MOTION OF A LINE SEGMENT WHOSE ENDPOINT PATHS HAVE EQUAL ARC LENGTH

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Abstract. The following geometric problem originating from an engineering task is being addressed: How can you move a rod in space so that its endpoint paths have equal length? Trivial examples of motions in the Euclidean plane and in Euclidean 3-space where two points A and B have paths of equal arc length are curved translations or screw motions. In the first case all point paths are congruent by translation and in the second all points on a right cylinder coaxial with the screw motion have congruent point paths. It turns out that in the plane there exists only one non-trivial type: If A and B have paths of equal arc length the motion is generated by the rolling of a straight line, namely the bisector n of AB on an arbitrary curve. In 3-space there is a nice relation to the ruled surface Φ generated by the line AB: The path of the midpoint S of AB is the striction curve on Φ.

This is also the key to the solution to the following interpolation problem: Given a set of discrete positions A_iB_i of the segment AB find a smooth motion that moves AB through the given positions and additionally guarantees that the paths of A and B have equal arc length.

Keywords: space kinematics, line geometry, paths of equal arc length, motion of a line, ruled surface, striction curve, projection theorem

1 Introduction

We will investigate the problem of moving a rod AB via a Euclidean motion μ in a way that its endpoints A and B follow paths of equal arc length (cf. [3]). The planar and spatial cases are treated in Section 2 and 3, respectively. The main part of the paper (Section 4) is the investigation of the following interpolation problem:

Given a set of discrete positions A_iB_i of a straight line segment AB find a smooth motion of AB that interpolates the positions A_iB_i with the side condition that the paths of A and B have the same length. This will lead us to the task of constructing a ruled surface with given striction curve (cf. [1] and [2]).

In the following we always assume that all occurring functions are C^2.

2 The planar case

Let t denote the time and $\mathbf{a}(t)$ and $\mathbf{b}(t)$ be the position vectors of the endpoints A and B of a straight line segment moved in the plane. From

$$d \;:=\; \operatorname{dist}(A, B\,) \;=\; const.$$

we obtain

$$\langle \dot{\mathbf{a}}, \mathbf{b} - \mathbf{a} \rangle \quad = \quad \langle \dot{\mathbf{b}}, \mathbf{b} - \mathbf{a} \rangle$$

where "$\dot{}$" means differentiation w.r.t. time t and "$\langle ., . \rangle$" denotes the Euclidean scalar product. This means that

$$\angle(\vec{AB}, \dot{\mathbf{a}}) \quad = \quad \pm\angle(\vec{AB}, \dot{\mathbf{b}}).$$

If

$$\angle(\vec{AB}, \dot{\mathbf{a}}) \quad = \quad \angle(\vec{AB}, \dot{\mathbf{b}})$$

holds on in interval $[t_0, t_1]$ then the motion under consideration is a *curved translation*. The instantaneous pole is always at infinity and all points have paths congruent by translation (Fig. 1, left). If contrary

$$\angle(\vec{AB}, \dot{\mathbf{a}}) \quad = \quad -\angle(\vec{AB}, \dot{\mathbf{b}}) \tag{1}$$

then the pole always lies on the bisector n of AB (Fig. 1, right), which therefore has to be the moving polhode if the condition (1) holds on an interval $[t_0, t_1]$. If S denotes the midpoint of AB and s its path then μ is the motion of the Frenet frame along s (Fig. 2). The fixed polhode is the evolute s^* of s, the instantaneous pole being the center S^* of curvature of the curve s.

(a) Planar case A: The velocity vectors of A and B are identical.

(b) Planar case B: $\angle(\vec{AB}, \dot{\mathbf{a}}) = -\angle(\vec{AB}, \dot{\mathbf{b}})$.

Figure 1: The two cases occurring in the plane.

We summarize in

Proposition 1. *If two points A and B are moved w.r.t. a planar Euclidean motion μ so that their paths a and b have equal arc length then μ is either a curved translation or the motion of the Frenet frame along a curve s. In the second case A and B lie symmetric w.r.t. the normal n of s.*

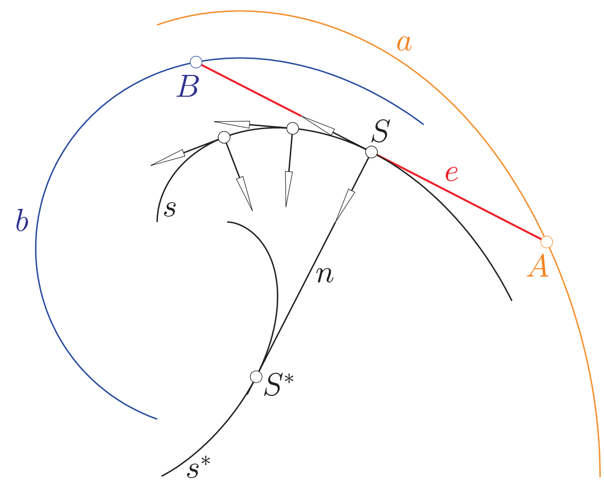

Figure 2: General planar motion where 2 points A and B have paths of equal length: The bisector n of the segment AB is the moving polhode.

