tilde{\Psi}_P$ is parallel to the axis $\mathcal{L}$: In this case *any* point of the moving system sticks to such a plane under γ. This is the only kind of equiform Darboux motion which is 2-parametric[6]; so in the following we will call this sort of motion *2-parametric Darboux motion.*

Case 2. The orbit plane $\tilde{\Psi}_P$ is not parallel to $\mathcal{L}$ and hence intersects $\mathcal{L}$ in a proper point $\tilde{O}$. Due to Theorem 2, (a) the points lying in the same level plane as P have planar orbits. As we will show in the following, the orbit $\tilde{\Psi}_X$ of any other point X (not lying on $\mathcal{L}$) is a quadratic cone, having its vertex in $\tilde{O}$

Without loss of generality[7] let P have the coordinates $(1, 0, 0)^t$ in Σ and $\tilde{\Psi}_P$ be the plane with the equation $\tilde{z} = k \cdot \tilde{x}$, $k \in \mathbb{R}$ in $\tilde{\Sigma}$; then via eq. (1) the condition

[6]In A. Karger's classification [7] this type is given by eq. (20).

[7]This always can be reached by suitable coordinate transformations.

that P sticks to $\tilde{\Psi}_P$ yields

$$t = k \cdot s \cdot \cos u. \tag{3}$$

Using this we obtain for the equation of the orbit $\tilde{\Psi}_X$ of a point $X(1, 0, z_0)$ on the line parallel to $\mathcal{L}$ through P:

$$\tilde{\Psi}_X : \; z_0^2 \cdot (\tilde{x}^2 + \tilde{y}^2) - (\tilde{z} - k \cdot \tilde{x})^2 = 0 \tag{4}$$

This is a quadratic cone with vertex $\tilde{O}$. The rest follows by the help of Theorem 1.

The quadratic cones given by eq. (4) become cones of revolution with axis $\mathcal{L}$ if and only if $k = 0$, which means that the orbit plane $\tilde{\Psi}_P$ of P is perpendicular to $\mathcal{L}$ (a level plane). In this case the point O is fixed throughout the motion. Conversely, if a point $O \in \mathcal{L}$ is fixed throughout a motion $\gamma \subset \Gamma_3$ then the plane through O perpendicular to the axis is fixed and any point outside this plane moves on the cone of revolution through this point with vertex O and axis $\mathcal{L}$. The transformations in Γ_3 which additionally fix a point $O \in \mathcal{L}$ build up a 2-parametric subgroup Γ_2 of Γ_3 which for instance is described by putting $t \equiv 0$ in eq. (1).

On the other hand the group Γ_2 can be filtered out of Γ_3 by the condition that the orbit $\tilde{\Psi}_P$ of one point P is a cone of revolution with axis $\mathcal{L}$: Under this assumption the vertex $\tilde{O}$ of $\tilde{\Psi}_P$ has to be a fixed point throughout the motion as the angle between $\mathcal{L}$ and the line $\tilde{O}P$ can not be changed by an equiform displacement and hence this line always has to pass through $\tilde{O}$. So, in this case all points are moved on cones of revolution with axis $\mathcal{L}$ and vertex O.

Due to Theorem 2, (b) the assumption, that one point P sticks to a cylinder of revolution with axis parallel to $\mathcal{L}$ under a motion of Γ_3, yields a (2-parametric) motion where *any* point of the moving system stays on such a cylinder during the motion.

Definition 1. *In the following we will call a motion within* Γ_2 *an FP-motion*[8]*, whereas we will speak of a CR-motion*[9] *if one point in* $\Sigma \setminus \mathcal{L}$ *is forced to move on a cylinder of revolution with generators parallel to the axis* $\mathcal{L}$*.*

It is well known from classic algebraic and descriptive geometry that the intersection curve c of two quadrics Ψ_1 and Ψ_2 of revolution with parallel axes $\mathcal{L}_1$, $\mathcal{L}_2$ either consists of planar parts (conic sections or straight lines) or is a spherical fourth

[8]"FP" stands for fixed point.
[9]"CR" stands for cylinder of revolution.

order curve[10] (that means it is algebraic of fourth order and lies on a ball).

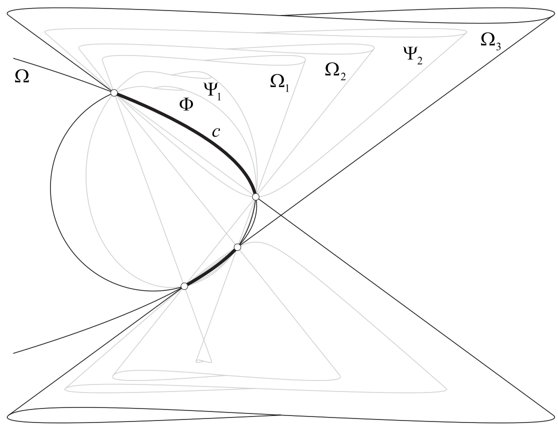

Figure 1. Pencil of quadrics of revolution with parallel axes (orthogonal projection onto the plane Π containing the axes). The pencil is spanned by each two of the quadrics, e.g. the ellipsoid Ψ_1 and the 2-sheet hyperboloid Ψ_2. The singular quadrics of the pencil are three cones of revolution Ω_1, Ω_2, Ω_3 and one parabolic cylinder Ω with generators orthogonal to Π. Due to the symmetry of all quadrics with respect to Π the projection of their common intersection curve c is part of the parabola which is the projection of Ω onto Π. Moreover the pencil contains one sphere Φ.

In the second case (c is a spherical fourth order curve, see figure 1) the only improper points on c are the absolute points of the planes perpendicular to $\mathcal{L}_1$, $\mathcal{L}_2$; the plane at infinty is tangent to c in these two points. We will call a curve of this type (intersection curve of two quadrics Ψ_1 and Ψ_2 of revolution with parallel axes which does not degenerate into two common planar sections of Ψ_1 and Ψ_2) a *special spherical quartic.*[11] The orthogonal projection of c into the direction of $\mathcal{L}_1$, $\mathcal{L}_2$ is a planar curve c^n which in general has algebraic order 4 again. Moreover c^n contains the absolute points of the planes orthogonal to $\mathcal{L}_1$, $\mathcal{L}_2$ as cusps. Such a curve is called *Cartesian oval.* The only exceptional case occurs if one of the planes perpendicular to $\mathcal{L}_1$, $\mathcal{L}_2$ contains the centres M_1, M_2 of *both* quadrics. Then this is a symmetry plane of c and the algebraic order of c^n reduces to 2 and c^n is a circle. With one exception any quadric Ψ belonging to the pencil of quadrics determined by Ψ_1 and Ψ_2 is a quadric of revolution with an axis $\mathcal{L}$ parallel to $\mathcal{L}_1$, $\mathcal{L}_2$ lying in the plane $\Pi := \mathcal{L}_1\mathcal{L}_2$. The exception is a parabolic cylinder Ω, whose generators are orthogonal to Π. Beside Ω the pencil contains three more singular quadrics Ω_1, Ω_2, Ω_3, which are cones of revolution.[12] One of these cones can "degenerate" into

[10]The case that their intersection consists of a cubic c^* and a straight line c^{**} is impossible: In this case the intersection c_{iu} of Ψ_i with the plane of infinity, $i \in \{1, 2\}$ has to contain the point C_u at infinity of the line c^{**}. As c^{**} must be a real line in this case and hence C_u cannot lie on the absolute circle this yields $c_{1u} = c_{2u}$. This means that this conic section is part of the intersection of Ψ_1 and Ψ_2 in contradiction to the assumptions.

[11]It may happen that c does not contain any real point.

