Geometry and Kinematics of the Mecanum Wheel

A. Gfrerrer

Graz University of Technology, Institute of Geometry, Kopernikusgasse 24, 8010 Graz, Austria

Abstract

Mecanum wheels are used when omnidirectional movability of a vehicle is desired. That means that the vehicle can move along a prescribed path and at the same time rotate arbitrarily around its center. A Mecanum wheel consists of a set of rolls arranged around the wheel axis. In this paper we describe in detail the geometry of these rolls. We derive simple canonical parameterizations of the roll generating curve and the roll surface itself. These parametric representations reveal the geometry of the roll. With their help we can easily find an approximation of the roll surface by a torus for manufacture purposes. Based on the roll parametrization we study the kinematics of a vehicle featured with Mecanum wheels.

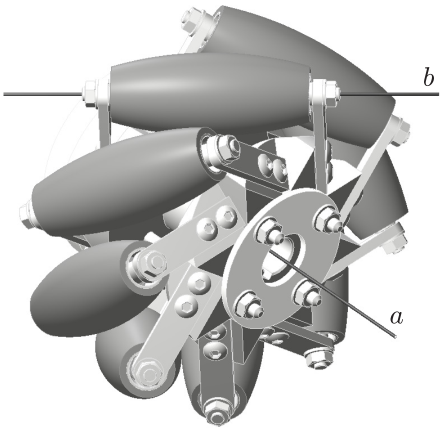
Key words: Mecanum wheel, forward kinematics, inverse kinematics

1 Introduction

The Mecanum wheel (Fig. 1, left) was invented by the Swedish engineer Bengt Ilon in 1973. [1] It consists of a set of k congruent rolls placed symmetrically around the wheel body. The face of each roll is part of a surface of revolution $\mathcal{R}$ whose axis b is skew to the wheel axis a. Usually an angle δ between a and b of $\pm 45°$ is chosen. Fig. 1, right, shows (the setup of) a mobile robot furnished with three wheels of that kind. Each of them is driven by a separate motor which gives the vehicle the three degrees of freedom necessary for an omnidirectional movement on level ground. The advantage of this architecture is that none of the wheels needs to be steerable. The wheel rolls rotate passively around their axes.

Email address: gfrerrer@tugraz.at (A. Gfrerrer).

[1] See the US patent 3,876,255 [Ilon (1975)].



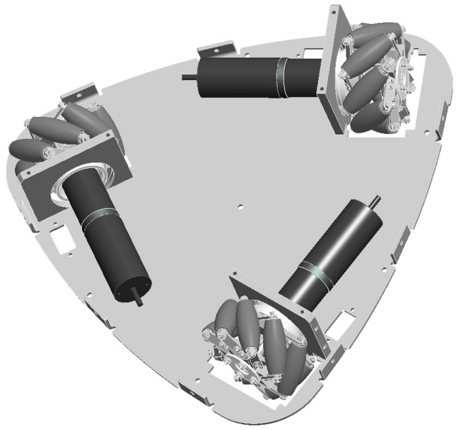


Fig. 1. Left: Mecanum wheel Right: Vehicle with 3 Mecanum wheels

The parametrization in [Dickerson & Lapin (1991)] of the roll generating curve is rather involved and does not reveal the geometry of the roll. With the help of Descriptive Geometry we derive a pretty natural parametrization of this curve which also yields simple parametric representations of the roll surface and its meridian (section 2). In section 3 we use these parametrizations to replace the roll by an approximating torus surface. Moreover, we derive the exact velocity equation for a kinematic system with a Mecanum wheel (section 4, Eq. (15)). With *"exact"* we mean that the position of the contact point C of the roll and the terrain is also taken into account. In the literature on the kinematics of Mecanum wheels it is (as a simplification) always assumed that C at any moment lies exactly beneath the wheel center (cf. for example with [Viboonchaicheep et al. (2003)] or [Siegwart & Nourbakhsh (2004), page 59]). Using this simplification we finally study the case of a vehicle supplied with three Mecanum wheels (section 4.1) and give a nice geometric characterization for the solvability of the forward kinematics of a such a robot.

2 Roll geometry of the Mecanum wheel

The roll axes of a Mecanum wheel establish a set of k equidistant generators belonging to a regulus on a one-sheet hyperboloid $\mathcal{H}$ of revolution with axis a. If the wheel moves on a plane terrain π its axis a remaining parallel to π then at each moment at least one roll touches the ground. Hereby a passive (non driven) rotation around the roll axis b is induced to the respective roll by the motion. Of course, it is desired to avoid vibration or jiggling of the vehicle throughout the motion, which means that the wheel axis a must keep a constant distance to the plane π:

$$dist(\pi, a) = r = const \tag{1}$$

Hence, the question is how to construct the roll surface $\mathcal{R}$ so that condition (1) is fulfilled. Fig. 2, left, shows the situation in ground view (first projection) and corresponding front view (second projection)[2] both, the wheel axis a and the roll axis b, being parallel to the first projection plane. The rays for the second projection are parallel to a, i.e., the second image a'' of a is a point.

