On the Construction of Rational Curves on Hyperquadrics

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Contents

1 Introduction

Due to their simple parametrization rational curves and surfaces play an important role in the field of Computer Aided Geometric Design (CAGD). In the beginning of the development research-work in this area focused on curves and surfaces with *integral (polynomial)* parametric representation in affine coordinates. But soon the interest turned to curves and surfaces with *fractional rational* parametrizations as they display a higher flexibility (compare e. g. with [Farin 1983], [Böhm 1984], [Piegl 1986], [Piegl 1987], [Piegl 1988], [Rösch 1991], [Rösch 1991 a]). Splinecurves and -surfaces too are often composed of rational parts (so called subsplines). They are simply called *rational splines* then.
As quadrics have nice geometric properties and a simple mathematical representation, they are also part of almost all CAGD-packages. Besides the classical literature on quadrics (see for instance [Bert 1924, chapters VI and VII] or [Hodge 1952, chapter XIII] or [Gier 1982, chapter IV]) there is also much new research on their use in CAGD (compare e. g. with [Levin 1976], [Sabin 1976], [Mill 1988], [Piegl 1986 a], [Geise 1991], [Kleij 1991], [Böhm 1992], [Böhm 1993]).
The most common way to generate curves and surfaces in CAGD is the following: Starting from a control-structure consisting of points (a polyline or a net of points), the curve or surface is generated with the help of a geometrical subdivision-algorithm. Famous examples are the algorithms of Aitken, de Casteljau and Cox-de Boor for generating Lagrange interpolants, Bézier curves and B-splines, respectively (see e. g. [Böhm 1999]).
A different approach is in use to generate rational curves on quadrics: A rational curve or spline is mapped to the given quadric via a rational transformation, for instance a stereographic projection (see e. g. [Jütt 1993], [Dietz 1993], [Dietz 1995]). The advantage of this method is that one can use the standard classes of rational curves in CAGD (Bézier curves, B-Splines, ...). But there are also two disadvantages:

- The resulting curve strongly depends on the transformation used, e. g. on the center of the stereographic projection.
- In general the algebraic order of the curve is increased by the transformation (in case of a stereographic projection the algebraic order is doubled!).

These considerations suggest to investigate the possibility of constructing rational curves on quadrics in a "more direct way". This means to construct such a curve directly out of the control points lying on the given quadric. These investigations are part of the paper. In treating the subject the restriction of the dimension $d = 3$ of the underlying projective space turned out to be redundant. Thus the results are also valid in real projective spaces of *arbitrary* (but finite) dimension. This brings us closer to additional applications, as by the help of appropriate transformations (which at least are rational in the inverse direction) hyperquadrics can be used as images of various geometries and transformation groups. We sketch the following examples:

- A special hyperquadric $\mathcal{M}^4$ in 5-dimensional projective space $\mathbb{P}^5$, called *Klein's quadric*, is a model for the 4-parametric set of straight lines in 3-dimensional projective space $\mathbb{P}^3$.

- The hyper-(unit-)sphere $\mathcal{S}^3$ in 4-dimensional projective space $\mathbb{P}^4$ is a model for the 3-parametric group $SO(3)$ of Euclidean rotations in 3-space (compare with [Blasch 1960]).
- A special quadratic hypercylinder $\mathcal{Z}^3$ in 4-dimensional projective space $\mathbb{P}^4$ is a model for the 3-parametric group $SE(2)$ of direct planar Euclidean displacements.
- *Study's quadric* $\mathcal{M}^6$, which is a special hyperquadric in 7-dimensional projective space $\mathbb{P}^7$, is a model for the 6-parametric group $SE(3)$ of direct Euclidean displacements in 3-space (compare with [Study 1903], [Weiss 1935], [Blasch 1960]).

Thus, for instance, rational interpolation of a (finite) sequence of points on Klein's quadric $\mathcal{M}^4$ means interpolation of the corresponding sequence of straight lines in 3-space by a rational ruled surface. Analogously, rational interpolation of a sequence of points on Study's quadric $\mathcal{M}^6$ means interpolation of the corresponding positions (discrete transformations in $SE(3)$) by a one-parametric rational motion.
In section 2 we repeat some preliminaries needed in the sequel. Hyperquadrics are introduced, being defined as zero-sets of quadratic forms (section 2.1). Cross-ratios for point- and line-quadrupels are discussed (section 2.2). Finally, in section 2.3, we deal with univariate rational interpolation in projective space.
Section 3 is dedicated to rational interpolation on a hyperquadric. Here the input consists of a sequence $\{A_i\}_{i\in\{0,\ldots,n\}}$ of points on a given hyperquadric $\mathcal{Q}^{d-1}$ and a sequence $\{t_i\}_{i\in\{0,\ldots,n\}}$ of corresponding parameter-values. What we want to find are all univariate rational interpolants with an algebraic order $\leq n$, satisfying this data and with the additional property of being contained by $\mathcal{Q}^{d-1}$. In subsection 3.1 we show, that this problem is a linear one (theorem 3.3). The skew-symmetric shape of the coefficient-matrix belonging to the corresponding linear-equation-system implies that in case of $n \equiv 0 (\mathrm{mod}\ 2)$ we in general have exactly one solution curve[1], whereas in case of $n \equiv 1 (\mathrm{mod}\ 2)$ we in general have none. Additionally, the set of solution-curves is connected with the control points in a projectively-invariant way (theorem 3.7).
Subsection 3.2 presents an algebraic algorithm, which in the main case gives the user the possibility to compute the solution curve. Within the algorithm the *weights* of the homogeneous coordinate vectors of the control points are determined in a way that the resulting interpolation curve is part of the given hyperquadric $\mathcal{Q}^{d-1}$. Especially, if $\mathcal{Q}^{d-1}$ is an oval hyperquadric[2], then there either is exactly one or none solution curve. In this case one can, by the help of the given algorithm, also decide if a solution curve exists or not and in the case of its existence determine its algebraic order (theorem 3.12).
Subsection 3.3 contains a *geometric* algorithm for the construction of rational interpolants on hyperquadrics. The algorithm can be considered as a projective generalization of Aitken's algorithm for the construction of Lagrange interpolants in affine space. It uses repeated subdivision on conic sections with the help of cross-ratios.

[1] In the further we will call this case the *main case*.
[2] This means that $\mathcal{Q}^{d-1}$ is projectively-equivalent with a hypersphere.

In subsection 3.4 we in detail discuss our interpolation problem for the case $n = 3$. We in general can not expect a solution here (see above). Theorem 3.14 lists up the cases in which solution curves exist yet, in detail it describes the configurations of four points A_0, A_1, A_2, A_3 on a given hyperquadric $\mathcal{Q}^{d-1}$ and the four corresponding parameter values t_0, t_1, t_2, t_3 which make a solution possible. We also have cases with a one-parametric set of solution cubics. The discussion of the various cases is done in a geometric way, strongly using the cross-ratio(s) of the four points A_i on the hyperquadric and that one of the corresponding parameter values t_i.
Section 4 is dedicated to an additional construction of rational curves on a hyperquadric. We start from the observation that the only difference between Aitken's algorithm for constructing integral Lagrange interpolants and that one of Casteljau for constructing ordinary Bézier curves is that the first uses subdivision with *varying* ratio whereas the second uses subdivision with *constant* ratio. This gives us the idea to carry out the algorithm introduced in subsection 4.3 with a constant cross-ratio. With this method we arrive at a new class of curves, so-called QB-curves[3]. A QB-curve is completely determined by its control points $A_0, \ldots, A_n$ on the given hyperquadric. The natural number n has again to be even for this construction. QB-curves display poperties, which are similar to that one of ordinary Bézier curves (see the theorems 4.4, 4.5, 4.6, 4.7).