[12]Some of these cones can be null cones.

a cylinder of revolution, which again characterizes the case that c^n is a circle. The curve c is rational if and only if one of the vertices of the cones Ω_i is on c. This point is then the singularity[13] of c. In this case the curve c is called *hippopede* or *horse fetter*.

As a conclusion of Theorem 2, (c) we also have

Theorem 3. *Let $P, Q \notin \mathcal{L}$ be two points of the moving system Σ which are lying in the same level plane Π but not on the same circle with axis $\mathcal{L}$. Let us moreover assume that the orbits of both points are lying on balls $\tilde{\Psi}_P$, $\tilde{\Psi}_Q$. Then the orbit of* any *point $X \in \Pi \setminus \mathcal{L}$ either is*

(a) on a special spherical quartic or

(b) planar.

Proof. Let λ_P, λ_Q denote the transformations in Λ_2 with $\lambda_P(P) = X, \lambda_Q(Q) = X$. Then due to the proof of Theorem 2, (c) the orbit of X is on the intersection curve $\tilde{c}_X$ of $\lambda_P(\tilde{\Psi}_P)$ and $\lambda_Q(\tilde{\Psi}_Q)$, which are two distinct[14] ellipsoides of revolution[15] with axes $\tilde{\mathcal{L}}_1$, $\tilde{\mathcal{L}}_2$ parallel to $\mathcal{L}$. Hence their intersection either is a special spherical quartic or degenerates into planar parts.[16] □

Definition 2. A motion determined by the condition that one point P of the moving system has an orbit on a circle $\tilde{c}$ in a level plane is called a *circular motion*[17].

Theorem 4. *Let a circular motion γ be given by the condition that $P \in \Sigma$, $P \notin \mathcal{L}$ moves on the circle $\tilde{c}_P$ in a level plane $\tilde{\Pi} \subset \tilde{\Sigma}$. Then the orbit of any point $X \notin \mathcal{L}$, lying in the same level plane Π as P, is on a circle in $\tilde{\Pi}$.*
Moreover we have:

(a) If the axis of the circle $\tilde{c}_P$ is $\mathcal{L}$ then the orbit of any other point is a circle with $\mathcal{L}$ as axis.

(b) If the axis of the circle $\tilde{c}_P$ is different from $\mathcal{L}$ then the orbit of any other point $X \notin \Pi$ is on a special spherical quartic which lies on a ball with center in Π.

[13] A fourth-order algebraic space curve can only have one singularity. This happens if and only if the curve is rational.

[14] $\lambda_P(\tilde{\Psi}_P)$ and $\lambda_Q(\tilde{\Psi}_Q)$ cannot be equal as this would imply that P, Q lie on the same circle with axis $\mathcal{L}$.

[15] One of these ellipsoides becomes a ball if X is on the circle with axis $\mathcal{L}$ through either P or Q.

[16] If the intersection of $\lambda_P(\tilde{\Psi}_P)$ and $\lambda_Q(\tilde{\Psi}_Q)$ consists of planar parts these must either be common conic sections or straight lines of the two ellipsoides. If straight lines are part of the intersection, they cannot be real, of course.

[17] Compare also with [4, pages 870–880].

Proof. There exists a displacement $\lambda \in \Lambda_2$ with $\lambda(P) = X$. Due to the commutativity of Γ_3 and Λ_2 we then have: $\gamma(X) = \gamma(\lambda(P)) = \lambda(\gamma(P))$. As $\gamma(P)$ is (part of) the circle $\tilde{c}_P$ in the level plane $\tilde{\Pi}$ and a circle in a level plane is transformed into a circle in the same level plane by λ, we have the stated result for points in $\Pi \setminus \mathcal{L}$. Obviously γ is a CR-motion.

Furthermore the level plane Π and with it the point $O := \Pi \cap \mathcal{L}$ is fixed throughout the motion, so γ also is an FP-motion.

If the axis of the circle $\tilde{c}_P$ is $\mathcal{L}$ then we trivially have a rotation around $\mathcal{L}$. Thus, the orbit of any point not lying on $\mathcal{L}$ is a circle with axis $\mathcal{L}$.

If the axis of the circle $\tilde{c}_P$ is different from $\mathcal{L}$ then the orbit of a point $X \notin \Pi, \mathcal{L}$ is on a cone of revolution with vertex O and axis parallel to $\mathcal{L}$ (as γ is an FP-motion) as well as on a cylinder of revolution with an axis parallel to $\mathcal{L}$ (as γ is a CR-motion). □

Theorem 4 shows us that circular motions are Bricard motions. By the next theorem we will see that this type of an axial equiform Bricard motion is only a special case in a larger class of such motions.

Theorem 5. *Let γ be an FP-motion and let $\Pi = \tilde{\Pi}$ denote the level plane through the fixed point O on $\mathcal{L}$. If one point $P \in \Sigma$, $P \notin \mathcal{L}, \Pi$ moves on a ball $\tilde{\Phi}_P$ with center $\tilde{M}_P \in \tilde{\Sigma} \setminus \mathcal{L}$, so that this ball intersects the orbit cone $\tilde{\Psi}_P$ under γ in a special spherical quartic, then* any *point $X \in \Sigma$ with $X \notin \mathcal{L}, \Pi$ moves on a special spherical quartic $\tilde{c}_X$ on a ball $\tilde{\Phi}_X$.*
Moreover we have

- *(a) If $\tilde{M}_P$ lies in $\tilde{\Pi}$, then the orbit of any point $Y \in \Pi$, $Y \neq O$ is on a circle. Additionally the centers $\tilde{M}_X$ of the balls $\tilde{\Phi}_X$ all lie in $\tilde{\Pi}$.*
- *(b) If $\tilde{M}_P$ does not lie in $\tilde{\Pi}$, then the orbit of any point $Y \in \Pi$, $Y \neq O$ is on a Cartesian oval and none of the centers $\tilde{M}_X$ lies in $\tilde{\Pi}$.*

Proof. The orbit of P is part of special spherical quartic, which ist the intersection $\tilde{c}_P$ of the ball $\tilde{\Phi}_P$ and the cone (of revolution) $\tilde{\Psi}_P$ (this cone has $\mathcal{L}$ as axis and the O as vertex). As one can verify by direct computation the displacements of Γ_2 commute with the ones of the group

$$\Lambda_3 : \begin{pmatrix} \tilde{x} \\ \tilde{y} \\ \tilde{z} \end{pmatrix} = \begin{pmatrix} w \cdot \cos v & -w \cdot \sin v & 0 \\ w \cdot \sin v & w \cdot \cos v & 0 \\ 0 & 0 & f \end{pmatrix} \cdot \begin{pmatrix} x \\ y \\ z \end{pmatrix} \tag{5}$$

with $f, v, w \in \mathbb{R}$ and $f, w \neq 0$. These transformations are compositions of axial dilatations, rotations around $\mathcal{L}$ and dilatations with center O. For any point X in

Σ with $X \notin \mathcal{L}, \Pi$ there is exactly one transformation $\lambda \in \Lambda_3$ with $\lambda(P) = X$. Hence the orbit of such a point X lies on the curve $\tilde{c}_X := \lambda(\tilde{c}_P)$ which is again a special spherical quartic. This gives us the stated result for the points $X \notin \mathcal{L}, \Pi$.

If $\tilde{M}$ is in $\tilde{\Pi}$, then due to the symmetry of $\tilde{c}_P$ with respect to $\tilde{\Pi} = \Pi$ the orthogonal projection of $\tilde{c}_P$ is a circle (see above), which contains the orbit of the ortogonal projection Q of P on Π. So γ is a circular motion and the rest follows with the help of Theorem 4.