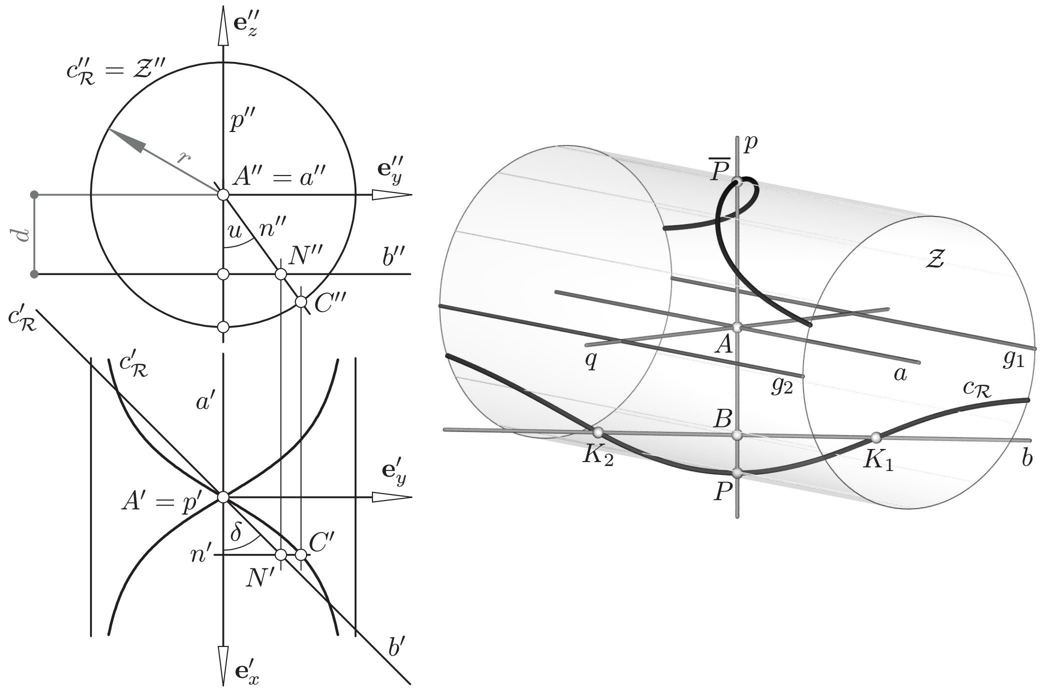


Fig. 2. Curve $c_{\mathcal{R}}$ generating the rolls

Geometrically condition (1) means that the curve $c_{\mathcal{R}}$ that generates the roll surface $\mathcal{R}$ has to be a part of the cylinder $\mathcal{Z}$ of revolution with axis a and radius r. This curve is the locus of contact points of $\mathcal{R}$ with the plane π. The roll $\mathcal{R}$ and the cylinder $\mathcal{Z}$ are tangent to each other along $c_{\mathcal{R}}$.

If $C'' \in c_{\mathcal{R}}''$ is the second image of a point $C \in c_{\mathcal{R}}$ we can easily construct its first image C':

- Let n denote the surface normal of $\mathcal{R}$ running through C. Since the circle $c_{\mathcal{R}}''$ is the second silhouette of $\mathcal{R}$ n'' is the diameter of $c_{\mathcal{R}}''$ containing C''.
- Because n is a surface normal in a contour point w.r.t. the second projection it lies parallel to the plane of this projection and hence its first projection n' is a horizontal line.
- As $\mathcal{R}$ is a surface of revolution n has to intersect b in a point N. In this way, the first image n' of n and with it the first image C' of C is fixed.

The above construction also yields a simple parametrization of $c_{\mathcal{R}}$. Let p denote the common perpendicular of a and b and let A be the intersection point of

[2] We mark the first (second) image of an object by one (two) prime(s).

p and a, i.e., the wheel center. We denote the distance and angle of a and b by d and δ and introduce a coordinate system $S := \{A; \mathbf{e}_x, \mathbf{e}_y, \mathbf{e}_z\}$ whose first and third unit vector $\mathbf{e}_x$ and $\mathbf{e}_z$ is on a and p, respectively. As parameter we use the angle u between p and n. With the help of Fig. 2 we derive

$$\mathbf{x}(u) = \begin{pmatrix} x(u) \\ y(u) \\ z(u) \end{pmatrix} = \begin{pmatrix} d\ \cot\delta\ \tan u \\ r \sin u \\ -r\cos u \end{pmatrix} \tag{2}$$

as parametrization of $c_{\mathcal{R}}$. Since the two axes a and b are skew $\cot\delta \neq \infty$ is guaranteed.

