Abstract. The following geometric problem originating from an engineering task is being addressed: How can you move a rod in space so that its endpoint paths have equal length? Trivial examples of motions in the Euclidean plane and in Euclidean 3-space where two points $A$ and $B$ have paths of equal arc length are curved translations or screw motions. In the first case all point paths are congruent by translation and in the second all points on a right cylinder coaxial with the screw motion have congruent point paths. It turns out that in the plane there exists only one non-trivial type: If $A$ and $B$ have paths of equal arc length the motion is generated by the rolling of a straight line, namely the bisector $n$ of $AB$ on an arbitrary curve. In 3-space there is a nice relation to the ruled surface $\Phi$ generated by the line $AB$: The path of the midpoint $S$ of $AB$ is the striction curve on $\Phi$.

This is also the key to the solution to the following interpolation problem: Given a set of discrete positions $A_iB_i$ of a straight line segment $AB$ find a smooth motion that moves $AB$ through the given positions and additionally guarantees that the paths of $A$ and $B$ have equal arc length.

Keywords: space kinematics, line geometry, paths of equal arc length, motion of a line, ruled surface, striction curve, projection theorem

1 Introduction

We will investigate the problem of moving a rod $AB$ via a Euclidean motion $\mu$ in a way that its endpoints $A$ and $B$ follow paths of equal arc length (cf. [3]). The planar and spatial cases are treated in Section 2 and 3, respectively. The main part of the paper (Section 4) is the investigation of the following interpolation problem:

Given a set of discrete positions $A_iB_i$ of a straight line segment $AB$ find a smooth motion of $AB$ that interpolates the positions $A_iB_i$ with the side condition that the paths of $A$ and $B$ have the same length. This will lead us to the task of constructing a ruled surface with given striction curve (cf. [1] and [2]).

In the following we always assume that all occurring functions are $C^2$.

2 The planar case

Let $t$ denote the time and $a(t)$ and $b(t)$ be the position vectors of the endpoints $A$ and $B$ of a straight line segment moved in the plane. From

\[ d := \text{dist}(A, B) = \text{const}. \]
we obtain
\[ \langle \dot{a}, b - a \rangle = \langle \dot{b}, b - a \rangle \]

where "\( \dot{\cdot} \)" means differentiation w.r.t. time \( t \) and "\( \langle \cdot, \cdot \rangle \)" denotes the Euclidean scalar product. This means that
\[ \angle(\vec{AB}, \dot{a}) = \pm \angle(\vec{AB}, \dot{b}). \]

If
\[ \angle(\vec{AB}, \dot{a}) = \angle(\vec{AB}, \dot{b}) \]

holds on in interval \([t_0, t_1]\) then the motion under consideration is a \textit{curved translation}. The instantaneous pole is always at infinity and all points have paths congruent by translation (Fig. 1, left). If contrary
\[ \angle(\vec{AB}, \dot{a}) = -\angle(\vec{AB}, \dot{b}) \quad (1) \]

then the pole always lies on the bisector \( n \) of \( \vec{AB} \) (Fig. 1, right), which therefore has to be the moving polhode if the condition (1) holds on an interval \([t_0, t_1]\). If \( S \) denotes the midpoint of \( \vec{AB} \) and \( s \) its path then \( \mu \) is the motion of the Frenet frame along \( s \) (Fig. 2). The fixed polhode is the evolute \( s^* \) of \( s \), the instantaneous pole being the center \( S^* \) of curvature of the curve \( s \).