If $\tilde{M}$ is not in $\tilde{\Pi}$, then the orthogonal projection of $\tilde{c}_P$ is a Cartesian oval. The transformation $\lambda \in \Lambda_2$ which transforms the orthogonal projection Q of P into the orthogonal projection Y of X also transforms the orbit of Q into the orbit of Y. As the restriction of λ to $\Pi = \tilde{\Pi}$ is an equiform displacement, namely the composition of a rotation and a dilation with center O, we see that the orbit of Y also is a Cartesian oval. None of the orbit-ball centers $\tilde{M}_X$ can lie in $\tilde{\Pi}$ as in this case we again would arrive at a circular motion with *all* orbit-ball centers lying in $\tilde{\Pi}$. □

With the help of Theorem 5 we have obtained all possible axial equiform Bricard motions which are also FP-motions or as we also might say, all possible axial equiform Bricard motions where a point is fixed. Are there any other axial equiform Bricard motions? The answer is positive as the following example will show.

In 1962 W. Wunderlich [17] studied the one-parametric motion:

$$\gamma_W : \begin{pmatrix} \tilde{x} \\ \tilde{y} \\ \tilde{z} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ h \cdot \sin u \end{pmatrix} + \cos u \cdot \begin{pmatrix} \cos u & -\sin u & 0 \\ \sin u & \cos u & 0 \\ 0 & 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} x \\ y \\ z \end{pmatrix} \quad (6)$$

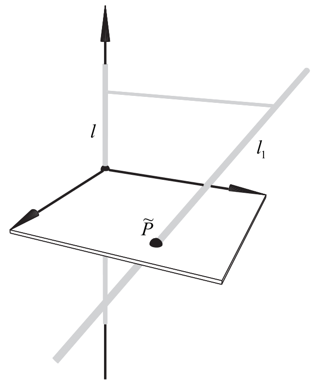

Figure 2. Wunderlich-Bricard motion. The moving system is represented by the grey lines l and l_1, the fixed system by the black coordinate frame. The line l is translated along itself, whereas the line l_1 always contains the point $\tilde{P}$ of the fixed system.

Geometrically this motion can be determined by requiring that one line l (the z-axis) of the moving system Σ is only translated along itself and another line $l_1 \subset \Sigma$ which is skew to l always contains a point $\tilde{P}$ of the fixed system $\tilde{\Sigma}$ (see figure 2). Among others the motion has the following properties:

(a) Each point $X \in \Sigma$ has its orbit on the intersection curve $\tilde{c}_X$ of a cylinder of revolution $\tilde{\Psi}_X$ which contains $z = \tilde{z}$ as generator and a ball $\tilde{\Phi}_X$, where $\tilde{\Phi}_X$ and $\tilde{\Psi}_X$ are tangent to each other in a point $\tilde{T}_X \notin \tilde{z}$.

(b) Since all orbit balls $\tilde{\Phi}_X$ contain the points $\tilde{H}_{\pm} \ldots (0, 0, \pm h)$ on $\tilde{z}$ this is also true for the orbit curves $\tilde{c}_X$.

Remark 1.

(a) Any orbit curve $\tilde{c}_X$ has a double point: $\tilde{T}_X$. Hence, $\tilde{c}_X$ is a hippopede.[18]

(b) The motion is also determined by the postulates that one point $X \in \Sigma$ has its orbit on the intersection of a cylinder of revolution which contains l as generator and a ball, with the additional property that cylinder and ball are tangent to each other in one of their common points, which is not lying on l.

(c) For $u = (2 \cdot k + 1) \cdot \frac{\pi}{2}$, $k \in \mathbb{Z}$ the transformation gets singular, i.e. the moving system Σ degenerates into one of the points H_+, H_-.

Definition 3. We will call a motion defined by the postulates in remark 1, (b) a *Wunderlich-Bricard- motion.*

3 A useful pair of mappings

The equation of a sphere $\tilde{\Phi}$ in the fixed space $\tilde{\Sigma}$ is given by

$$\tilde{\Phi}: \quad a \cdot (\tilde{x}^2 + \tilde{y}^2 + \tilde{z}^2) + 2 \cdot b \cdot \tilde{x} + 2 \cdot c \cdot \tilde{y} + 2 \cdot d \cdot \tilde{z} + e \quad = \quad 0 \tag{7}$$

Here $a, b, c, d, e \in \mathbb{R}$ and at least one of the first four coefficients a, b, c, d has to be different from zero. Moreover $\tilde{\Phi}$ degenerates into a plane iff $a = 0$.

The condition that the image of a point $X(x, y, z) \in \Sigma \setminus \mathcal{L}$ under a displacement $\gamma(s, t, u)$ in Γ_3 (see eq. (1)) lies on the sphere $\tilde{\Phi}$ with the equation (7) reads as follows:

$$\begin{array}{rcll} s^2 & \cdot & a \cdot (x^2 + y^2 + z^2) & + \\ t^2 & \cdot & a & + \\ 2 \cdot t \cdot s & \cdot & a \cdot z & + \\ 2 \cdot s \cdot \sin u & \cdot & (c \cdot x - b \cdot y) & + \\ 2 \cdot s \cdot \cos u & \cdot & (b \cdot x + c \cdot y) & + \\ 2 \cdot t & \cdot & d & + \\ 2 \cdot s & \cdot & d \cdot z & + \\ 1 & \cdot & e & = \; 0. \end{array} \tag{8}$$

[18]The rationality of the orbit-curves $\tilde{c}_X$ can already be seen from the parametric representation of the motion γ_W.

We will now define a pair of mappings $\mathfrak{K}$ and $\mathfrak{C}$, which will be of good use for our further investigations.

The first one is a kinematic mapping:

$$\mathfrak{K}: \left\{ \begin{array}{ccc} \Gamma_3 & \longrightarrow & \mathbb{P}_7 \\ \gamma(s,t,u) & \longrightarrow & G\ldots(g_0:\ldots:g_7)^t \end{array} \right\}, \tag{9}$$

where $\mathbb{P}_7$ denotes the 7-dimensional real projective space and g_i are the homogeneous coordinates of the image-point G in this space, calculated as follows:

$$g_0:\ldots:g_7 = s^2 : t^2 : 2ts : 2s\sin u : 2s\cos u : 2t : 2s : 1 \tag{10}$$

The second one is defined as follows:

$$\mathfrak{C}: \left\{ \begin{array}{ccc} \Sigma\times\mathbf{S} & \longrightarrow & \mathbb{P}_7 \\ (X,\tilde{\Phi}) & \longrightarrow & C\ldots(c_0:\ldots:c_7)^t \end{array} \right\}. \tag{11}$$

It maps a pair consisting of a point $X \in \Sigma \setminus \mathcal{L}$ with the coordinates (x,y,z) and a sphere[19] $\tilde{\Phi} \subset \tilde{\Sigma}$ with the equation (7) onto a point C in $\mathbb{P}_7$ with the homogeneous coordinates

$$c_0:\ldots:c_7 = a\cdot(x^2+y^2+z^2) : a : az : (cx-by) : (bx+cy) : d : dz : e. \tag{12}$$

The coordinates of a point G in the $\mathfrak{K}$-image of Γ satisfy the equations

$$\left. \begin{array}{rcl} g_3^2 + g_4^2 - 4\cdot g_0\cdot g_7 &=& 0 \\ g_6^2 - 4\cdot g_0\cdot g_7 &=& 0 \\ g_5\cdot g_6 - 2\cdot g_2\cdot g_7 &=& 0 \\ g_2^2 - 4\cdot g_0\cdot g_1 &=& 0 \end{array} \right\} \tag{13}$$

These equations represent a certain 3-dimensional algebraic variety $\mathcal{G}_3$ in $\mathbb{P}_7$. As each of the equations is quadratic, $\mathcal{G}_3$ is intersection of four hyperquadrics in $\mathbb{P}_7$.