3 The spatial case

Let Φ be the ruled surface generated by the straight line $e := AB$ via a Euclidean motion μ:

$$\mathbf{y}(t,u) \quad = \quad \mathbf{x}(t) + u\mathbf{e}(t)$$

Here $\mathbf{e}$ is a normalized direction vector of the the line e, i.e.,

$$\langle \mathbf{e}(t), \mathbf{e}(t) \rangle \quad \equiv \quad 1 \tag{2}$$

and t denotes the time. Then the two points A and B have position vectors $\mathbf{a}(t) = \mathbf{y}(t,a)$ and $\mathbf{b}(t) = \mathbf{y}(t, a+d)$ where $d := \mathrm{dist}(A,B)$ and a are constants. Let us moreover assume that Φ is not a cylinder, which means that $\mathbf{e}$ is not a constant vector.

From

$$|\dot{\mathbf{a}}| \quad = \quad |\dot{\mathbf{b}}|$$

we easily derive that

$$a + \frac{d}{2} \quad = \quad -\frac{\langle \dot{\mathbf{x}}, \dot{\mathbf{e}} \rangle}{\langle \dot{\mathbf{e}}, \dot{\mathbf{e}} \rangle}.$$

Hence we have

Proposition 2. *If two points A and B are moved via a spatial Euclidean motion μ so that their paths a and b have equal arc length then the midpoint S of the straight line segment AB is the* $\left\{ \begin{array}{c} \text{striction point} \\ \text{point of regression} \\ \text{vertex} \end{array} \right\}$ *on e, in case of Φ being a* $\left\{ \begin{array}{c} \text{skew ruled surface} \\ \text{tangent surface} \\ \text{cone} \end{array} \right\}$.

4 An interpolation problem

We consider the following interpolation problem in 3-space: Given a set of discrete positions A_iB_i, $i = 1, \dots, n$ of the segment AB find a smooth motion that moves AB through the given positions and additionally guarantees that the paths of A and B have equal arc length. Being aware of Proposition 2 we suggest to solve this problem in two steps:

Step 1: Determine an interpolation curve $s \dots \mathbf{s}(t)$ of the midpoint series S_i of A_iB_i, $i = 1, \dots, n$.

Step 2: Construct a ruled surface Φ that interpolates $e_i = A_iB_i$ and whose striction curve is s.

Whereas the first step is a standard task the second needs some additional considerations. Let

$$\mathbf{s} \;=\; \mathbf{s}(\tau)$$

be the arclength parametrization of s and

$$\mathbf{e} \;=\; \mathbf{e}(\tau)$$

the direction vector of the ruled surface's generator e which we have to determine. We assume that $\mathbf{e}$ is normalized:

$$\langle \mathbf{e}, \mathbf{e} \rangle \;\overset{\tau}{\equiv}\; 1 \tag{3}$$

Denoting derivatives w.r.t. the arclength τ of s by $'$, $''$, ... and introducing the striction

$$\sigma \;:=\; \angle(\mathbf{s}', \mathbf{e})$$

of Φ we have

$$\langle \mathbf{s}', \mathbf{e} \rangle \;=\; \cos\sigma \tag{4}$$

Moreover,

$$\langle \mathbf{s}', \mathbf{e}' \rangle \;=\; 0,$$

because s is the striction line on Φ. Thus, differentiating (4) we obtain

$$\langle \mathbf{s}'', \mathbf{e} \rangle \;=\; -\sigma' \cdot \sin\sigma. \tag{5}$$

Let $\varkappa$ be the curvature and $\{\mathbf{t} = \mathbf{s}', \mathbf{h} = \frac{1}{\varkappa}\mathbf{s}'', \mathbf{b} = \mathbf{t} \times \mathbf{h}\}$ denote the Frenet frame of s. Then (4), (5) can be rewritten as

$$\langle \mathbf{t}, \mathbf{e} \rangle \;=\; \cos\sigma, \tag{6}$$

$$\langle \mathbf{h}, \mathbf{e} \rangle \;=\; -\frac{\sigma' \cdot \sin\sigma}{\varkappa} \tag{7}$$

which together with (3) yields

$$\mathbf{e} \;=\; \cos\sigma \cdot \mathbf{t} \;-\; \frac{\sigma' \cdot \sin\sigma}{\varkappa} \cdot \mathbf{h} \;\pm\; \sin\sigma \sqrt{1 - \frac{\sigma'^2}{\varkappa^2}} \cdot \mathbf{b}. \tag{8}$$

We give a geometric interpretation of the formulæ above (Fig. 3). Considering $\mathbf{e}$ as unknown position vector of a point, Eq. (4) represents a plane ε with normal vector $\mathbf{s}'$ and distance $|\cos\sigma|$ from the origin. For running τ we obtain a one parametric set of such planes. The envelope of these planes is a developable surface Ψ whose equation can be determined by eliminating τ from the two equations Eq. (4) and Eq. (5). The latter represents another plane ε_1 perpendicular to ε. In order to find suitable vectors $\mathbf{e}$ we have to intersect the generators $g = \varepsilon \cap \varepsilon_1$ of Ψ with the unit sphere represented by Eq. (3):

The spherical generator image of Φ lies in the intersection of the developable surface Ψ and the unit sphere.