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{figure1}
\caption{(a) Planar case A: The velocity vectors of \( A \) and \( B \) are identical. (b) Planar case B: \( \angle(\vec{AB}, \dot{a}) = -\angle(\vec{AB}, \dot{b}) \).}
\end{figure}

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{figure2}
\caption{Figure 1: The two cases occurring in the plane.}
\end{figure}

We summarize in

\textbf{Proposition 1.} If two points \( A \) and \( B \) are moved w.r.t. a planar Euclidean motion \( \mu \) so that their paths \( a \) and \( b \) have equal arc length then \( \mu \) is either a curved translation or the motion of the Frenet frame along a curve \( s \). In the second case \( A \) and \( B \) lie symmetric w.r.t. the normal \( n \) of \( s \).
3 The spatial case

Let $\Phi$ be the ruled surface generated by the straight line $e := AB$ via a Euclidean motion $\mu$:

$$\mathbf{y}(t, u) = \mathbf{x}(t) + u\mathbf{e}(t)$$

Here $\mathbf{e}$ is a normalized direction vector of the the line $e$, i.e.,

$$\langle \mathbf{e}(t), \mathbf{e}(t) \rangle \equiv 1 \quad (2)$$

and $t$ denotes the time. Then the two points $A$ and $B$ have position vectors $\mathbf{a}(t) = \mathbf{y}(t, a)$ and $\mathbf{b}(t) = \mathbf{y}(t, a + d)$ where $d := \text{dist}(A, B)$ and $a$ are constants. Let us moreover assume that $\Phi$ is not a cylinder, which means that $\mathbf{e}$ is not a constant vector.

From

$$|\dot{\mathbf{a}}| = |\dot{\mathbf{b}}|$$

we easily derive that

$$a + \frac{d}{2} = -\frac{\langle \dot{\mathbf{x}}, \dot{\mathbf{e}} \rangle}{\langle \dot{\mathbf{e}}, \dot{\mathbf{e}} \rangle}.$$

Hence we have

**Proposition 2.** If two points $A$ and $B$ are moved via a spatial Euclidean motion $\mu$ so that their paths $a$ and $b$ have equal arc length then the midpoint $S$ of the straight line segment $AB$ is the

\[
\begin{cases}
\text{striction point} \\
\text{point of regression} \\
\text{vertex}
\end{cases}
\quad \text{on } e, \text{ in case of } \Phi \text{ being a }
\begin{cases}
\text{skew ruled surface} \\
\text{tangent surface} \\
\text{cone}
\end{cases}
\]
4 An interpolation problem

We consider the following interpolation problem in 3-space: Given a set of discrete positions $A_i B_i$, $i = 1, \ldots, n$ of the segment $AB$ find a smooth motion that moves $AB$ through the given positions and additionally guarantees that the paths of $A$ and $B$ have equal arc length. Being aware of Proposition 2 we suggest to solve this problem in two steps:

**Step 1:** Determine an interpolation curve $s \ldots s(t)$ of the midpoint series $S_i$ of $A_i B_i$, $i = 1, \ldots, n$.

**Step 2:** Construct a ruled surface $\Phi$ that interpolates $e_i = A_i B_i$ and whose striction curve is $s$.

Whereas the first step is a standard task the second needs some additional considerations. Let

$$s = s(\tau)$$

be the arclength parametrization of $s$ and

$$e = e(\tau)$$

the direction vector of the ruled surface’s generator $e$ which we have to determine. We assume that $e$ is normalized:

$$\langle e, e \rangle = 1$$  \hspace{1cm} (3)

Denoting derivatives w.r.t. the arclength $\tau$ of $s$ by $'$, $''$, ... and introducing the striction

$$\sigma := \langle s', e \rangle$$

of $\Phi$ we have

$$\langle s', e \rangle = \cos \sigma$$  \hspace{1cm} (4)

Moreover,

$$\langle s', e' \rangle = 0,$$

because $s$ is the striction line on $\Phi$. Thus, differentiating (4) we obtain

$$\langle s'', e \rangle = -\sigma' \cdot \sin \sigma.$$  \hspace{1cm} (5)