The coordinates of a point C in the $\mathfrak{C}$-image of $\Sigma\times\mathbf{S}$ satisfy

$$c_1\cdot c_6 - c_2\cdot c_5 = 0, \tag{14}$$

which is the equation of a singular hyperquadradic $\mathcal{C}_6$ in $\mathbb{P}_7$. This hypercone has the 3-space $\mathcal{V}_3 := [O_0, O_3, O_4, O_7]_p$ as its vertex and contains 5-dimensional subspaces as (maximal-dimensioned) generators. Each of these generators connects the vertex $\mathcal{V}_3$ with one of the 1-dimensional generators (lines) of the basis-quadric, which is the annular quadric, given by eq. (14), interpreting this as the equation of a quadric in

[19] With $\mathbf{S}$ we denote the set of all spheres in $\tilde{\Sigma}$.

the 3-subspace $[O_1, O_2, O_5, O_6]_p$. The bilinear form belonging to the hypercone $\mathcal{C}_6$ is given by

$$\langle \mathbf{x}, \mathbf{y} \rangle = x_1 \cdot y_6 + x_6 \cdot y_1 - x_2 \cdot y_5 - x_5 \cdot y_2. \quad (15)$$

Here $\mathbf{x} = (x_0, \ldots x_7)$ and $\mathbf{y} = (y_0, \ldots y_7)$ denote the homogeneous coordinate vectors of two points X, Y in $\mathbb{P}_7$. X and Y are conjugate with respect to the hypercone $\mathcal{C}_6$ if and only if $\langle \mathbf{x}, \mathbf{y} \rangle = 0$.

Let $C \ldots (c_0, \ldots, c_7)$ be the $\mathfrak{C}$-image of a pair consisting of a point $X \ldots (x, y, z)$ in $\Sigma \setminus \mathcal{L}$ and a sphere $\tilde{\Phi} \ldots a : b : c : d : e$ in $\mathbf{S}$. Then we have to distinguish the following three cases:

(a) If $a \neq 0$ ($\tilde{\Phi}$ is a ball), then

$$c_1 \neq 0, \quad c_0 \cdot c_1 - c_2^2 \neq 0. \quad (16)$$

The second inequality holds as $X \notin \mathcal{L}$.

(b) If $a = 0$, $d \neq 0$ ($\tilde{\Phi}$ is a plane not parallel to the axis $\mathcal{L}$), then

$$c_0 = c_1 = c_2 = 0, \quad c_5 \neq 0. \quad (17)$$

(c) If $a = 0$, $d = 0$ and $b^2 + c^2 \neq 0$ ($\tilde{\Phi}$ is a plane not parallel to the axis $\mathcal{L}$), then

$$c_0 = c_1 = c_2 = c_5 = c_6 = 0, \quad c_3^2 + c_4^2 \neq 0. \quad (18)$$

The second condition holds as $x^2 + y^2, b^2 + c^2 \neq 0$.

Let us now determine the $\mathfrak{C}$-preimage of a point $C \ldots (c_0, \ldots, c_7)$ on $\mathcal{C}_6$. A necessary condition for the existence of a preimage $(X, \tilde{\Phi})$ with $X \ldots (x, y, z)$ in Σ and $\tilde{\Phi} \ldots a : b : c : d : e$ in $\mathbf{S}$ is that either (16) or (17) or (18) holds.

(a) Let the conditions (16) be fulfilled. Then we get

$$z = \tfrac{c_2}{c_1}, \quad x^2 + y^2 = \tfrac{c_0 \cdot c_1 - c_2^2}{c_1^2}, \quad (19)$$

which means that the points X belonging to the "condition" C lie on a circle in Σ with center $(0, 0, \frac{c_2}{c_1})$ on the axis $\mathcal{L}$ and radius r where $r^2 = \frac{c_0 \cdot c_1 - c_2^2}{c_1^2}$.

The ball corresponding to such a point X is then uniquely determined by:

$$\begin{aligned} a : b : c : d : e = {} & c_1(x^2 + y^2) : c_4 x - c_3 y : \\ & c_3 x + c_4 y : c_5(x^2 + y^2) : c_7(x^2 + y^2) \end{aligned} \quad (20)$$

This fits well to our result that if one point X moves on a sphere, then any other point on the circle through X with axis $\mathcal{L}$ also has to (compare with Theorem 2, (c)).

(b) Let us now assume that the conditions (17) hold. Then the z-coordinate of the corresponding points X is uniquely determined:

$$z = \frac{c_6}{c_5}, \tag{21}$$

whereas x and y can be arbitrarily chosen with $x^2 + y^2 \neq 0$. This means that the points X belonging to the condition C lie in a level plane. The coefficients in the equation of the corresponding orbit plane for such a point X are then determined by putting $c_1 = 0$ in (20). This is in accordance with the fact that if the orbit of one point P is in a plane not parallel to $\mathcal{L}$, this holds for any point of the level plane determined by P (Theorem 2, (a)).

(c) Let now (18) be satisfied for $C \in \mathcal{C}_6$. This yields (beside the general assumption $x^2 + y^2 \neq 0$ as $X \notin \mathcal{L}$) no condition for the coordinates x, y, z of the points X, which means that they can be arbitrarily chosen in $\Sigma \setminus \mathcal{L}$. The coefficients of the plane corresponding to such a point are then given again by (20) after putting $c_1 = c_5 = 0$. This yields $a = d = 0$. Morover $b^2 + c^2 \neq 0$ is guaranteed as $x^2 + y^2, c_3^2 + c_4^2, \neq 0$. So the plane is parallel to the axis $\mathcal{L}$. The result is in accordance with the fact, that it suffices to require that *one* point moves in a plane parallel to $\mathcal{L}$ to guarantee this for all points in Σ (2-parametric Darboux motion, see section 2).

Let now γ denote a displacement in Γ_3 and $G := \mathfrak{K}(\gamma) \ldots (g_0, \ldots, g_7)$ its $\mathfrak{K}$-image. Let moreover X be a point in Σ and $\tilde{\Phi} \in S$ be a sphere and let $C := \mathfrak{C}(X, \tilde{\Phi}) \ldots (c_0, \ldots, c_7)$. Then we have due to eq. (8)

$$\gamma(X) \in \tilde{\Phi} \iff \sum_{i=0}^{7} g_i \cdot c_i = 0. \tag{22}$$

So, it seems to make sense to introduce the following bilinear form in $\mathbb{P}_7$:

$$\begin{aligned} \mathbb{P}_7 \times \mathbb{P}_7 &\longrightarrow \mathbb{R} \\ (G, C) &\longrightarrow \sum_{i=0}^{7} g_i \cdot c_i. \end{aligned} \tag{23}$$

This bilinear form is regular and induces an orthogonality relation in $\mathbb{P}_7$:

$$G \perp C \iff \sum_{i=0}^{7} g_i \cdot c_i = 0. \tag{24}$$

In the following we will denote the orthogonal complement of a subset S of $\mathbb{P}_7$ by $S^\perp$:

$$S^\perp \quad := \quad \{Y \in \mathbb{P}_7 | \forall X \in S : X \perp Y\} \tag{25}$$

$S^\perp$ is always a projective subspace of $\mathbb{P}_7$. Because of eqs. (22) and (24) we have the following

Theorem 6.