Eq. (2) tells us that $c_{\mathcal{R}}$ in general is a rational 4th order space curve. This follows for instance by re-parameterizing $c_{\mathcal{R}}$ via $\tau = \tan\frac{u}{2}$.

Only in case of $b \perp a$ ($\delta = \pm\frac{\pi}{2}$) $c_{\mathcal{R}}$ is an ordinary circle with radius r. Wheels of that type are often called *"Swedish wheels"* in the literature.

Fig. 2, right, gives an impression of the curve $c_{\mathcal{R}}$ which consists of two branches. It has the axis a of the cylinder $\mathcal{Z}$ and the common perpendicular p of a and b as symmetry axes. Thus, the common perpendicular q of a and p is also a symmetry axis of $c_{\mathcal{R}}$. The curve intersects p in the points $P(0,0,-r)$, $\overline{P}(0,0,r)$ and if $\delta \neq \pm\frac{\pi}{2}$ it has the generators $g_{1,2} \ldots z = 0, y = \pm r$ of $\mathcal{Z}$ as asymptotes. The axis b and $c_{\mathcal{R}}$ meet in the common points $K_{1,2}(\pm\sqrt{r^2-d^2}\cot\delta, \pm\sqrt{r^2-d^2}, -d)$ of b and $\mathcal{Z}$.

Since the curve $c_{\mathcal{R}}$ consists of two parts in case of $\delta \neq \pm\frac{\pi}{2}$, the same is true for the roll surface $\mathcal{R}$. Fig. 3 shows the part of $\mathcal{R}$ generated by the branch of $c_{\mathcal{R}}$ running through P. The roll $\mathcal{R}$ is a surface of revolution with axis b and a symmetry plane σ through p and orthogonal to b. The common point B of b and p is the symmetry center of $\mathcal{R}$. The circle $e_P \subset \sigma$ through P and centered in B is an equator on $\mathcal{R}$, that means it has locally maximal radius, namely $r - d$. The two points $K_{1,2}$ are conical knots on $\mathcal{R}$.

Remark: The surface normals n used in the construction above intersect the lines a, b and the line at infinity of the yz-plane. Hence, if $\delta \neq \pm\frac{\pi}{2}$ they establish a generator set on a hyperbolic paraboloid $\mathcal{P}$. As one can easily check $\mathcal{P}$ has the equation

$$x\ z + d\ \cot\delta\ y = 0.$$

The y-axis of the coordinate system is the axis of $\mathcal{P}$ and A is its vertex. The second generator set on $\mathcal{P}$ consists of lines parallel to the xy-plane. Thus, $\mathcal{P}$

intersects the plane at infinity in the two lines at infinity of the xy- and yz-plane. The curve $c_{\mathcal{R}}$ is the intersection curve of $\mathcal{P}$ and the cylinder $\mathcal{Z}$. From this fact, we can see again (in a purely geometric way) that $c_{\mathcal{R}}$ is a rational fourth order curve with the asymptotes described above. The singular point of $c_{\mathcal{R}}$ is the point X_∞ at infinity of the x-axis since this point is the vertex of the cylinder $\mathcal{R}$ and at the same time lies on the paraboloid $\mathcal{P}$. The tangent plane τ of $\mathcal{P}$ in X_∞ is the xy-plane and intersects the cylinder in the two tangents $g_{1,2}$ of $c_{\mathcal{R}}$ in its singular point X_∞.