2 Preliminaries

2.1 Hyperquadrics

The theory of hyperquadrics is a classical field of projective geometry and we will only emphasize a few facts which are important for our investigations. For a more detailed information and for the proofs we refer the reader to [Hodge 1952, chapter XIII] or [Bert 1924, chapters VI and VII] or [Gier 1982, chapter IV].
Let $\mathbb{P}^d$ denote the d-dimensional real projective space. We will use a projective coordinate system $\mathfrak{S} = \{O_0, \ldots, O_d; E\}$. With respect to $\mathfrak{S}$ any point X of $\mathbb{P}^d$ is represented by a homogeneous coordinate vector $(x_0, \ldots, x_d)^t =: \mathbf{x}$, with $\mathbf{x} \neq (0, \ldots, 0)^t$. This will be abbreviated by $\mathbf{x} \triangleq X$. Two homogeneous coordinate vectors $\mathbf{x}$ and $\overline{\mathbf{x}}$ represent the same point X, iff there exists a factor $f \neq 0$ with $\mathbf{x} = f \cdot \overline{\mathbf{x}}$. This we will denote by $\mathbf{x} \sim \overline{\mathbf{x}}$.
Any hyperquadric $\mathcal{Q}^{d-1} \subset \mathbb{P}^d$ can be represented by an equation of the form

$$\mathbf{x}^t \cdot \mathbf{M} \cdot \mathbf{x} = 0, \tag{1}$$

where $\mathbf{M}$ is a symmetric $(d+1) \times (d+1)$ matrix with rank $\mathbf{M} > 0$ and $\mathbf{x}$ is the homogeneous coordinate vector of a point lying on $\mathcal{Q}^{d-1}$. The corresponding bilinear form will be denoted by

$$\langle \mathbf{x}, \mathbf{y} \rangle := \mathbf{x}^t \cdot \mathbf{M} \cdot \mathbf{y}. \tag{2}$$

Thus (1) can also be written in the form

$$\langle \mathbf{x}, \mathbf{x} \rangle = 0. \tag{3}$$

[3]The "Q" stands for quadric, the "B" for Bézier.

A hyperquadric is called regular if $\det \mathbf{M} \neq 0$, else it is called singular.
Two hyperquadrics $\mathcal{Q}^{d-1}$ and $\overline{\mathcal{Q}}^{d-1}$ are called projectively equivalent, if there exists an autocollineation κ in $\mathbb{P}^d$ with $\kappa(\mathcal{Q}^{d-1}) = \overline{\mathcal{Q}}^{d-1}$.
If $d = 3$ then up to projective equivalence we have eight types of real quadrics:

- The *oval quadric* is a regular quadric, which contains no real straight lines; the Euclidean sphere is an example.
- The *annular quadric* is also regular and contains two one-parametric sets $\mathfrak{R}$ and $\overline{\mathfrak{R}}$ of real straight lines. These sets are called the two reguli of the hyperboloid. Each two lines of the same regulus are skew, lines of different reguli intersect each other. As examples we have the one-sheet rotational hyperboloid or the hyperbolic paraboloid.
- The *null quadric* is a regular quadric without any real point.
- The *real quadratic cone* is a singular quadric with one singular point, the vertex S; it contains a one-parametric set of real lines - each of the lines passing through S. Here we have rank $\mathbf{M} = 3$.
- The *quadratic null cone* is a singular quadric, which contains only one real point, namely its vertex. Again rank $\mathbf{M} = 3$.
- A *pair of real planes.*
- A *pair of complex conjugate planes*; the intersection line l of these two planes is real. The only real points of the quadric are those coincident with l.
- A *double-counted real plane.*

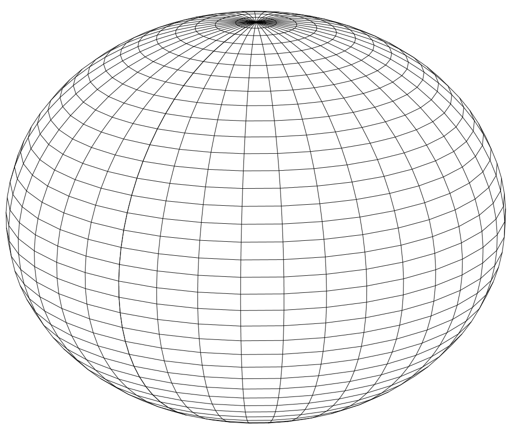

Figure 2.1 *Oval quadric.*

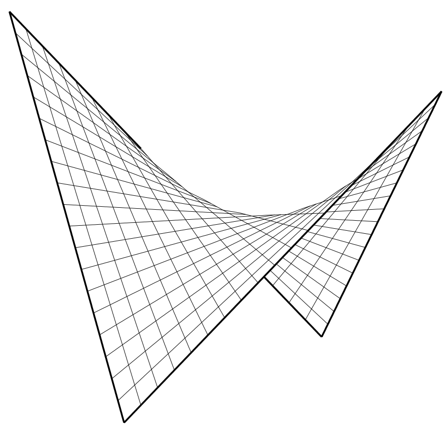

Figure 2.2 *Annular quadric.*

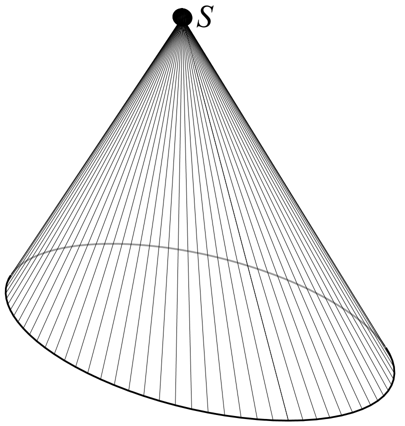

Figure 2.3 *Quadratic cone.*

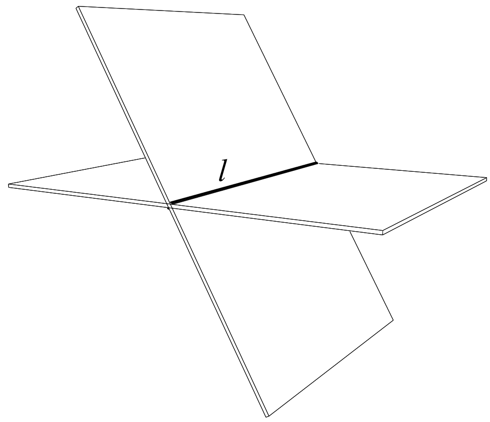

Figure 2.4 *Pair of planes.*

If the quadric consists of a pair of real planes or of a pair of complex conjugate planes then rank $\mathbf{M} = 2$, if it is a double-counted plane then rank $\mathbf{M} = 1$. So these three types are also singular.

Definition 2.1 *Two points A and B of $\mathbb{P}^d$ are called "conjugate with respect to the hyperquadric" $\mathcal{Q}^{d-1}$, if $\langle \mathbf{a}, \mathbf{b} \rangle = 0$, where $\mathbf{a} \overset{\wedge}{=} A$ and $\mathbf{b} \overset{\wedge}{=} B$. If two points are conjugate with respect to the hyperquadric $\mathcal{Q}^{d-1}$ we will denote this by $A \overset{\mathcal{Q}^{d-1}}{\sim} B$.*

Remark 2.1 *(a) To be conjugate with respect to a hyperquadric is a symmetric relation on the set of points in $\mathbb{P}^d$.*

(b) To be conjugate with respect to the $\mathcal{Q}^{d-1}$ has geometric meaning: If π is an autocollineation of $\mathbb{P}^d$ with $\kappa(A) = \overline{A}$, $\kappa(B) = \overline{B}$, $\kappa(\mathcal{Q}^{d-1}) = \overline{\mathcal{Q}}^{d-1}$, then $A \overset{\mathcal{Q}^{d-1}}{\sim} B \Longleftrightarrow \overline{A} \overset{\overline{\mathcal{Q}}^{d-1}}{\sim} \overline{B}$.

(c) If $A, B \in \mathcal{Q}^{d-1}$ we have: $A \overset{\mathcal{Q}^{d-1}}{\sim} B$ if and only if $A = B$ or the line $[A, B]_p$ is part of $\mathcal{Q}^{d-1}$.