(a) Let γ be a (1- or 2-parametric) motion in Γ_3 and $\mathfrak{K}(\gamma)$ its $\mathfrak{K}$-image and let moreover M_γ denote the set of all pairs $(X, \tilde{\Phi})$ in $\Sigma \times \mathbf{S}$ with $\gamma(X) \subset \tilde{\Phi}$, then

$$\mathfrak{C}(M_\gamma) \subset \mathfrak{K}(\gamma)^\perp \cap \mathcal{C}_6. \tag{26}$$

(b) Given conversely a subset M of $\Sigma \times \mathbf{S}$, the $\mathfrak{K}$-image of the maximal motion γ, satisfying these "conditions" has to lie in $\mathfrak{C}(M)^\perp \cap \mathcal{G}_3$:

$$\mathfrak{K}(\gamma) \subset \mathfrak{C}(M)^\perp \cap \mathcal{G}_3. \tag{27}$$

Definition 4. Let $X \in \Sigma \setminus \mathcal{L}$ and $\tilde{\Phi} \in \mathbf{S}$, then we will call $C := \mathfrak{C}(X, \tilde{\Phi})$ a *sphere-condition.* If $\tilde{\Phi}$ degenerates into a plane, we will also speak of a *plane-condition* or *degenerated sphere-condition.*

4 The lines on $\mathcal{C}_6$

We now want to investigate the geometric meaning of a line $\mathcal{L}_\mathcal{C}$, which is spanned by 2 sphere-conditions and moreover lies on the hypercone $\mathcal{C}_6$. Let 2 sphere-conditions $C_i = \mathfrak{C}(X_i, \tilde{\Phi}_i)$, with $X_i \ldots (x_i, y_i, z_i)$ and $\tilde{\Phi}_i \ldots a_i : b_i : c_i : d_i : e_i$, $i \in \{1, 2\}$ be given:

$$c_{i0} : \ldots : c_{i7} = a_i(x_i^2+y_i^2+z_i^2) : a_i : a_i z_i : (c_i x_i - b_i y_i) : (b_i x_i + c_i y_i) : d_i : d_i z_i : e_i \tag{28}$$

The line $\mathcal{L}_\mathcal{C} := [C_1, C_2]_p$ is parametrized by

$$\mathcal{L}_\mathcal{C} \ldots (c_0, \ldots, c_7)^t \quad = \quad f_1 \cdot (c_{10}, \ldots, c_{17})^t \quad + \quad f_2 \cdot (c_{20}, \ldots, c_{27})^t \tag{29}$$

By the two conditions "X_i sticks to $\tilde{\Phi}_i$, $i \in \{1, 2\}$" a certain 1-parametric motion γ is filtered out of Γ_3. The line $\mathcal{L}_\mathcal{C}$ lies on $\mathcal{C}_6$ if and only if the points C_1, C_2 are conjugate with respect to $\mathcal{C}_6$. Thus by using (15) we obtain the condition

$$(a_1 \cdot d_2 - a_2 \cdot d_1) \cdot (z_2 - z_1) \quad = \quad 0. \tag{30}$$

We distinguish the following two cases

- Both spheres degenerate into planes: $a_1 = a_2 = 0$ (case 1).
- At least one of the two spheres is a ball: $a_1 \neq 0$ (case 2).

Case 1. $a_1 = a_2 = 0$. We now have

$$c_{i0} : \ldots : c_{i7} = 0 : 0 : 0 : (c_i x_i - b_i y_i) : (b_i x_i + c_i y_i) : d_i : d_i z_i : e_i \tag{31}$$

Case 1.1. If additionally $z_1 = z_2$ then any point in the level plane $\Pi \ldots z = z_1 = z_2$ has to stick to two different planes (Theorem 2, (a)) and thus to their intersection line. This shows us that the motion can be obtained by requiring, that one line (the axis $\mathcal{L}$) is only translated along itself and one point P (outside the axis) moves on a line $\tilde{\mathcal{L}}$. Hence, in case that $\mathcal{L}$ and $\tilde{\mathcal{L}}$ are skew, the motion γ is the inverse to a Wunderlich-Bricard motion (see section 2). As can be easily shown, the points outside Π and $\mathcal{L}$ move on hyperbolas in planes parallel to $\mathcal{L}$.

Case 1.2. Let now $z_1 \neq z_2$. If $d_1 = 0$ or $d_2 = 0$ then one of the points, let us say X_1, has to move in a plane parallel to $\mathcal{L}$. But then (see section 2) each point of the moving system has to move in such a plane. The point X_2 is fixed to *two* different planes and we again obtain the previous case. So, without loss of generality, we can assume $d_1, d_2 \neq 0$. Any point C on $\mathcal{L}_\mathcal{C}$ except for the one with $f_1 : f_2 = -d_2 : d_1$ is a plane-condition as (17) is satisfied. The z-coordinates of the corresponding points $X \in \Sigma$ are computed via

$$z = \frac{f_1 \cdot d_1 \cdot z_1 + f_2 \cdot d_2 \cdot z_2}{f_1 \cdot d_1 + f_2 \cdot d_2} \tag{32}$$

and thus take all values in $\mathbb{R}$. Hence, due to Theorem 2, (a), *all* points have planar orbits.

Case 2. Let now $a_1 \neq 0$. Then the intersection of $\mathcal{L}_\mathcal{C}$ with the hyperplane $c_1 = 0$ is empty or a point. The same is true for the intersection of $\mathcal{L}_\mathcal{C}$ with the 4-space $\mathcal{S}_4 \ldots c_0 = c_1 = c_2 = 0$. Moreover $\mathcal{L}_\mathcal{C}$ can not be contained in the hyperquadric determined by $c_0 \cdot c_1 - c_2^2 = 0$, as by assumption C_1 is not on this hyperquadric. Hence, with exception of at most three points, any point C on $\mathcal{L}_\mathcal{C}$ is a non-degenerated sphere condition (see (16). Moreover only one of the points C on $\mathcal{L}_\mathcal{C}$ can represent a plane-condition. Therefore we are allowed to assume $a_1, a_2 \neq 0$ and can moreover normalize: $a_1 = a_2 := 1$. Then according to eq. (30) we either have $d_1 = d_2$ or $z_1 = z_2$.

Case 2.1. $d_1 \neq d_2$. Then z_1 and z_2 have to be equal. The z-coordinate belonging to an arbitrary point C on $\mathcal{L}_\mathcal{C}$ is again computed by eq. (32), which yields now $z = z_1 = z_2 = const.$ This means that the concerned points X all lie in the same level plane. So, γ turns out to be one of the motions described in Theorem 3 or (if X_1 and X_2 lie on the same circle with axis $\mathcal{L}$) a motion, where one point sticks to a circle in a plane not parallel to the axis $\mathcal{L}$.

Case 2.2. $d_1 = d_2$. The z-coordinates of the centers of the balls $\tilde{\Phi}$ corresponding to the points C on $\mathcal{L}_\mathcal{C}$ are computed by

$$-\frac{d}{a} = -\frac{c_5}{c_1} = -\frac{f_1 \cdot d_1 + f_2 \cdot d_2}{f_1 + f_2}, \tag{33}$$

which due to $d_1 = d_2$ is a constant value. Hence all these centers lie in the same level plane. What is the locus of the corresponding points $X \dots (x, y, z)$? According to eqs. (19)

$$z = \frac{f_1 \cdot z_1 + f_2 \cdot z_2}{f_1 + f_2}, \tag{34}$$

$$x^2 + y^2 = \frac{[f_1(x_1^2 + y_1^2 + z_1^2) + f_2(x_2^2 + y_2^2 + z_2^2)](f_1 + f_2) - (f_1 z_1 + f_2 z_2)^2}{(f_1 + f_2)^2}. \tag{35}$$

If $z_1 = z_2$ then eq. (34) implies $z = z_1 = z_2$, which means that the locus is a level plane.