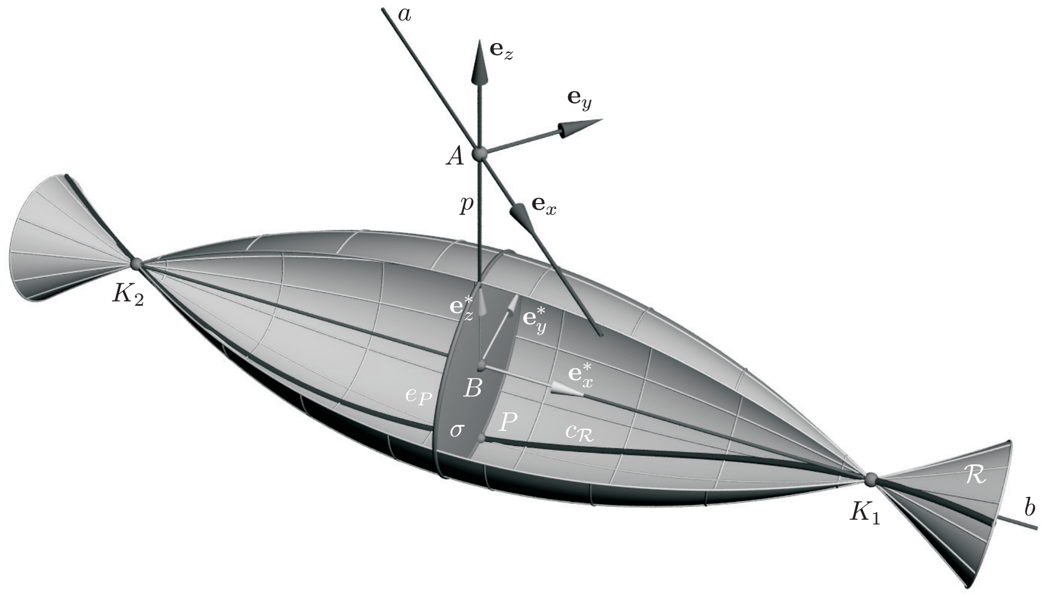


Fig. 3. The roll surface $\mathcal{R}$

As a surface of revolution generated by a rational curve $c_{\mathcal{R}}$ $\mathcal{R}$ itself is also rational. If $\delta \neq \pm\frac{\pi}{2}$ the algebraic order of $\mathcal{R}$ is 8, i.e., twice the order of its generating curve $c_{\mathcal{R}}$. In the special case of $\delta = \pm\frac{\pi}{2}$ the roll surface $\mathcal{R}$ is a torus whose meridian circle $c_{\mathcal{R}}$ intersects its axis b in $K_{1,2}$.

Of course, for the physical roll of the Mecanum wheel only a certain part of $\mathcal{R}$ lying between the knots $K_{1,2}$ is taken.

To obtain a suitable representation of the roll surface $\mathcal{R}$ we use a new coordinate system $S^* := \{B; \mathbf{e}^*_x, \mathbf{e}^*_y, \mathbf{e}^*_z\}$ with origin in B, x-axis $x^* = b$ and the new z-axis coincident with the old one (Fig. 3). W.r.t. this system the curve $c_{\mathcal{R}}$ has the parametrization

$$\mathbf{x}^*(u) = \begin{pmatrix} d\,\frac{\cos^2\delta}{\sin\delta}\,\tan u + r\,\sin\delta\,\sin u \\ \cos\delta\,\tan u\,(r\,\cos u - d) \\ d - r\,\cos u \end{pmatrix}. \tag{3}$$

By rotation around the x^*-axis with angle v we find

$$\mathbf{y}^*(u,v) = \begin{pmatrix} x^*(u,v) \\ y^*(u,v) \\ z^*(u,v) \end{pmatrix} = \begin{pmatrix} d\,\frac{\cos^2\delta}{\sin\delta}\ \tan u + r\ \sin\delta\ \sin u \\ (r\ \cos u - d)(\cos\delta\ \tan u\ \cos v + \sin v) \\ (r\ \cos u - d)(\cos\delta\ \tan u\ \sin v - \cos v) \end{pmatrix} \tag{4}$$

as parametrization of $\mathcal{R}$.

Putting $y^* = 0$ we obtain $\tan v = -\cos\delta\ \tan u$ which after substitution into the third line of (4) yields together with the first line a parametrization of the meridian curve $m_{\mathcal{R}}$ of the roll that lies in the x^*z^*-plane (parameter u):

$$m_{\mathcal{R}} \ldots \left\{ \begin{array}{l} x^*(u) = d\,\frac{\cos^2\delta}{\sin\delta}\ \tan u + r\ \sin\delta\ \sin u \\ z^*(u) = -\sqrt{\cos^2\delta\ \tan^2 u + 1}\ (r\ \cos u - d) \end{array} \right\} \tag{5}$$

More accurately: (5) is the parametrization of **one** of the two branches of the meridian curve; the other branch is symmetric to the first one w.r.t. the axis $b = x^*$ of revolution and one gets its parametrization by changing the sign in front of the square root.

3 Approximation of the roll by a torus

As we have seen in the previous section the roll surface $\mathcal{R}$ of a Mecanum wheel is algebraic of order 8 generated by a fourth order space curve $c_{\mathcal{R}}$. The natural parametrizations (Eq. 4) of the roll surface and its meridian curve (Eq. 5) can be used for manufacturing the roll precisely. But since the rolls usually have a flexible rubber coat it is sufficient to use a less complicated surface which approximates $\mathcal{R}$ sufficiently accurate. For instance, one could approximate the meridian curve by a suitable conic section or a rational freeform curve. As an example, we will construct an approximating torus surface $\mathcal{T}$ for the roll $\mathcal{R}$.

Problem 1 *Construct a torus $\mathcal{T}$ with axis b so that $\mathcal{T}$ and the roll surface $\mathcal{R}$ have contact of order 2 along the equator circle e_P.*

Due to the symmetry with respect to the plane σ of e_P the center of the wanted torus $\mathcal{T}$ must be the point B. Fig. 4, left, shows the situation in the x^*z^*-plane: One of the two meridian circles of $\mathcal{R}$ in this plane has to osculate the roll meridian $m_{\mathcal{R}}$ (Eq. (5)) at P. Let us denote this meridian circle by $m_{\mathcal{T}}$.