(d) If A is a point on $\mathcal{Q}^{d-1}$ then the points X with $A \overset{\mathcal{Q}^{d-1}}{\sim} X$ either fulfill a hyperplane (the "hyperplane tangent to $\mathcal{Q}^{d-1}$ in A") or the whole projective space. In the first (second) case A is called a "regular" ("singular") point of $\mathcal{Q}^{d-1}$. Regular hyperquadrics are those which contain only regular points. They are characterized by rank $\mathbf{M} = d + 1$.

2.2 Cross-ratios

In this section we repeat some definitions of classic projective geometry. For a more detailed information or proofs we refer the reader to any textbook on this subject.

It is well known that four collinear points A_0, A_1, A_2 and A_3 of the real projective space $\mathbb{P}^d$ uniquely determine a cross-ratio $(A_0\ A_1\ A_2\ A_3) \in \mathbb{R} \cup \infty$. If

$$\mathbf{x} \;=\; b \cdot \mathbf{b} + c \cdot \mathbf{c} \tag{4}$$

is a homogeneous parametrization of the line containing the points A_i and $b_i : c_i$ are the homogeneous parameters belonging to these points, then

$$(A_0\ A_1\ A_2\ A_3) \;=\; \frac{\begin{vmatrix} b_0 & b_2 \\ c_0 & c_2 \end{vmatrix} \cdot \begin{vmatrix} b_1 & b_3 \\ c_1 & c_3 \end{vmatrix}}{\begin{vmatrix} b_0 & b_3 \\ c_0 & c_3 \end{vmatrix} \cdot \begin{vmatrix} b_1 & b_2 \\ c_1 & c_2 \end{vmatrix}} \tag{5}$$

Cross-ratios can also be defined for other quadrupels of points or lines:

- If four lines l_0, l_1, l_2, l_3 of a planar line pencil are given (see figure 2.5), we intersect them with an arbitrarily chosen line l of the same plane, which does not contain the vertex of the pencil: $L_i := l_i \cap l$ and define $(l_0\ l_1\ l_2\ l_3) := (L_0\ L_1\ L_2\ L_3)$. This definition is independent of the choice of l.

- Now let four points A_0, A_1, A_2, A_3 on a regular second-order curve c be given: If S is an arbitrarily chosen point on c (see figure 2.6) and l_i denotes[4] the line $[S, A_i]_p$, then $(A_0\ A_1\ A_2\ A_3) := (l_0\ l_1\ l_2\ l_3)$. This definition is independent of the choice of S on c.

  If $\mathbf{x}(t) = (x_0(t), \ldots, x_d(t))^t$ is an arbitrary rational second-order parametrization[5] of c and furthermore $\mathbf{x}(t_i) \triangleq A_i$ for $i \in \{0, 1, 2, 3\}$, then

$$(A_0\ A_1\ A_2\ A_3) \;=\; (t_0\ t_1\ t_2\ t_3) \;:=\; \frac{\begin{vmatrix} 1 & 1 \\ t_0 & t_2 \end{vmatrix} \cdot \begin{vmatrix} 1 & 1 \\ t_1 & t_3 \end{vmatrix}}{\begin{vmatrix} 1 & 1 \\ t_0 & t_3 \end{vmatrix} \cdot \begin{vmatrix} 1 & 1 \\ t_1 & t_2 \end{vmatrix}} \tag{6}$$

- Let l_0, l_1, l_2, l_3 be lines of a regulus $\mathfrak{R}$ and let the line $\bar{l}$ belong to the complementary regulus $\overline{\mathfrak{R}}$ of an annular quadric $\mathcal{H}^2$ (see figure 2.7). If $L_i := l_i \cap \bar{l}$ we define $(l_0\ l_1\ l_2\ l_3) := (L_0\ L_1\ L_2\ L_3)$. The definition is independent of the choice of $\bar{l}$.

- Now we consider four real points A_0, A_1, A_2, A_3 on an annular quadric $\mathcal{H}^2 \subset \mathbb{P}^3$ (see figure 2.7). Let l_i and $\bar{l}_i$ denote the two generators passing through A_i, l_i belonging to the first regulus on $\mathcal{H}^2$ and $\bar{l}_i$ to the second one. We now can define two cross-ratios for the quadrupel of points: $(A_0\ A_1\ A_2\ A_3) := (l_0\ l_1\ l_2\ l_3)$ and $\overline{(A_0\ A_1\ A_2\ A_3)} := (\bar{l}_0\ \bar{l}_1\ \bar{l}_2\ \bar{l}_3)$.

[4] If S is chosen equal to one of the points A_i then the tangent to c in S has to be taken as line l_i.

[5] The coordinate functions $x_i(t)$ are polynomials of degree ≤ 2.

Let $\langle \mathbf{x}, \mathbf{x} \rangle = 0$ be the equation of the annular quadric; then if $\mathbf{a}_i \triangleq A_i$, $i \in \{0,1,2,3\}$ and $a_{\{i,j\}} := \langle \mathbf{a}_i, \mathbf{a}_j \rangle$ the following equation holds:

$$\frac{a_{\{0,2\}} \cdot a_{\{1,3\}}}{a_{\{0,3\}} \cdot a_{\{1,2\}}} = (A_0\ A_1\ A_2\ A_3) \cdot \overline{(A_0\ A_1\ A_2\ A_3)}. \qquad (7)$$

- For four generators l_0, l_1, l_2, l_3 on a quadratic cone $\mathcal{C}^2 \subset \mathbb{P}^3$ (see figure 2.8) we define a cross-ratio in the following way: Let c be the intersection line of $\mathcal{C}^2$ with an arbitrarily chosen plane not containing the vertex of the cone - c is a regular second-order curve - and let $L_i := l_i \cap c$, then $(l_0\ l_1\ l_2\ l_3) := (L_0\ L_1\ L_2\ L_3)$. This cross-ratio is independent of the choice of the intersection plane.

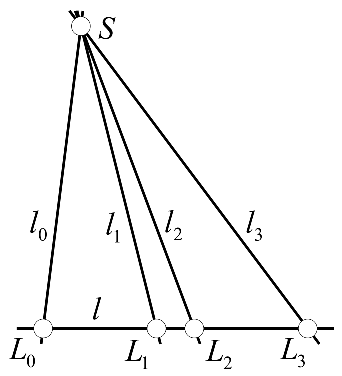

Figure 2.5 *The cross-ratio of four lines l_0, l_1, l_2, l_3 of a line pencil is reduced to the cross-ratio of four points L_0, L_1, L_2, L_3 on a line.*

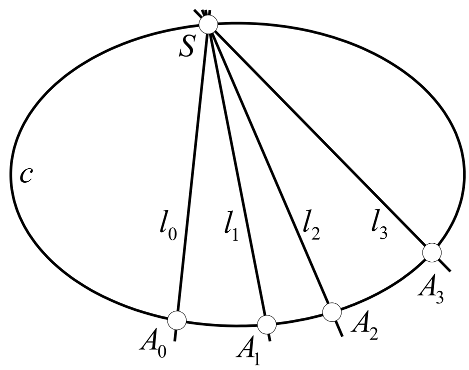

Figure 2.6 *The cross-ratio of four points A_0, A_1, A_2, A_3 on a conic section c is reduced to the cross-ratio of four lines l_0, l_1, l_2, l_3 of a line pencil.*