If $z_1 \neq z_2$ we - after having eliminated $f_1 : f_2$ from the equations (34, 35) - obtain the equation of the ball with center on the axis $\mathcal{L}$ that contains X_1, X_2:

$$x^2 + y^2 + z^2 = \frac{(x_1^2 + y_1^2 + z_1^2)(z_2 - z) - (x_2^2 + y_2^2 + z_2^2)(z_1 - z)}{z_2 - z_1}. \tag{36}$$

We summarize the result in

Theorem 7. *Let $\mathcal{L}_\mathcal{C} = [C_1, C_2]_p$ be a line on the quadratic hypercone $\mathcal{C}_6$, where C_1, C_2 are two sphere-conditions. Then the 1-parametric motion $\gamma \subset \Gamma_3$ satisfying the conditions represented by $\mathcal{L}_\mathcal{C}$*

(a) either is an axial Darboux motion or

(b) moves the points of a level plane on spherical orbits or

(c) moves the points of a ball with center on the axis $\mathcal{L}$ on balls with centers in a common level plane.

Additionally we have

Theorem 8.

(a) If a motion $\gamma \subset \Gamma_3$ moves two points in planes, then all *points of the moving space have planar orbits (case 1.2).*

(b) If a motion $\gamma \subset \Gamma_3$ moves two points X_1, X_2, not lying in the same level plane, on balls with centers in the same level plane $\tilde{\Pi}$, then all points lying on the ball with center on the axis $\mathcal{L}$, that contains X_1, X_2, move on balls with centers in $\tilde{\Pi}$ (case 2.2).

5 The planes on $\mathcal{C}_6$

Let $\Pi_\mathcal{C}$ denote a projective 2-space (plane), which is spanned by 3 sphere-conditions $C_i = \mathfrak{C}(X_i, \tilde{\Phi}_i)$, with $X_i \dots (x_i, y_i, z_i)$ and $\tilde{\Phi}_i \dots a_i : b_i : c_i : d_i : e_i$, $i \in \{1, 2, 3\}$ and is moreover contained in $\mathcal{C}_6$. Let us additionally assume, that none of the 3 spheres $\tilde{\Phi}_i$ degenerates into a plane, which means that we can put $a_i = 1$ without loss of generality. The plane $\Pi_\mathcal{C} := [C_1, C_2, C_3]_p$ lies on $\mathcal{C}_6$ if and only if the points C_i are mutually conjugate with respect to $\mathcal{C}_6$, which gives us the conditions

$$(d_2 - d_1) \cdot (z_2 - z_1) = (d_1 - d_3) \cdot (z_1 - z_3) = (d_3 - d_2) \cdot (z_3 - z_2) = 0 \tag{37}$$

by using (15). The plane $\Pi_\mathcal{C}$ is parametrized by

$$\Pi_\mathcal{C} \dots (c_0, \dots, c_7)^t = f_1 \cdot (c_{10}, \dots, c_{17})^t + f_2 \cdot (c_{20}, \dots, c_{27})^t + f_3 \cdot (c_{30}, \dots, c_{37})^t. \tag{38}$$

$\Pi_\mathcal{C}$ is not part of the hyperplane $c_1 = 0$, as C_1, C_2, C_3 are not in this hyperplane; so it intersects it in a line. Out of analogous considerations the intersection of $\Pi_\mathcal{C}$ with the 4-space $c_0 = c_1 = c_2 = 0$ is either a line or a point or the empty set. Moreover the hyperquadric given by $c_0 \cdot c_1 - c_2^2 = 0$ does not contain $\Pi_\mathcal{C}$ as C_1, C_2, C_3 are not on it, which means that this hyperquadric intersects $\Pi_\mathcal{C}$ in a second order curve. So, with exception of a at most 1-parametric set of points any point in $\Pi_\mathcal{C}$ is a (non-degenerating) sphere-condition (see (16)).

We now can discuss in the following way: The points X_1, X_2, X_3 lie in the same level plane ($z_1 = z_2 = z_3$) (case 1) or at most two of these points have the same z-coordinate (case 2), for instance $z_1 \neq z_2, z_3$, which implies $d_1 = d_2 = d_3$.

Case 1. $z_1 = z_2 = z_3$. Then the eqs. (37) are satisfied. Let X be a point in the moving system Σ corresponding to a non-degenerating sphere-condition $C \in \Pi_\mathcal{C}$. The z-coordinate of X is constant:

$$z = \frac{f_1 \cdot z_1 + f_2 \cdot z_2 + f_3 \cdot z_3}{f_1 + f_2 + f_3} = z_1 = z_2 = z_3. \tag{39}$$

So all these points lie in the same level plane.

Case 2. $z_1 \neq z_2, z_3$. Then the conditions (37) only hold, if $d_1 = d_2 = d_3$. Let C again denote a non-degenerating sphere-condition in $\Pi_{\mathcal{C}}$. Then the corresponding ball $\tilde{\Phi}$ has a center with z-coordinate

$$-\frac{d}{a} = -\frac{c_5}{c_1} = -\frac{f_1 \cdot d_1 + f_2 \cdot d_2 + f_3 \cdot d_3}{f_1 + f_2 + f_3} = -d_1 = -d_2 = -d_3 = const. \quad (40)$$

Hence all these balls have centers in the same level plane.

We summarize in

Theorem 9. *Let* $\Pi_{\mathcal{C}} = [C_1, C_2, C_3]_p$ *be a plane contained in the quadratic hypercone* $\mathcal{C}_6$*, where* C_1*,* C_2*,* C_3 *are three non-degenerated sphere-conditions. Then either*

(a) the corresponding points X *of the moving system* Σ *or*

(b) the centers of the corresponding orbit-balls $\tilde{\Phi}$ *lie in a common level-plane.*

Remark 2. *In general the intersection of* $\Pi^{\perp}$ *with* $\mathcal{G}_3$ *will only consist of a finite set of points, as* $\dim \Pi^{\perp} = 4$*. This means that in general there will only be a finite set of transformations in* Γ_3*, satisfying the sphere-conditions* C_1*,* C_2*,* C_3*.*

6 Bricard motions within Γ_3

The aim of this section is to show that each axial equiform Bricard motion, which is not a Euclidean Bricard motion, either is an FP-motion or a Wunderlich-Bricard motion. To prove this we will - beside elementary calculations - also make use of the methods developed in section 3.

In a first step we will show that an axial equiform Bricard motion, where some points have planar orbits, necessarily is an FP-motion. Let γ be an axial equiform Bricard motion, where the orbit of a point $P \in \Sigma$ is in a plane $\tilde{\Pi}$ not parallel[20] to the axis $\mathcal{L}$. Let moreover Π denote the level plane that contains P. Any other point in Π also has a planar orbit (Theorem 2, (a)). None of the points X not lying in Π or on $\mathcal{L}$ can have a planar orbit, as this would lead to a Darboux motion (Theorem 8, (a)). On the one hand the orbit of such a point X has to be a special spherical quartic $\tilde{c}_X$ (Theorem 3) and on the other hand this orbit has to lie on a quadratic cone $\tilde{\Psi}_X$ which has its center $\tilde{O}$ on $\mathcal{L}$ (see section 2). The special spherical quartic $\tilde{c}_X$ can only be part of $\tilde{\Psi}_X$ if $\tilde{\Psi}_X$ is a cone of revolution. But this implies that γ is an FP-motion. Moreover $\Pi = \tilde{\Pi}$ and this plane is perpendicular to the axis $\mathcal{L}$ and intersects $\mathcal{L}$ in the fixed point $O = \tilde{O}$. So we have

[20]If the orbit plane of P is parallel to the axis $\mathcal{L}$, then γ is part of a 2-parametric Darboux motion, which is not of interest here.

Theorem 10. *An axial equiform Bricard motion γ, which moves one point in a plane, is an FP-motion.*[21]

The second step will be to show that any non-Euclidean axial equiform Bricard motion, where no point has a planar orbit, is a CR-motion.