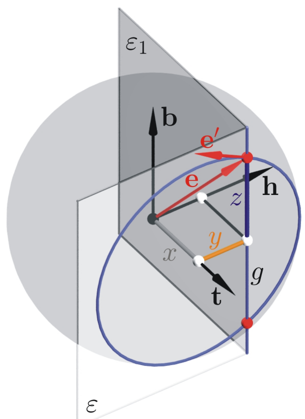

Figure 3: Spherical image of a generator e

Making use of this we can now tackle Step 2 by constructing a function $\sigma = \sigma(\tau)$ which fulfills

$$\sigma(\tau_i) = \arccos\langle \mathbf{s}'(\tau_i), \mathbf{e}_i\rangle, \tag{9}$$

$$\sigma'(\tau_i) = -\frac{\langle \mathbf{s}''(\tau_i), \mathbf{e}_i\rangle}{\sin\sigma(\tau_i)}, \tag{10}$$

$$\sigma'^2(\tau) \leq \varkappa^2(\tau). \tag{11}$$

Here τ_i is the arc length parameter value belonging to the midpoint S_i of the given segment A_iB_i, $i = 1, \ldots, n$ and $\mathbf{e}_i := \frac{\overrightarrow{A_iB_i}}{|\overrightarrow{A_iB_i}|}$. After having fixed the function $\sigma = \sigma(\tau)$ the direction vector $\mathbf{e} = \mathbf{e}(\tau)$ is determined via Eq. (8).

The ruled surface Φ in Fig. 4 was constructed by the method outlined above. In this example four generators $e_i = A_iB_i$, $i = 1, 2, 3, 4$ were given. The striction curve s was then constructed as interpolant of the midpoints S_1, S_2, S_3, S_4 (Step 1) and reparametrized w.r.t. arclength. Afterwards a suitable striction function $\sigma = \sigma(\tau)$ was constructed (Step 2) as Hermite interpolant fulfilling the conditions (9), (10) and (11).

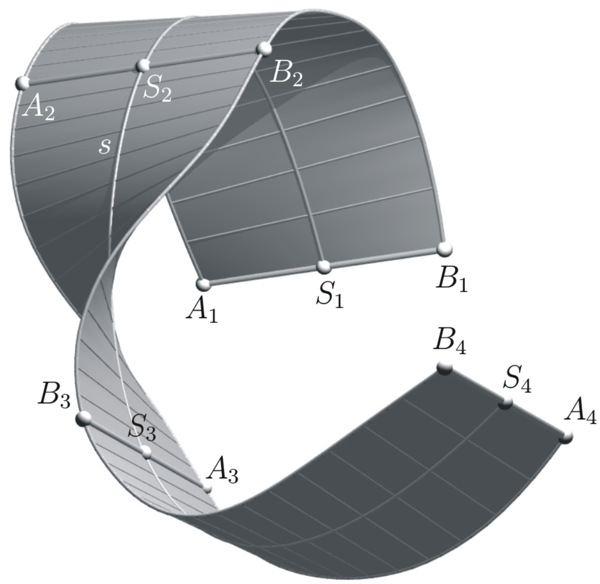

Figure 4: Ruled surface Φ interpolating the segments A_iB_i; the endpoints A and B are symmetric w.r.t. the striction curve s and run on curves of equal length.

Remarks:

(a) As condition (3) is quadratical the proposed method can fail if the sign chosen in front of the square root in (8) differs for the prescribed generators e_i, $i = 1, \ldots, n$.

(b) Eq. (8) can already be found in [1] where it is derived in another way.

(c) In [2] a method to construct ruled surfaces Φ from a given striction curve $s \ldots \mathbf{s} = \mathbf{s}(t)$ is suggested: As the generators of a ruled surface are geodesically parallel along the striction curve one can take any developable surface Δ through s, develop it into a plane π, then choose an arbitrary direction in π and draw the lines $g(t)$ parallel to this direction. Bringing these lines back into space by means of the inverse developing mapping one gets the generators of a solution surface Φ. This method is not appropriate to solve the task in Step 2 as we are given a set of prescribed generators $e_i = A_iB_i$, $i = 1, \ldots, n$.

References

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