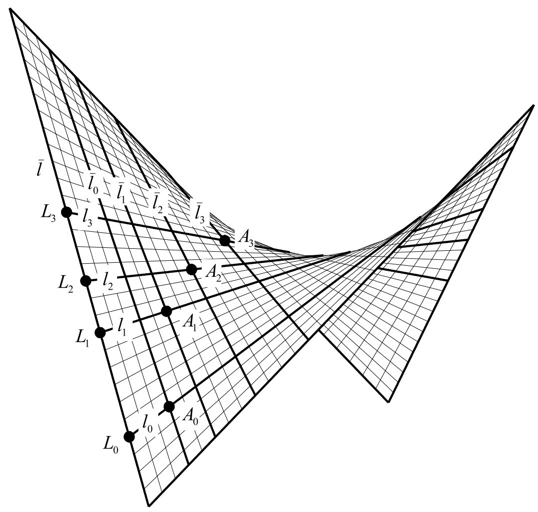

Figure 2.7 *The cross-ratio of four generators l_0, l_1, l_2, l_3 and the cross-ratios of four points A_0, A_1, A_2, A_3 on an annular quadric is reduced to the cross-ratio of four points L_0, L_1, L_2, L_3 on a generator.*

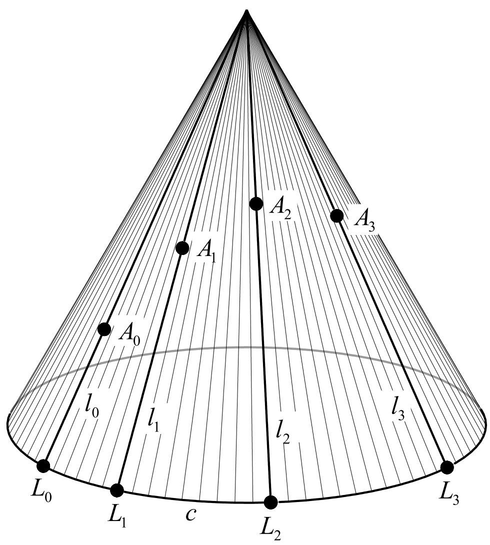

Figure 2.8 *The cross-ratio of four generators l_0, l_1, l_2, l_3 and cross-ratio of four points A_0, A_1, A_2, A_3 on a quadratic cone is reduced to the cross-ratio of four points L_0, L_1, L_2, L_3 on a conic section c.*

- Finally we consider four points A_0, A_1, A_2, A_3 on a quadratic cone $\mathcal{C}^2$, none of them being equal to the vertex (see figure 2.8). Let l_i be the generator passing through A_i; then the cross-ratio of these quadrupel is defined by $(A_0\ A_1\ A_2\ A_3) := (l_0\ l_1\ l_2\ l_3)$.

  Let $\langle \mathbf{x}, \mathbf{x} \rangle = 0$ be the equation of the cone; then if $\mathbf{a}_i \triangleq A_i$, $i \in \{0, 1, 2, 3\}$ and $a_{\{i,j\}} := \langle \mathbf{a}_i, \mathbf{a}_j \rangle$ the following equation holds:

$$\frac{a_{\{0,2\}} \cdot a_{\{1,3\}}}{a_{\{0,3\}} \cdot a_{\{1,2\}}} = (A_0\ A_1\ A_2\ A_3)^2. \tag{8}$$

Any of the defined cross-ratios is projectively invariant.

2.3 Rational interpolation curves

Let $n \geq 1$ be an integer, $J = \{j_0, \dots, j_n\}$ be a set of positive integers and $t_{j_0}, \dots, t_{j_n}$ be $n + 1$ pairwise different real values; then we define

$$p_J(t) := \prod_{k \in J} (t - t_k), \tag{9}$$

$$p_{J\setminus\{i\}}(t) := \prod_{k \in J\setminus\{i\}} (t - t_k), \quad i \in J, \tag{10}$$

$$p_{J\setminus I}(t) := \prod_{k \in J\setminus I} (t - t_k) \ \text{ for } \ I := \{i_0, \dots, i_l\} \subset J. \tag{11}$$

We will use the following (trivial) properties of the polynomials (9), (10), (11):

Theorem 2.1 *(a)* $\deg p_J = n + 1$, $\deg p_{J\setminus\{i\}} = n$, $\deg p_{J\setminus I} = n - l$.

(b) $p_{J\setminus\{i\}}(t_j) = 0$, *if* $i, j \in J$ *and* $i \neq j$.

(c) $p_{J\setminus\{i\}}(t_j) \neq 0$, *if* $i, j \in J$ *and* $i = j$.

(d) The polynomials $p_{J\setminus\{i\}}$ *form a basis of the* $(n + 1)$*-dimensional vector space of all polynomials of degree* $\leq n$.

(e) $p_{J\setminus\{i_0\}}(t) \cdot p_{J\setminus\{i_1\}}(t) = p_{J\setminus\{i_0,i_1\}}(t) \cdot p_J(t)$ *if* $i_0, i_1 \in J$; $i_0 \neq i_1$.

Definition 2.2 *Let* $n + 1$ *parameter values* $t_0, \dots, t_n \in \mathbb{R}$ *(pairwise distinct) and* $n + 1$ *points* $A_0, \dots, A_n \in \mathbb{P}^d$ *(at least two of them being distinct) be given. Any rational curve* c *with a parametrization*

$$c \ \dots \ \mathbf{x} = \mathbf{x}(t) = (x_0(t), \dots, x_d(t))^t, \tag{12}$$

where $x_k(t)$ *are polynomials and* $\forall i \in J = \{0, \dots, n\} : \mathbf{x}(t_i) \triangleq A_i$*, is called "rational interpolation curve" for the control points* A_i *and the corresponding parameter-values* t_i.

Definition 2.3 *Let* $\mathbf{x} = \mathbf{x}(t) = (x_0(t), \ldots, x_d(t))^t$, *where* x_i *are polynomials, then we define* $\deg \mathbf{x} := \max\{\deg x_i | i \in \{0, \ldots, d\}\}$.

We have to distinguish accurately between the algebraic order $o(c)$ of c and the degree $\deg \mathbf{x}$ of a parametrization $\mathbf{x} = \mathbf{x}(t)$ of c; the relation

$$o(c) \leq \deg \mathbf{x} \tag{13}$$

is valid however.[6]

Because of theorem 2.1, (d) the parametrization of a rational interpolation curve c with $o(c) \leq n$ can be written in the form

$$c \ \ldots \ \mathbf{x}(t) \ = \ \sum_{i \in J} p_{J \setminus \{i\}}(t) \cdot \mathbf{b}_i \tag{14}$$

with some vector-coefficients $\mathbf{b}_i$. As $\mathbf{x}(t_i) = p_{J\setminus\{i\}}(t_i) \cdot \mathbf{b}_i \sim \mathbf{a}_i$ we get

$$\forall i \in J \ \exists w_i \in \mathbb{R} : \mathbf{b}_i \ = \ w_i \cdot \mathbf{a}_i. \tag{15}$$

So as a result we have

Theorem 2.2 *Every rational interpolation curve* c *with* $o(c) \leq n$ *for the points* A_i *and the corresponding parameter-values* t_i *has a parametrization of the form*

$$c \ \ldots \ \mathbf{x}(t) \ = \ \sum_{i \in J} p_{J\setminus\{i\}}(t) \cdot w_i \cdot \mathbf{a}_i, \tag{16}$$

where $w_i \in \mathbb{R}$, $\mathbf{a}_i \stackrel{\wedge}{=} A_i$ *and* $i \in J := \{0, \ldots, n\}$.

A trivial conclusion is

Remark 2.2 *Every rational interpolation curve* c *with* $o(c) \leq n$ *spans the same projective space as its control points:* $[c]_p = [A_0, \ldots, A_n]_p$

Any choice of the *"weights"* w_i, $w_i \neq 0$ in (16) yields a rational interpolation curve with $o(c) \leq n$ for the series of points A_i and the corresponding series of parameter values t_i; so in general we have an n-parametric set of rational interpolation curves[7] with $o(c) \leq n$ for this given interpolation problem[8].

In general the interpolation curves are of algebraic order n. Sometimes the interpolation problem can also have solution curves of different algebraic order, as the following example illustrates:

[6] Lüroths theorem (see e. g. [Bert 1924, pages 318–321]) yields an algorithm to find the algebraic order of a rational curve given by its parametrization.