Let $\kappa$ be the curvature and \{t = s', h = $\frac{1}{\kappa}s''$, b = t × h\} denote the Frenet frame of $s$. Then (4), (5) can be rewritten as

$$\langle t, e \rangle = \cos \sigma,$$  \hspace{1cm} (6)

$$\langle h, e \rangle = -\frac{\sigma' \cdot \sin \sigma}{\kappa}.$$  \hspace{1cm} (7)

which together with (3) yields

$$e = \cos \sigma \cdot t - \frac{\sigma' \cdot \sin \sigma}{\kappa} \cdot h \pm \sin \sigma \sqrt{1 - \frac{\sigma'^2}{\kappa^2}} \cdot b.$$  \hspace{1cm} (8)
We give a geometric interpretation of the formulae above (Fig. 3). Considering \( e \) as unknown position vector of a point, Eq. (4) represents a plane \( \varepsilon \) with normal vector \( s' \) and distance \( |\cos \sigma| \) from the origin. For running \( \tau \) we obtain a one parametric set of such planes. The envelope of these planes is a developable surface \( \Psi \) whose equation can be determined by eliminating \( \tau \) from the two equations Eq. (4) and Eq. (5). The latter represents another plane \( \varepsilon_1 \) perpendicular to \( \varepsilon \). In order to find suitable vectors \( e \) we have to intersect the generators \( g = \varepsilon \cap \varepsilon_1 \) of \( \Psi \) with the unit sphere represented by Eq. (3):

The spherical generator image of \( \Phi \) lies in the intersection of the developable surface \( \Psi \) and the unit sphere.

[Diagram: Figure 3: Spherical image of a generator \( e \)]

Making use of this we can now tackle Step 2 by constructing a function \( \sigma = \sigma(\tau) \) which fulfills

\[
\begin{align*}
\sigma(\tau_i) & = \arccos\langle s'(\tau_i), e_i \rangle, \\
\sigma'(\tau_i) & = -\frac{\langle s''(\tau_i), e_i \rangle}{\sin \sigma(\tau_i)}, \\
\sigma'^2(\tau) & \leq \kappa^2(\tau).
\end{align*}
\]

Here \( \tau_i \) is the arc length parameter value belonging to the midpoint \( S_i \) of the given segment \( A_iB_i, \ i = 1, \ldots, n \) and \( e_i := \frac{A_iB_i}{|A_iB_i|} \). After having fixed the function \( \sigma = \sigma(\tau) \) the direction vector \( e = e(\tau) \) is determined via Eq. (8).

The ruled surface \( \Phi \) in Fig. 4 was constructed by the method outlined above. In this example four generators \( e_i = A_iB_i, \ i = 1, 2, 3, 4 \) were given. The striction curve \( s \) was then constructed as interpolant of the midpoints \( S_1, S_2, S_3, S_4 \) (Step 1) and reparametrized w.r.t. arclength. Afterwards a suitable striction function \( \sigma = \sigma(\tau) \) was constructed (Step 2) as Hermite interpolant fulfilling the conditions (9), (10) and (11).
Figure 4: Ruled surface $\Phi$ interpolating the segments $A_iB_i$; the endpoints $A$ and $B$ are symmetric w.r.t. the striction curve $s$ and run on curves of equal length.

Remarks:

(a) As condition (3) is quadratical the proposed method can fail if the sign chosen in front of the square root in (8) differs for the prescribed generators $e_i, i = 1, \ldots, n$.

(b) Eq. (8) can already be found in [1] where it is derived in another way.

(c) In [2] a method to construct ruled surfaces $\Phi$ from a given striction curve $s \ldots s = s(t)$ is suggested: As the generators of a ruled surface are geodesically parallel along the striction curve one can take any developable surface $\Delta$ through $s$, develop it into a plane $\pi$, then choose an arbitrary direction in $\pi$ and draw the lines $g(t)$ parallel to this direction. Bringing these lines back into space by means of the inverse developing mapping one gets the generators of a solution surface $\Phi$. This method is not appropriate to solve the task in Step 2 as we are given a set of prescribed generators $e_i = A_iB_i, i = 1, \ldots, n$.

References