Vice versa, if $m_{\mathcal{R}}$ and $m_{\mathcal{T}}$ have contact of order k in P than the same is true for the generated surfaces $\mathcal{R}$ and $\mathcal{T}$ along e_P.

The equation of $m_{\mathcal{T}}$ can be set up as

$$F(x^*, z^*) := x^{*2} + (z^* - r_l)^2 - r_m^2 = 0. \tag{6}$$

Here r_l and r_m denote the yet unknown radii of the center circle l and of the torus meridian circle $m_{\mathcal{T}}$.

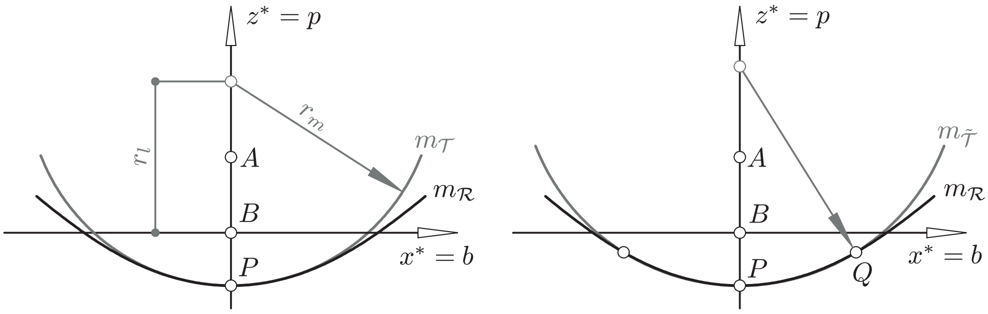


Fig. 4. Two ways to construct an approximating torus

By substitution of (5) into (6) we obtain the function

$$\begin{aligned} f(u) := {} & \sin^2 u \, \left(d \, \frac{\cos^2 \delta}{\sin \delta \, \cos u} + r \, \sin \delta \right)^2 \\ & + \left(\sqrt{\cos^2 \delta \, \tan^2 u + 1} \, (r \, \cos u - d) + r_l \right)^2 - r_m^2 \end{aligned} \tag{7}$$

in u. Since both of the curves $m_{\mathcal{R}}$ and $m_{\mathcal{T}}$ are symmetric w.r.t. z^* f is an even function. Hence, all derivatives of odd order vanish at $u = 0$:

$$\left. \frac{\partial f}{\partial u} \right|_{u=0} = \left. \frac{\partial^3 f}{(\partial u)^3} \right|_{u=0} = \ldots = 0$$

This is true for arbitrary values of r_l, r_m.

Now we determine r_l and r_m so that the two additional conditions

$$f(0) = (r_l - d + r)^2 - r_m^2 = 0,$$

$$\left. \frac{\partial^2 f}{(\partial u)^2} \right|_{u=0} = 2 \frac{(r \, \sin^2 \delta + d \, \cos^2 \delta)(d - r_l \, \sin^2 \delta)}{\sin^2 \delta} = 0$$

are fulfilled. The solution is

$$r_l = \frac{d}{\sin^2\delta}, \tag{8}$$
$$r_m = r + d\cot^2\delta. \tag{9}$$

For these values of r_l and r_m all derivatives of f up to order 3 are zero at $u = 0$. Therefore, the roll meridian curve $m_{\mathcal{R}}$ and the meridian circle $m_{\mathcal{T}}$ of the torus have contact of order 3 in the point P. The same is true for the generated surfaces of revolution along their common equator e_P:

Theorem 1 *The roll surface $\mathcal{R}$ and the coaxial torus $\mathcal{T}$ with center circle $l \subset \sigma$ (center B, radius $r_l = \frac{d}{\sin^2\delta}$) and meridian circle radius $r_m = r + d\cot^2\delta$ have contact of order 3 along their common equator circle e_P.*

Exactly in case of Swedish wheels ($\delta = \pm\frac{\pi}{2}$) the torus surface $\mathcal{T}$ and the roll surface $\mathcal{R}$ are identical.

If especially $\delta = \pm\frac{\pi}{4}$ (the case that mainly occurs in praxis) the radii of the torus are

$$r_l = 2d, \quad r_m = r + d.$$

Theorem 1 says that close to their common equator e_P the torus $\mathcal{T}$ approximates the roll surface $\mathcal{R}$ well. Fig. 4, left, shows both roll meridian $m_{\mathcal{R}}$ and torus meridian circle $m_{\mathcal{T}}$. On the other hand this figure also reveals that at some distance from P the approximation is not satisfying. So, if a Mecanum wheel is supplied with rolls of bigger length it may be advantageous to use a torus $\tilde{\mathcal{T}}$ for the approximation whose meridian circle $m_{\tilde{\mathcal{T}}}$ is tangent to $m_{\mathcal{R}}$ at P and additionally contains another point Q of $m_{\mathcal{R}}$ (Fig. 4, right). The point Q can be computed with the help of (5). The approximation order of $\tilde{\mathcal{T}}$ along e_P is only C^1 but at the outside regions one obtains a better approximation.