[7] If $\overline{w}_i = \lambda \cdot w_i$ holds for two $n+1$-tupels $(w_0, \ldots, w_n)$ and $(\overline{w}_0, \ldots, \overline{w}_n)$ they represent the same interpolation curve and parametrization.

[8] This is no longer the case if we require that the points A_i and the whole curve c have to be part of a hyperquadric. In section 2.1 we will see that in this case the interpolation problem in general has a unique solution curve if $n \equiv 0 (\mathrm{mod}\ 2)$ and none if $n \equiv 1 (\mathrm{mod}\ 2)$.

Let $d = 2$, then for

$$\begin{aligned} t_0 &= -\sqrt{\tfrac{a-1}{2} + \sqrt{D}}, \\ t_1 &= -\sqrt{\tfrac{a-1}{2} - \sqrt{D}}, \\ t_2 &= \sqrt{\tfrac{a-1}{2} - \sqrt{D}}, \\ t_3 &= \sqrt{\tfrac{a-1}{2} + \sqrt{D}} \text{ with} \\ D &= \tfrac{a^2 - 6 \cdot a + 1}{4} \end{aligned} \tag{17}$$

the curves[9]

$$c \ \ldots \ \mathbf{x}(t) = \begin{pmatrix} 1 + t^2 \\ -2 \\ -2 \cdot t \end{pmatrix} \tag{18}$$

and

$$\bar{c} \ \ldots \ \bar{\mathbf{x}}(t) = \begin{pmatrix} a \\ t^2 - a \\ t \cdot (t^2 - a) \end{pmatrix} \tag{19}$$

interpolate the same four points A_0, A_1, A_2, A_3 (see figure 2.9).

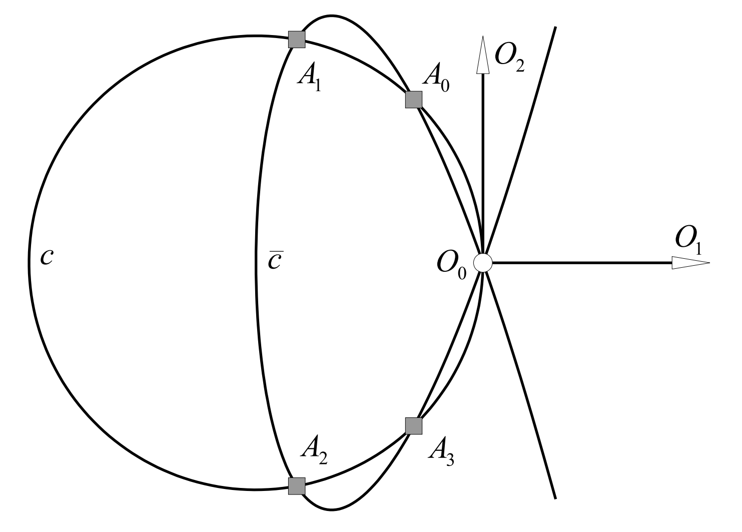

Figure 2.9 *Interpolation of four points by a circle c and a Newton node $\bar{c}$.*

[9]After having introduced affine coordinates $x := \frac{x_1}{x_0}$, $y := \frac{x_2}{x_0}$ and a Euclidean metric c and $\bar{c}$ turn out to be a circle and a Newton node, respectively.

3 Rational interpolation on a hyperquadric

3.1 The problem

We intend to investigate the following interpolation problem IP_J:
Let[10] $n \geq 2$ and let $n+1$ parameter values $t_0, \ldots, t_n \in \mathbb{R}$ (pairwise distinct) and $n+1$ points $A_0, \ldots, A_n$ (at least two of them distinct) on a hyperquadric $\mathcal{Q}^{d-1}$ (3) be given.
Problem: Find a rational interpolation curve $c \ldots \mathbf{x} = \mathbf{x}(t)$ of algebraic order $o(c) \leq n$ with $c \subset \mathcal{Q}^{d-1}$ and $\mathbf{x}(t_i) \triangleq A_i$.
We will call any curve c, solving this problem, *solution curve of* IP_J. Here $J := \{0, \ldots, n\}$.
Any solution curve has a parametric representation of the form[11]

$$c \ldots \mathbf{x}(t) \;=\; \sum_{i \in J} p_{J\setminus\{i\}}(t) \cdot p_{J\setminus\{i\}}(t_i) \cdot w_i \cdot \mathbf{a}_i, \tag{20}$$

where $\mathbf{a}_i \triangleq A_i$ for $i \in J$. We will call this the *Lagrange-representation* of c. Because of theorem 2.1 (d) this representation is uniquely determined up to multiplication of the weights w_i with a common factor $f \neq 0$.
As c has to be part of $\mathcal{Q}^{d-1}$

$$\left\langle \sum_{i \in J} p_{J\setminus\{i\}}(t) \cdot p_{J\setminus\{i\}}(t_i) \cdot w_i \cdot \mathbf{a}_i, \sum_{i \in J} p_{J\setminus\{i\}}(t) \cdot p_{J\setminus\{i\}}(t_i) \cdot w_i \cdot \mathbf{a}_i \right\rangle \;\equiv\; 0 \tag{21}$$

must hold. Using theorem 2.1, e) and $\langle \mathbf{a_i}, \mathbf{a_i} \rangle = 0$ equation (21) reduces to

$$p_J(t) \cdot \underbrace{\sum_{\substack{i,j \in J \\ i<j}} a_{\{i,j\}} \cdot p_{J\setminus\{i,j\}}(t) \cdot p_{J\setminus\{i\}}(t_i) \cdot p_{J\setminus\{j\}}(t_j) \cdot w_i \cdot w_j}_{:=\; f(t)} \;\equiv\; 0, \tag{22}$$

where $a_{\{i,j\}} := \langle \mathbf{a}_i, \mathbf{a}_j \rangle$. As the polynomial $p_J(t) \not\equiv 0$, the second factor $f(t)$ has to vanish for all values t, especially for t_k, $k \in J$. Thus, by using $p_{J\setminus\{i,j\}}(t_k) = 0$ for $i, j \neq k$ the series $w_0, \ldots, w_n$ has to be a solution of the system

$$Q_J: \quad p_{J\setminus\{k\}}(t_k) \cdot w_k \cdot \sum_{\substack{i \in J \\ i \neq k}} a_{\{k,i\}} \cdot p_{J\setminus\{k,i\}}(t_k) \cdot p_{J\setminus\{i\}}(t_i) \cdot w_i \;=\; 0, \;\; k \in J \tag{23}$$

of $n+1$ homogeneous *quadratic* equations. As $\deg f \leq n-1$, we also have the converse statement: If $(w_0, \ldots, w_n)$ is a solution of the system Q_J, then $f(t)$ vanishes identically.

[10] If $n = 1$ a rational interpolation curve on $\mathcal{Q}^{d-1}$ for the points A_0, A_1 exists, iff $\langle \mathbf{a}_0, \mathbf{a}_1 \rangle = 0$, where $\mathbf{a}_0 \triangleq A_0$ and $\mathbf{a}_1 \triangleq A_1$; this means (compare with remark 2.1) either

- that the line $[A_0, A_1]_p$ is part of $\mathcal{Q}^{d-1}$ - in this case this line is the solution curve - or
- that $A_0 = A_1$; then the solution curve is this point.

[11] Compare with theorem 2.2.

If we furthermore assume $\forall k \in J: \; w_k \neq 0$ the system

$$L_J: \quad \sum_{\substack{i \in J \\ i \neq k}} a_{\{k,i\}} \cdot p_{J \setminus \{k,i\}}(t_k) \cdot p_{J \setminus \{i\}}(t_i) \cdot w_i \;\; = \;\; 0, \;\; k \in J \tag{24}$$

of $n+1$ homogeneous *linear* equations has to be fulfilled by $w_0, \ldots, w_n$. The coefficient matrix $\mathbf{C}_J$ of this system has the form

$$\begin{aligned} \mathbf{C}_J &= (c_{ij})_{i,j \in J} \;\text{ with} \\ c_{ij} &= a_{\{i,j\}} \cdot p_{J \setminus \{i,j\}}(t_i) \cdot p_{J \setminus \{j\}}(t_j) \;\text{ for }\; i \neq j \;\text{ and} \\ c_{ii} &= 0. \end{aligned} \tag{25}$$

The system L_J has a non-trivial solution $(w_0, \ldots, w_n) \neq (0, \ldots, 0)$, if and only if the determinant Δ_J of $\mathbf{C}_J$ is equal to zero. The following theorem shows that in case of n even, one always has a non-trivial solution:

Theorem 3.1 *(a)* $\mathbf{C}_J$ *is a skew-symmetric matrix.*

(b) rank $\mathbf{C}_J$ *is even.*

Proof.