Let γ be a non-Euclidean axial equiform Bricard motion with no planar orbits; then any of the ∞^3 points in the moving system Σ has its orbit on exactly one ball.[22] As the points X belonging to the same circle with axis $\mathcal{L}$ yield the same sphere-condition, this means that there is exactly a 2-parametric set of non-degenerated sphere-conditions. This set has to lie in the intersection of the quadratic hypercone $\mathcal{C}_6$ with a projective subspace $\mathcal{S}$ of $\mathbb{P}_7$ (Theorem 6, (a)). Thus $\mathcal{S}$ either is a 3-space, which intersects $\mathcal{C}_6$ in a (2-dimensional) quadric (case 1) or $\mathcal{S}$ is a plane contained in $\mathcal{C}_6$ (case 2).

Case 1. Let $\mathcal{S}$ be a 3-space; then the intersection of $\mathcal{S}$ with the 4-space $\mathcal{S}_4 \ldots c_0 = c_1 = c_2 = 0$ at least contains one point $C \ldots (0, 0, 0, c_3, c_4, c_5, c_6, c_7)$. Moreover

$$c_5 \;=\; 0 \tag{41}$$

has to hold, as else C would represent a plane-condition. Due to the linearization process described in section 3 any point in $\mathcal{S}$ represents a condition on the motion parameters. In case of the point C this condition reads as follows:

$$s \;\equiv\; -\frac{c_7}{2 \cdot (c_3 \cdot \sin u + c_4 \cdot \cos u + c_6)}. \tag{42}$$

With condition (42) we have filtered out a 2-parametric motion

$$\gamma(t,u) : \begin{pmatrix} \tilde{x} \\ \tilde{y} \\ \tilde{z} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ t \end{pmatrix} + s \cdot \begin{pmatrix} \cos u & -\sin u & 0 \\ \sin u & \cos u & 0 \\ 0 & 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} x \\ y \\ z \end{pmatrix} \tag{43}$$

within Γ_3. As can be seen from this representation, the orbit of an arbitrary point $X \ldots (x, y, z)$ under $\gamma(t, u)$ is a quadratic cylinder with generators parallel to $\mathcal{L}$. As in case of a Bricard motion this quadratic cylinder has to contain a special spherical quartic, it must be a cylinder of revolution, which means that the curves parametrized by

$$\begin{pmatrix} \tilde{x} \\ \tilde{y} \end{pmatrix} = s \cdot \begin{pmatrix} \cos u & -\sin u \\ \sin u & \cos u \end{pmatrix} \cdot \begin{pmatrix} x \\ y \end{pmatrix} \tag{44}$$

[21] For the properties of such motions see Theorem 5.

[22] Due to Theorem 3 this orbit is a special spherical quartic.

have to be circles. After having derived the equation of the curves given by (44) one immediately sees, that this can only happen if $c_3 = c_4 = 0$, which means that s has to be constant. So, this case only leads us to *Euclidean motions* in contradiction to the assumption. This shows us, that $\mathcal{S}$ cannot be a 3-space.

Case 2. Let $\mathcal{S}$ be a plane contained in $\mathcal{C}_6$. As the corresponding points X fill out the complete moving space, the centers of the corresponding balls $\tilde{\Phi}$ all lie in a common level plane $\tilde{\Pi}$ (Theorem 9). As this especially holds for each pair (X, Y) of points lying in the same level plane, the orbit $\tilde{c}_X$ of each point X lies on the intersection of a sphere $\tilde{\Phi}$ and an ellipsoid of revolution $\tilde{\Psi}$ both having centers in the level plane $\tilde{\Pi}$. Hence, this curve is symmetric with respect to $\tilde{\Pi}$ and therefore lies on a cylinder of revolution. So we have a CR-motion.

As a result we have:

Lemma 1. *A non-Euclidean axial equiform Bricard motion γ, which does not move any point in a plane, is a CR-motion. Moreover the centers of all orbit-balls lie in a common level plane.*

Armed with Lemma 1 we are now ready to prove the following

Theorem 11. *A non-Euclidean axial equiform Bricard motion γ, which does not move any point in a plane, is a Wunderlich-Bricard motion.*

Proof. Let γ be a non-Euclidean axial equiform Bricard motion, which does not move any point in a plane. Then due to Lemma 1 the orbit $\tilde{c}_X$ of a point $X \in \Sigma \setminus \mathcal{L}$ is on a special spherical quartic, which is the intersection of a ball $\tilde{\Phi}_X$ and a cylinder of revolution $\tilde{\Psi}_X$ with generators parallel to $\mathcal{L}$. Moreover the centers $\tilde{M}_X$ of all orbit balls $\tilde{\Phi}_X$ lie in a common level plane $\tilde{\Pi}$. Without loss of generality we can choose the coordinate frame $\{\tilde{O}; \tilde{x}, \tilde{y}, \tilde{z}\}$ in the fixed system $\tilde{\Sigma}$ in a way that $\tilde{\Pi}$ is given by $\tilde{z} = 0$. Let $E_1 \ldots (1, 0, 0)$ denote the unit point on the x-axis of the moving system and let moreover the orbit-ball $\tilde{\Phi}_{E_1}$ of E_1 be given by its equation

$$\tilde{\Phi}_{E_1} : \ (\tilde{x} - m)^2 + (\tilde{y} - n)^2 + \tilde{z}^2 - r^2 \ = \ 0 \tag{45}$$

and the orbit-cylinder $\tilde{\Psi}_{E_1}$ of E_1 by the parametrization

$$\tilde{\Psi}_{E_1} : \ \begin{pmatrix} \tilde{x} \\ \tilde{y} \\ \tilde{z} \end{pmatrix} = \begin{pmatrix} m_1 + r_1 \cdot \cos v \\ r_1 \cdot \sin v \\ w \end{pmatrix}, \ \ v \in [0, 2 \cdot \pi), \ w \in \mathbb{R} \tag{46}$$

So, $\tilde{\Phi}_{E_1}$ is the ball with center $\tilde{M}_{E_1} \ldots (m, n, 0)$ and radius r and $\tilde{\Psi}_{E_1}$ is the cylinder of revolution with an axis in the $\tilde{x}\tilde{z}$-plane[23] parallel to $\mathcal{L} = z = \tilde{z}$ and radius r_1. Substitution of (46) into (45) yields

$$w \ = \ \pm\sqrt{g_1 \cdot \cos v + h_1 \cdot \sin v + k_1}, \tag{47}$$

[23]This can be guaranteed by an appropriate rotation around $\mathcal{L}$.

where $g_1 := 2\cdot(m-m_1)\cdot r_1$, $h_1 := 2\cdot n\cdot r_1$, $k_1 := r^2 - r_1^2 - (m-m_1)^2 - n^2$. Hence, the intersection curve $\tilde c_{E_1}$ of $\tilde\Phi_{E_1}$ and $\tilde\Psi_{E_1}$ is parametrized by

$$\tilde c_{E_1}: \begin{pmatrix}\tilde x\\ \tilde y\\ \tilde z\end{pmatrix} = \begin{pmatrix} m_1 + r_1\cdot\cos v\\ r_1\cdot\sin v\\ \pm\sqrt{g_1\cdot\cos v + h_1\cdot\sin v + k_1}\end{pmatrix}. \tag{48}$$

Let now $X\dots(1,0,c)$ be an arbitray point on the line through E_1 parallel to the axis $\mathcal{L}$. Such a point too has its orbit on the cylinder $\tilde\Psi_{E_1}$. If these points have spherical orbits, this is true for *all* points in $\Sigma\setminus\mathcal{L}$ (Theorem 2, (c)). So it suffices to concentrate on these points. Consider the triangle OE_1X. (figure 3). In any position of the motion the displaced triangle $\tilde O\tilde E_1\tilde X$ and the original one are homothetic. Hence, we have

$$\frac{\overline{\tilde E_1\tilde X}}{\overline{\tilde O\tilde E_1}} = \frac{\overline{E_1X}}{\overline{OE_1}} = c, \tag{49}$$

which gives us

$$\overline{\tilde E_1\tilde X} = \pm c\cdot\overline{\tilde O\tilde E_1} = \pm c\cdot\sqrt{g_2\cdot\cos v + k_2} \tag{50}$$

with $g_2 := 2\cdot m_1\cdot r_1$, $k_2 := m_1^2 + r_1^2$.