4 Kinematics of the Mecanum wheel

We consider a vehicle moving on level ground and furnished with Mecanum wheels like the one in (Fig. 1, right). Let us analyze the situation for one of the wheels at a certain moment t (Fig. 5). Four systems are involved: the terrain Σ_0, the vehicle Σ_1, the wheel Σ_2 and the roll Σ_3 which at that moment touches the ground at a certain point C (contact point). Note that this point always lies beneath the axis a of the wheel Σ_2: It is the intersection point of the orthogonal projections of the wheel axis a and the roll axis b in Σ_0. Only in case of b being in a horizontal position C lies beneath the wheel center A!

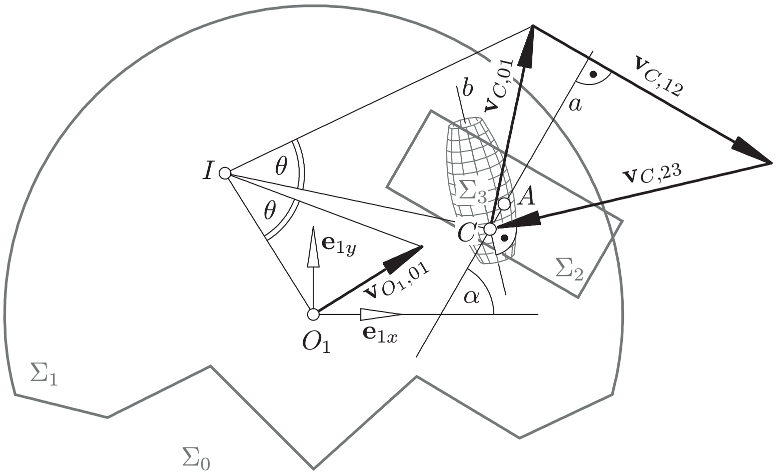


Fig. 5. Velocities for a vehicle with Mecanum wheels

For the analytical description we choose an arbitrary point O_1 (*"vehicle center"*) in Σ_1 as origin of a coordinate system $S_1 := \{O_1; \mathbf{e}_{1x}, \mathbf{e}_{1y}, \mathbf{e}_{1z}\}$ connected with the vehicle Σ_1, the x- and y-axis being parallel to the ground. The wheel center A may have x- and y-coordinates a_x and a_y w.r.t. S_1 and α may denote the angle between $\mathbf{e}_{1x}$ and the wheel axis a. Then

$$\mathbf{a} = \begin{pmatrix} \cos\alpha \\ \sin\alpha \\ 0 \end{pmatrix}$$

is the direction vector of a. The direction vector $\mathbf{b}$ of the roll axis depends on the rotation angle u of the wheel as follows:

$$\mathbf{b} = \begin{pmatrix} \cos\alpha\ \cos\delta - \sin\alpha\ \sin\delta\ \cos u \\ \sin\alpha\ \cos\delta + \cos\alpha\ \sin\delta\ \cos u \\ \sin\delta\ \sin u \end{pmatrix} =: \begin{pmatrix} b_x \\ b_y \\ b_z \end{pmatrix} \tag{10}$$

W.r.t. S_1 the contact point C has the x- and y-coordinates

$$\left.\begin{aligned} c_x &= a_x - d\ \cos\alpha\ \cot\delta\ \tan u \\ c_y &= a_y - d\ \sin\alpha\ \cot\delta\ \tan u \end{aligned}\right\} . \tag{11}$$

In the following considerations we can neglect the z-coordinates since the occurring velocity vectors are all parallel to the xy-plane.

Let ω be the angular velocity of the motion Σ_1/Σ_0 (vehicle/ground) and

$\mathbf{v}_{O_1,01} = (v_x, v_y)^\top$ be the velocity vector of O_1 for that motion at the instant t. Then the vectorial velocity of the contact point $C(c_x, c_y)$ w.r.t. the motion Σ_1/Σ_0 is[3]

$$\mathbf{v}_{C,01} = \begin{pmatrix} v_x - \omega c_y \\ v_y + \omega c_x \end{pmatrix}. \tag{12}$$

The motion Σ_2/Σ_1 (wheel/vehicle) is a simple rotation around the axis a, hence, the velocity vector of C for this motion is

$$\mathbf{v}_{C,12} = \dot{u}\, r \begin{pmatrix} -\sin\alpha \\ \cos\alpha \end{pmatrix} \tag{13}$$

where $\dot{u} = \frac{du}{dt}$ is the angular velocity of Σ_2/Σ_1.