(a)

$$c_{ij} = a_{\{i,j\}} \cdot p_{J \setminus \{i,j\}}(t_i) \cdot p_{J \setminus \{j\}}(t_j) = a_{\{i,j\}} \cdot \prod_{\substack{k \in J \\ k \neq i,j}} (t_i - t_k) \cdot \prod_{\substack{l \in J \\ l \neq j}} (t_j - t_l)$$

$$= a_{\{i,j\}} \cdot (t_j - t_i) \cdot \prod_{\substack{k \in J \\ k \neq i,j}} (t_i - t_k) \cdot \prod_{\substack{l \in J \\ l \neq i,j}} (t_j - t_l) = -a_{\{j,i\}} \cdot \prod_{\substack{l \in J \\ l \neq i,j}} (t_j - t_l) \cdot \prod_{\substack{k \in J \\ k \neq j}} (t_i - t_k)$$

$$= -a_{\{j,i\}} \cdot p_{J \setminus \{j,i\}}(t_j) \cdot p_{J \setminus \{i\}}(t_i) = -c_{ji}$$

(b) is a consequence[12] of (a) $\diamond$

We now will study the most simple case: $n = 2$. Here we have

$$\mathbf{C}_{\{0,1,2\}} \;\; = \;\; (t_0 - t_1) \cdot (t_1 - t_2) \cdot (t_2 - t_0) \cdot \begin{pmatrix} 0 & a_{\{0,1\}} & -a_{\{0,2\}} \\ -a_{\{0,1\}} & 0 & a_{\{1,2\}} \\ a_{\{0,2\}} & -a_{\{1,2\}} & 0 \end{pmatrix}. \tag{26}$$

The following cases can occur:

[12]See any textbook on linear algebra, e. g. [Greub 1967, pages 217–219].

- $a_{\{0,1\}}$, $a_{\{0,2\}}$, $a_{\{1,2\}}$ are all different from zero.
  This means[13] that any two of the three points A_0, A_1 and A_2 are distinct and none of the lines $[A_i, A_j]_p$, $i, j \in \{0, 1, 2\}$, $i \neq j$ is part of $\mathcal{Q}^{d-1}$. Then $[A_0, A_1.A_2]_p$ is a plane and intersects $\mathcal{Q}^{d-1}$ in a regular second-order curve c. Thus c is the uniquely determined solution curve of IP_J. Moreover all 2×2-subdeterminants of the matrix $\mathbf{C}_{\{0,1,2\}}$ are different from zero in this case. So, rank $\mathbf{C}_{\{0,1,2\}} = 2$ and the weights of the solution curve c are computed as the uniquely determined homogeneous solution triple of $L_{\{0,1,2\}}$:

$$w_0 : w_1 : w_2 \quad = \quad a_{\{1,2\}} : a_{\{0,2\}} : a_{\{0,1\}} \tag{27}$$

  and the Lagrange-representation of the solution curve is

$$\begin{aligned} \mathbf{x}(t) \quad &= \quad p_{\{1,2\}}(t) \cdot p_{\{1,2\}}(t_0) \cdot a_{\{1,2\}} \cdot \mathbf{a}_0 + p_{\{0,2\}}(t) \cdot p_{\{0,2\}}(t_1) \cdot a_{\{0,2\}} \cdot \mathbf{a}_1 + \\ &+ \quad p_{\{0,1\}}(t) \cdot p_{\{0,1\}}(t_2) \cdot a_{\{0,1\}} \cdot \mathbf{a}_2. \end{aligned} \tag{28}$$

- One of the $a_{\{i,j\}}$ is equal to zero and another one is different from zero; e.g. $a_{\{0,1\}} = 0$ and $a_{\{0,2\}} \neq 0$. Two cases might occur:
  a) $A_0 = A_1$ no solution curve exists, as $[A_0, A_1, A_2]_p$ is a line not contained by $\mathcal{Q}^{d-1}$.
  b) $A_0 \neq A_1$; then the line $[A_0, A_1]_p$ is part of $\mathcal{Q}^{d-1}$ and $[A_0, A_1, A_2]_p$ is a plane which intersects $\mathcal{Q}^{d-1}$ in a singular second-order curve c. Again no solution curve exists.

- $a_{\{0,1\}}$, $a_{\{0,2\}}$, $a_{\{1,2\}}$ are all equal to zero; then the linear space $[A_0, A_1, A_2]_p$ is part of $\mathcal{Q}^{d-1}$. In this case the solution
  a) consists of a two parametric set of conic sections if dim $[A_0, A_1, A_2]_p = 2$,
  b) is the line $[A_0, A_1, A_2]_p$ if dim $[A_0, A_1, A_2]_p = 1$,
  c) is the point $[A_0, A_1, A_2]_p$ if dim $[A_0, A_1, A_2]_p = 0$.
  The matrix $\mathbf{C}_{\{0,1,2\}}$ is the null-matrix - so *any* set of weights trivially solves the linear equation system $L_{\{0,1,2\}}$.

As a result we have

Theorem 3.2 *(a) The interpolation problem $IP_{\{0,1,2\}}$ only has a solution if either*

- $a_{\{0,1\}} \neq 0$, $a_{\{0,2\}} \neq 0$, $a_{\{1,2\}} \neq 0$ *- the solution curve is uniquely determined and a conic section - or*
- $a_{\{0,1\}} = 0$, $a_{\{0,2\}} = 0$, $a_{\{1,2\}} = 0$. *If $[A_0, A_1, A_2]_p$ is a plane then we have a 2-parametric set of solution conics else the solution curve is uniquely determined and either a point or a line.*

(b) If a solution curve exists for $IP_{\{0,1,2\}}$ then the weights w_0, w_1, w_2 of its Lagrange-representation satisfy the linear equation system $L_{\{0,1,2\}}$.

[13]Compare with remark 2.1, (c).

Now we again return to the general case. We had to assume $\forall k \in J : \; w_k \neq 0$ to make sure that the weights $w_0, \ldots, w_n$ of a solution curve solve the linear equation system L_J. So we cannot be sure that the weights $w_0, \ldots, w_n$ of *any* solution curve of IP_J satisfy L_J, but we will prove this (see theorem 3.3). If c is a solution curve with weights $w_0, \ldots, w_n$ solving Q_J but not L_J, then some of them have to be zero. Assume $w_0 \neq 0, \ldots, w_{\bar{n}} \neq 0$, $w_{\bar{n}+1} = \ldots = w_n = 0$ for instance. In this case we have $o(c) \leq \bar{n} < n$, as the factor $\prod_{k=\bar{n}+1}^{n}(t - t_k)$ can be cancelled out of the Lagrange-representation. This yields

Lemma 3.1 *If c is a solution curve of IP_J with algebraic order $o(c) = n$, then the weights $w_0, \ldots, w_n$ of its Lagrange-representation (20) satisfy the linear equation system L_J.*

We are now prepared to prove the generalization of the previous theorem for an arbitrary solution curve of IP_J:

Theorem 3.3 *If c is a solution curve of IP_J then the weights $w_0, \ldots, w_n$ of its Lagrange-representation solve the linear equation system L_J.*

Proof. (Induction over n)
Initial step: $n = 2$; see theorem 3.2, (b).
Induction step:

- If $o(c) = n$ then the desired result is given via lemma 3.1.
- If $o(c) = \overline{n} < n$, then c is also a solution curve of $IP_{J\setminus\{n\}}$; so, it must have a representation of the form

$$\mathbf{x}^*(t) \quad = \sum_{i \in J\setminus\{n\}} p_{J\setminus\{i,n\}}(t) \cdot p_{J\setminus\{i,n\}}(t_i) \cdot w_i^* \cdot \mathbf{a}_i, \tag{29}$$

with $w_0^*, \ldots, w_{n-1}^*$ solving $L_{J\setminus\{n\}}$ (induction hypothesis). But then the $n+1$ values $w_0 := \frac{w_0^*}{t_0 - t_n}, \ldots, w_{n-1} := \frac{w_{n-1}^*}{t_{n-1} - t_n}, w_n := 0$ solve L_J: Simple substitution yields that the first n lines are satisfied. Furthermore

$$\mathbf{x}^*(t_n) \stackrel{\wedge}{=} A_n \Longrightarrow \mathbf{x}^*(t_n) \sim \mathbf{a}_n,$$

which yields

$$\langle \mathbf{x}^*(t_n), \mathbf{a}_n \rangle = \sum_{i \in J\setminus\{n\}} a_{\{n,i\}} \cdot p_{J\setminus\{i,n\}}(t_n) \cdot p_{J\setminus\{i\}}(t_i) \cdot w_i = 0.$$

Thus the last line of the system $L_{J\setminus\{n\}}$ is satisfied too by the weights w_i and

$$\mathbf{x}(t) := \sum_{i \in J} p_{J\setminus\{i\}}(t) \cdot p_{J\setminus\{i\}}(t_i) \cdot w_i \cdot \mathbf{a}_i$$

is the Lagrange-representation of c $\diamond$

Theorem 3.3 shows that the weights of a solution curve necessarily have to solve L_J, but we cannot assure that the converse statement is true: If $w_0, \ldots, w_n$ is a non-trivial solution of L_J, we get a corresponding parametrization of the form (20). The curve c represented by this parametrization is completely contained in $\mathcal{Q}^{d-1}$ as $\langle \mathbf{x}(t), \mathbf{x}(t) \rangle \equiv 0$ holds, but the conditions $\mathbf{x}(t_i) \triangleq A_i$ do not have to be fulfilled necessarily, as some of the weights might be zero. So we only have a weak form of a converse statement to theorem 3.3:

Theorem 3.4 *If $w_0, \ldots, w_n$ satisfy the linear equation system L_J and additionally $\forall i \in J: \ w_i \neq 0$ then $c \ldots \mathbf{x}(t) = \sum_{i \in J} p_{J \setminus \{i\}}(t) \cdot p_{J \setminus \{i\}}(t_i) \cdot w_i \cdot \mathbf{a}_i$ is a solution curve of IP_J.*

An estimation of the algebraic order of possible solution curves is given by the next theorem:

Theorem 3.5 *If c is a solution curve of IP_J then $o(c) \geq \operatorname{rank} \mathbf{C}_J$.*

Proof. Let c be a solution curve with algebraic order $o(c) = \overline{n} \leq n$.

- *$\overline{n} = n$:* If n is even due to theorem 3.1, (b) the statement is true. If n is odd $\operatorname{rank} \mathbf{C}_J \leq n$ also has to hold, as L_J has a non-trivial solution.
- *Now let $\overline{n} < n$:* Then c is also a solution curve of $IP_{\overline{J}}$, where $\overline{J}$ is any subset of J with $\overline{n}+1$ elements. E. g. c is a solution curve for IP_{J_j}, where $J_j := \{0, \ldots, \overline{n}-1, j\}$ with $j \in \{\overline{n}, \ldots, n\}$. So, with the help of theorem 3.3 we know, that c has $n - \overline{n} + 1$ distinct parametrizations

$$\mathbf{x}_j^*(t) \ = \ \sum_{i \in J_j} p_{J_j \setminus \{i\}}(t) \cdot p_{J_j \setminus \{i,\}}(t_i) \cdot w_{j,i}^* \cdot \mathbf{a}_i, \ \ j \in \{\overline{n}, \ldots, n\}, \tag{30}$$

$w_{j,0}^*, \ldots, w_{j,\overline{n}-1}^*, w_{j,j}^*$ being a solution of L_{J_j}.
We define

$$w_{i,j} \ := \ \left\{ \begin{array}{cl} \frac{w_{i,j}^*}{\prod\limits_{\substack{k=\overline{n} \\ k \neq j}}^{n} (t_k - t_j)} & \text{for } \ i \in J_j \\ 0 & \text{for } \ i \in \{\overline{n}, \ldots, n\} \setminus \{j\} \end{array} \right\}. \tag{31}$$

Then each line of the matrix

$$\begin{pmatrix} w_{\overline{n},0} & \ldots & w_{\overline{n},\overline{n}} & 0 & \ldots & 0 \\ w_{\overline{n}+1,0} & \ldots & 0 & w_{\overline{n}+1,\overline{n}+1} & \ldots & 0 \\ \vdots & \vdots & & & \ddots & \vdots \\ w_{n,0} & \ldots & 0 & 0 & \ldots & w_{n,n} \end{pmatrix} \tag{32}$$

satisfies[14] the linear equation system L_J. As moreover the $n - \overline{n} + 1$ lines of the matrix (32) are linearly independent, we have $\operatorname{rank} \mathbf{C}_J \leq n - (n - \overline{n}) = \overline{n}$ ◇

[14]This can be shown in the same way as in the proof of theorem 3.3.

Theorem 3.6 *The following statements are logically equivalent:*

- rank $\mathbf{C}_J = 0$
- $\forall i, j \in J$ *with* $i \neq j : a_{\{i,j\}} = 0$
- $[A_0, \ldots, A_n]_p \subset \mathcal{Q}^{d-1}$

Proof.

- If rank $\mathbf{C}_J = 0$ then $\mathbf{C}_J$ is the zero-matrix, which yields $a_{\{i,j\}} = 0 \ \forall i, j \in J$ with $i \neq j$.
- Let now $\forall i, j \in J, i \neq j : a_{\{i,j\}} = 0$. If $X \in [A_0, \ldots, A_n]_p$, then X is represented by a homogeneous coordinate vector $\mathbf{x}$ with

$$\mathbf{x} \;=\; \sum_{i \in J} \lambda_i \cdot \mathbf{a}_i, \;\text{ with }\; \lambda_i \in \mathbb{R} \;\text{ and }\; \mathbf{a}_i \stackrel{\wedge}{=} A_i.$$

  As

$$\langle \mathbf{x}, \mathbf{x} \rangle \;=\; \langle \sum_{i \in J} \lambda_i \cdot \mathbf{a}_i \;,\; \lambda_i \cdot \mathbf{a}_i \rangle \;=\; 2 \cdot \sum_{\substack{i,j \in J \\ i<j}} \lambda_i^2 \cdot a_{\{i,j\}} \;=\; 0,$$

  X must be on $\mathcal{Q}^{d-1}$.

- If $[A_0, \ldots, A_n]_p \subset \mathcal{Q}^{d-1}$ then *any* choice of the weights $w_0, \ldots w_n$ with $w_0 \neq 0, \ldots, w_n \neq 0$ yields a solution curve of IP_J; thus $\mathbf{C}_J$ is the zero-matrix $\diamond$

Another important property of the set of solution-curves of the interpolation problem IP_J is the projective invariance: Let κ be an autocollineation of $\mathbb{P}^d$ and $A_i^* := \kappa(A_i)$. Let furthermore IP_J^* denote the interpolation problem for the points A_i^* and the parameter-values t_i. Then the set of solution curves of IP_J^* is the κ-image of the set of solution curves of IP_J. Thus, we have

Theorem 3.7 *The set of solution-curves is invariantly combined with the series of points* $A_0, \ldots, A_n$ *with respect to autcollineations of* $\mathbb{P}^d$.