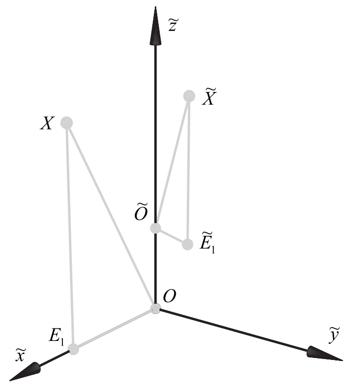

Figure 3. Two positions OE_1X, $\tilde O\tilde E_1\tilde X$ of a triangle under an aequiform motion are homothetic.

Hence, the orbit $\tilde c_X$ of X is represented by the parametrization

$$\tilde c_X: \begin{pmatrix}\tilde x\\ \tilde y\\ \tilde z\end{pmatrix} = \begin{pmatrix} m_1 + r_1\cdot\cos v\\ r_1\cdot\sin v\\ \pm\sqrt{g_1\cdot\cos v + h_1\cdot\sin v + k_1} \pm c\cdot\sqrt{g_2\cdot\cos v + k_2}\end{pmatrix}. \tag{51}$$

We now have to find out (necessary and sufficient) conditions on the coefficients g_1, h_1, k_1, g_2, k_2 that force $\tilde c_X$ on a ball $\tilde\Phi_X$ with center in $\tilde\Pi$:

$$\tilde\Phi_X: (\tilde x - m_c)^2 + (\tilde y - n_c)^2 + \tilde z^2 - r_c^2 = 0 \tag{52}$$

After having substituted (51) into (52) we obtain the condition

$$(g \cdot \cos v + h \cdot \sin v + k)^2 - 4 \cdot c^2 \cdot (g_1 \cdot \cos v + h_1 \cdot \sin v + k_1) \cdot (g_2 \cdot \cos v + k_2) = 0 \quad (53)$$

with $g := 2 \cdot r_1 \cdot (c^2 \cdot m_1 + m - m_c)$, $h := 2 \cdot r_1 \cdot (n - n_c)$, $k := (m_1 - m_c)^2 + n_c^2 + r^2 - r_c^2 - n^2 + c^2 \cdot (m_1 + r_1)^2$. As eq. (53) has to be satisfied identically in v, we - after having expanded this expression into the linear independent functions $1, \sin v, \cos v, \sin 2v, \cos 2v$ - obtain by comparison of coefficients five (quadratic) conditions

$$\left.\begin{array}{rcl} g^2 + k^2 &=& 4 \cdot c^2 \cdot (g_1 \cdot g_2 + k_1 \cdot k_2) \\ h^2 + k^2 &=& 4 \cdot c^2 \cdot k_1 \cdot k_2 \\ g \cdot h &=& 2 \cdot c^2 \cdot h_1 \cdot g_2 \\ g \cdot k &=& 2 \cdot c^2 \cdot (g_1 \cdot k_2 + k_1 \cdot g_2) \\ h \cdot k &=& 2 \cdot c^2 \cdot h_1 \cdot k_2 \end{array}\right\} \quad (54)$$

for g, h, k. A detailed investigation shows that the system (54) can only have a solution if either (case 1)

$$g_2 = k_2 = 0 \quad (55)$$

or (case 2)

$$h_1 = 0, \;\; g_1 \cdot k_2 - k_1 \cdot g_2 = 0 \quad (56)$$

or (case 3)

$$k_2^2 = g_2^2, \;\; k_1^2 = g_1^2 + h_1^2. \quad (57)$$

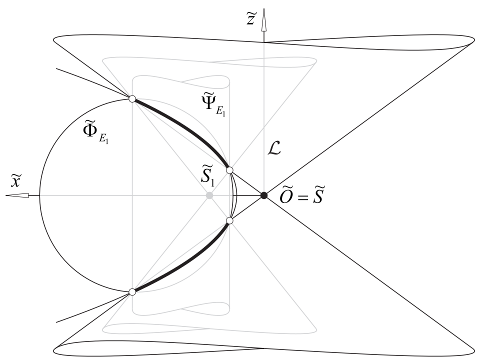

Figure 4. The situation in case 2 (orthogonal projection onto the $\tilde{x}\tilde{z}$-plane).

Case 1. $g_2 = k_2 = 0$. Due to the definition of g_2, k_2 this implies $m_1 = r_1 = 0$, which enforces the cylinder $\tilde{\Psi}_{E_1}$ to degenerate into the z-axis. Contradiction!

Case 2. $h_1 = 0$, $g_1 \cdot k_2 - k_1 \cdot g_2 = 0$. The first condition ($h_1 = n \cdot r_1 = 0$) implies $n = 0$, which means that the center of the ball $\tilde{\Phi}_{E_1}$ is on the $\tilde{x}$-axis. Using the second condition ($g_1 \cdot k_2 - k_1 \cdot g_2 = 0$) and taking into account the definition of g_1, k_1, g_2, k_2 we obtain

$$(m - m_1) \cdot (m_1^2 + r_1^2) - m_1 \cdot (r^2 - r_1^2 - (m - m_1)^2) \;=\; 0. \tag{58}$$

Moreover we are allowed to assume that $m \neq m_1$ (otherwise the center of the ball $\tilde{\Phi}_{E_1}$ would lie on the axis of the cylinder $\tilde{\Psi}_{E_1}$, which would yield circles as their intersection. Beside $\tilde{\Psi}_{E_1}$ and a parabolic cylinder the pencil of quadrics spanned by $\tilde{\Phi}_{E_1}$, $\tilde{\Psi}_{E_1}$ contains two cones of revolution[24]. Due to the symmetry of $\tilde{\Phi}_{E_1}$, $\tilde{\Psi}_{E_1}$ with respect to the $\tilde{x}\tilde{y}$-plane and to the $\tilde{x}\tilde{z}$-plane the vertices $\tilde{S}$, $\tilde{S}_1$ of these two cones lie on the $\tilde{x}$-axis (see figure 4). A short computation shows that condition (58) is equivalent to the property that one of these points, let us say $\tilde{S}$, coincides with the origin $\tilde{O}$. This implies that $\tilde{S}$ has to be a fixed point throughout the motion, as the angle between $\mathcal{L}$ and the line $\tilde{S}E_1$ can not be changed by an equiform displacement and hence this line always has to pass through $\tilde{S}$. As a conclusion, the plane through $\tilde{S}$ and perpendicular to the axis $\mathcal{L}$ is fixed, which is a contradiction to the assumption, that there is no point with a planar orbit.

Case 3. $k_2^2 = g_2^2$, $k_1^2 = g_1^2 + h_1^2$. From $m_1^2 + r_1^2 = k_2^2 = g_2^2 = 2 \cdot m_1 \cdot r_1$ we obtain

$$r_1^2 \;=\; m_1^2, \tag{59}$$

which means that the cylinder $\tilde{\Psi}_{E_1}$ contains the axis $\mathcal{L}$. The second condition ($k_1^2 = g_1^2 + h_1^2$) together with the definition of g_1, h_1, k_1 and (59) yields

$$r^2 \;=\; \left(m_1 \pm \sqrt{(m_1 - m)^2 + n^2}\right)^2, \tag{60}$$

which is equivalent to the condition that the ball $\tilde{\Phi}_{E_1}$ and the cylinder $\tilde{\Psi}_{E_1}$ are tangent to each other. This shows us that γ is a Wunderlich-Bricard motion (remark 1, (b)). □

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[24] Compare with section 2.

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