The motion Σ_3/Σ_2 (roll/wheel) is a rotation around b. Thus, the instantaneous vectorial velocity $\mathbf{v}_{C,23}$ of C is perpendicular to $\mathbf{b}$ (Eq. (10)):

$$\mathbf{v}_{C,23} = \lambda \begin{pmatrix} -b_y \\ b_x \end{pmatrix} \tag{14}$$

The velocity vector $\mathbf{v}_{C,03}$ of C for the motion Σ_3/Σ_0 (roll/ground) has to be zero since the (passive) roll moves on the ground without sliding. Using the additivity rule for velocities of composed motions we obtain the condition

$$\mathbf{v}_{C,01} + \mathbf{v}_{C,12} + \mathbf{v}_{C,23} = \mathbf{v}_{C,03} = \mathbf{o} = (0,0)^\top$$

which by substitution of (12), (13), (14) yields

$$\left.\begin{aligned} r\ \sin\alpha\ \dot{u} + b_y\ \lambda &= v_x - \omega\ c_y \\ r\ \cos\alpha\ \dot{u} + b_x\ \lambda &= -v_y - \omega\ c_x \end{aligned}\right\}.$$

By elimination of λ we get the differential equation

[3] One can construct the vector $\mathbf{v}_{C,01}$ from the input $\mathbf{v}_{O_1,01}$, ω as indicated in Fig. 5. Here I denotes the instantaneous pole and $\tan\theta = \omega$. Compare also with [Wunderlich (1970), page 22] or [Bottema & Roth (1990), page 258].

$$r(b_x \sin\alpha - b_y \cos\alpha)\ \dot{u} - b_x(v_x - \omega c_y) - b_y(v_y + \omega c_x) = 0 \tag{15}$$

ruling the connection between the vehicle motion and the wheel rotation. The terms b_x, b_y, c_x, c_y in this equation are functions in u according to the equations (10), (11) and u itself, of course, depends on time t.

If we study the motion globally, the situation is rather complicated. While one roll of the wheel is in contact with the ground the contact point C moves from the first side of the wheel to the second. When the turn is on the next roll C jumps back to the first side again. It follows that $b_x(u), b_y(u), c_x(u), c_y(u)$ are functions with jump discontinuities corresponding to the changes of the rolls.[4]

This is the reason that for practical purposes[5] it is assumed that the contact point C in the average lies beneath the wheel center A. By this simplification we can put $b_x = \cos(\alpha+\delta)$, $b_y = \sin(\alpha+\delta)$, $c_x = a_x$, $c_y = a_y$ in Eq. (15). Then we obtain

$$\dot{u} = -\frac{1}{r\sin\delta}\left[\sin(\alpha+\delta)\ (v_y + \omega a_x) + \cos(\alpha+\delta)\ (v_x - \omega a_y)\right]. \tag{16}$$

This formula allows to compute the (approximate) wheel velocity $\dot{u}$ for given vehicle velocity data v_x, v_y, ω.

4.1 Example: Kinematics of a vehicle with three Mecanum wheels

As an example we study the case of a vehicle supplied with three Mecanum wheels with wheel centers $A_i(a_{ix}, a_{iy})$ and wheel axis angles α_i, $i = 1, 2, 3$. If we denote the corresponding angular velocities of the wheels by ω_i then according to Eq. (16) we have

$$\begin{pmatrix} \omega_1 \\ \omega_2 \\ \omega_3 \end{pmatrix} = -\frac{1}{r\sin\delta}\ \mathbf{M} \begin{pmatrix} v_x \\ v_y \\ \omega \end{pmatrix} \tag{17}$$

[4] Moreover, to avoid vibration the rolls are arranged around the wheel body in a way that the silhouettes of adjacent rolls are slightly overlapping. This means that at the change of two rolls both of them are in contact with the ground for a short time interval, one close to the first side of the wheel and the other close to the second.

[5] See [Viboonchaicheep et al. (2003)] or [Siegwart & Nourbakhsh (2004), page 59].

with

$$\mathbf{M} = \begin{pmatrix} \cos(\alpha_1+\delta) & \sin(\alpha_1+\delta) & a_{1x}\sin(\alpha_1+\delta) - a_{1y}\cos(\alpha_1+\delta) \\ \cos(\alpha_2+\delta) & \sin(\alpha_2+\delta) & a_{2x}\sin(\alpha_2+\delta) - a_{2y}\cos(\alpha_2+\delta) \\ \cos(\alpha_3+\delta) & \sin(\alpha_3+\delta) & a_{3x}\sin(\alpha_3+\delta) - a_{3y}\cos(\alpha_3+\delta) \end{pmatrix}.$$

Eq. (17) is the solution to the inverse kinematic problem of the vehicle:

Inverse Kinematic Problem:
Given: Angular velocity ω of the vehicle Σ_1 and vectorial velocity $(v_x, v_y)^\top$ of the vehicle center O_1;
Wanted: Angular velocities ω_i of the wheels $i = 1, 2, 3$;

Conversely, we have the

Forward Kinematic Problem:
Given: Angular velocities ω_i of the wheels $i = 1, 2, 3$;
Wanted: Angular velocity ω of the vehicle Σ_1 and vectorial velocity $(v_x, v_y)^\top$ of the vehicle center O_1;

Clearly, a unique solution for the forward kinematic problem exists if and only if $\det \mathbf{M} \neq 0$, namely

$$\begin{pmatrix} v_x \\ v_y \\ \omega \end{pmatrix} = -r \sin\delta\, \mathbf{M}^{-1} \begin{pmatrix} \omega_1 \\ \omega_2 \\ \omega_3 \end{pmatrix}. \tag{18}$$

Our simplification from above means that the axis b_i of the roll that is in contact with the ground is assumed to be horizontal, roll center B_i and contact point C_i lying exactly beneath the wheel center $A_i(a_{ix}, a_{iy})$. In this position (the first projection of) b_i has the equation

$$-\sin(\alpha_i+\delta)\ (x - a_{ix}) + \cos(\alpha_i+\delta)\ (y - a_{iy}) = 0.$$

In other words, the line coordinates of b_i are

$$(-\sin(\alpha_i+\delta),\ \cos(\alpha_i+\delta),\ a_{ix}\ \sin(\alpha_i+\delta) - a_{iy}\ \cos(\alpha_i+\delta)).$$

If one multiplies the second column of $\mathbf{M}$ with -1 and additionally exchanges this column with the first one then the i-th row of the modified matrix is identical with that vector. Hence, $\det \mathbf{M} = 0$ is equivalent to the condition that the three roll axes meet in a common point I which can also be at infinity. As a result we have

Theorem 2 *The direct kinematics of a robot with three Mecanum wheels has a unique solution if and only if the wheels are arranged so that the roll axes are not concurrent or parallel.*

In a bad wheel arrangement with the roll axes running through a common point I the possible self motion of the vehicle is the rotation around I. In this case the three contact rolls rotate around their axes even if the wheel motors stand still. Such an unwanted self motion might be induced by slightly inclined terrain. Of course, this effect also shows up in case of vehicles with more than three Mecanum wheels.

5 Conclusions

In the paper I give some detailed geometric analysis of Mecanum wheels and work out natural parametrizations of the roll surface (Eq. 4) and its meridian curve (Eq. 5). The result can be used for manufacturing the rolls precisely. Alternatively I investigate suitable approximations of the roll surface by torus patches (section 3).

Moreover I show that the instantaneous contact point C of a roll moves from one side of the Mecanum wheel to the opposite as the wheel rotates. This is neglected in the standard literature which might be a reason for deviations between the real and the predicted motion of a vehicle on such wheels. I develop the differential equation ruling the connection between the vehicle velocity and the angular velocity of the wheel (Eq. 15). This could be the starting point for some more accurate analysis of the kinematics of Mecanum wheel vehicles in future research work. As a drawback the formula requires the knowledge of the rotation angle function $u = u(t)$ of the wheel which may not be available in practice.

Finally, by returning to the simplified equation between the vehicle and wheel velocities, I deliver some nice geometric characterization of singular wheel constellations (Theorem 2).

Acknowledgement

I cordially thank Mathias Brandstötter from the RoboCup Team at TU Graz for fruitful discussions and for providing the ProEngineer-files of the Mecanum wheel vehicle Krikkit$^{\gamma}$ developed by the team. From this CAD model I generated the images in Fig. 1.

I also thank the reviewers for their help to improve the paper.

References

[Bottema & Roth (1990)] Bottema, O. & Roth, B.: Theoretical Kinematics, Dover Publications, 1990.

[Dickerson & Lapin (1991)] Dickerson, S. & Lapin, B.D., Control of an Omni-Directional Robot Vehicle with Mecanum Wheels, 1991, National Telesystems Conference I, 323-328.

[Ilon (1975)] Ilon, B.E., Wheels for a Course Stable Selfpropelling Vehicle Movable in any Desired Direction on the Ground or Some Other Base, 1975, US Patents and Trademarks Office, Patent 3,876,255.

[Siegwart & Nourbakhsh (2004)] Siegwart, R. & Nourbakhsh, I.R.: Introduction to Autonomous Mobile Robots, MIT Press, Cambridge 2004.

[Viboonchaicheep et al. (2003)] Viboonchaicheep, P. & Shimada, A. & Kosaka, Y., Position Rectification Control for Mecanum Wheeled Omni-directional Vehicles, 2003, Proc. of the 29th Annual Conference of the IEEE, 854-859.

[Wunderlich (1970)] Wunderlich, W.: Ebene Kinematik, Bibliographisches Institut Mannheim, 1970.