3.2 A useful recursion formula

In the previous section we showed that interpolation on a hyperquadric $\mathcal{Q}^{d-1}$ is a linear problem: The weights of a solution curve for IP_J have to solve a system L_J of $n+1$ linear homogeneous equations (theorem 3.3). Furthermore we found out that the determinant of the coefficient matrix is equal to zero if $n \equiv 0 (\mathrm{mod}\ 2)$ (theorem 3.1, (b)). In the current section we will see that in general a solution curve exists and is uniquely determined if n is even. Moreover we will supply the user with a recursively defined formula for computing the solution curve in this case.

Definition 3.1 *Let $\mathcal{Q}^{d-1}$ be a hyperquadric in d-dimensional projective space $\mathbb{P}^d$ and let n be an odd positive integer; Let furthermore $J := \{j_0, \ldots, j_n\} \subset \mathbb{N}$ and let t_k be pairwise distinct values in $\mathbb{R}$ and let A_k be points on $\mathcal{Q}^{d-1}$ with homogeneous coordinate vectors $\mathbf{a}_k$ for $k \in J$; then we recursively define*

- *for $n = 1$:*

$$a_J = a_{\{j_0,j_1\}} \quad := \quad \langle \mathbf{a}_{j_0}, \mathbf{a}_{j_1} \rangle, \tag{33}$$

- *for $n \geq 3$:*

$$a_J \quad := \quad \sum_{\substack{k \in J \\ k \neq l}} p_{J\setminus\{k,l\}}(t_k) \cdot p_{J\setminus\{k,l\}}(t_l) \cdot a_{\{k,l\}} \cdot a_{J\setminus\{k,l\}}, \text{ where } l \in J. \tag{34}$$

At first sight the above definition seems to depend on the choice of $l \in J$ if $n \geq 3$ and on the order of sequence of the indices in J. We will prove immediately that this is not the case:

Theorem 3.8 *(a) Let $n \geq 3$; then the definition of a_J is independent of the choice of $l \in J$.*

(b) The value of a_J is not changed by any permutation on the index-set J.

Proof. (a) Let m be in J, $m \neq l$ and let

$$a_J^* := \sum_{\substack{j \in J \\ j \neq m}} p_{J\setminus\{j,m\}}(t_j) \cdot p_{J\setminus\{j,m\}}(t_m) \cdot a_{\{j,m\}} \cdot a_{J\setminus\{j,m\}}.$$

We have to show $a_J = a_J^*$;
Let first $n = 3$: Then *any* choice of l yields

$$a_{\{j_0,j_1,j_2,j_3\}} = f_{j_0j_1} \cdot a_{\{j_0,j_1\}} \cdot a_{\{j_2,j_3\}} + f_{j_0j_2} \cdot a_{\{j_0,j_2\}} \cdot a_{\{j_1,j_3\}} + f_{j_0j_3} \cdot a_{\{j_0,j_3\}} \cdot a_{\{j_1,j_2\}},$$

where the factors $f_{i_0i_1}$ are either computed by

$$f_{i_0i_1} \quad = \quad p_{\{i_0,i_1\}}(t_{i_2}) \cdot p_{\{i_0,i_1\}}(t_{i_3}) \tag{35}$$

or by

$$f_{i_0i_1} \quad = \quad p_{\{i_2,i_3\}}(t_{i_0}) \cdot p_{\{i_2,i_3\}}(t_{i_1}), \tag{36}$$

i_0, i_1, i_2, i_3 being pairwise distinct indices in $\{j_0, j_1, j_2, j_3\}$. As the right-hand sides of (35) and (36) are equal, the proof is completed for $n = 3$.
Now let $n \geq 5$: Then

$$a_J \quad = \quad p_{J\setminus\{l,m\}}(t_l) \cdot p_{J\setminus\{l,m\}}(t_m) \cdot a_{\{l,m\}} \cdot a_{J\setminus\{l,m\}} \ +$$

$$+ \sum_{\substack{k\in J\\ k\neq l,m}} p_{J\setminus\{k,l\}}(t_k)\cdot p_{J\setminus\{k,l\}}(t_l)\cdot a_{\{k,l\}}\cdot a_{J\setminus\{k,l\}} \quad =$$

$$= p_{J\setminus\{l,m\}}(t_l)\cdot p_{J\setminus\{l,m\}}(t_m)\cdot a_{\{l,m\}}\cdot a_{J\setminus\{l,m\}} \; +$$

$$+ \sum_{\substack{k\in J\\ k\neq l,m}} p_{J\setminus\{k,l\}}(t_k)\cdot p_{J\setminus\{k,l\}}(t_l)\cdot a_{\{k,l\}}$$

$$\times \sum_{\substack{j\in J\\ j\neq k,l,m}} p_{J\setminus\{j,k,l,m\}}(t_j)\cdot p_{J\setminus\{j,k,l,m\}}(t_m)\cdot a_{\{j,m\}}\cdot a_{J\setminus\{j,k,l,m\}}.$$

On the other hand

$$a_J^* \;=\; p_{J\setminus\{l,m\}}(t_l)\cdot p_{J\setminus\{l,m\}}(t_m)\cdot a_{\{l,m\}}\cdot a_{J\setminus\{l,m\}} \; +$$

$$+ \sum_{\substack{j\in J\\ j\neq l,m}} p_{J\setminus\{j,m\}}(t_j)\cdot p_{J\setminus\{j,m\}}(t_m)\cdot a_{\{j,m\}}\cdot a_{J\setminus\{j,m\}} \quad =$$

$$= p_{J\setminus\{l,m\}}(t_l)\cdot p_{J\setminus\{l,m\}}(t_m)\cdot a_{\{l,m\}}\cdot a_{J\setminus\{l,m\}} \; +$$

$$+ \sum_{\substack{j\in J\\ j\neq l,m}} p_{J\setminus\{j,m\}}(t_j)\cdot p_{J\setminus\{j,m\}}(t_m)\cdot a_{\{j,m\}}$$

$$\times \sum_{\substack{k\in J\\ k\neq j,l,m}} p_{J\setminus\{j,k,l,m\}}(t_k)\cdot p_{J\setminus\{j,k,l,m\}}(t_l)\cdot a_{\{k,l\}}\cdot a_{J\setminus\{j,k,l,m\}}.$$

As furthermore

$$p_{J\setminus\{k,l\}}(t_k)\cdot p_{J\setminus\{k,l\}}(t_l)\cdot p_{J\setminus\{j,k,l,m\}}(t_j)\cdot p_{J\setminus\{j,k,l,m\}}(t_m) \quad =$$

$$= \; p_{J\setminus\{j,m\}}(t_j)\cdot p_{J\setminus\{j,m\}}(t_m)\cdot p_{J\setminus\{j,k,l,m\}}(t_k)\cdot p_{J\setminus\{j,k,l,m\}}(t_l)$$

holds, (a) is proven.
(b) (Proof by induction.) Let π be a permutation on J and let a_J^* denote the value after having applied π.
If $n = 1$ then $a_J^* = a_J$ as $\langle .,. \rangle$ is a *symmetric* billinear form.
Now let n be an odd number ≥ 3. Without loss of generality we can assume that π is a transposition[15], e. g. exchanging the elements l and m of J and leaving the other ones unchanged. Then we have

$$a_J^* = \sum_{\substack{k\in J\\ k\neq l}} p_{J\setminus\{k,l\}}(t_k)\cdot p_{J\setminus\{k,l\}}(t_l)\cdot a_{\{k,l\}}\cdot a^*_{J\setminus\{k,l\}}$$

$$\overset{\text{using the induction hypothesis}}{=}$$

$$\sum_{\substack{k\in J\\ k\neq l}} p_{J\setminus\{k,l\}}(t_k)\cdot p_{J\setminus\{k,l\}}(t_l)\cdot a_{\{k,l\}}\cdot a_{J\setminus\{k,l\}} \quad = \quad a_J \quad \diamond$$

The connection between the terms a_J in definition 3.1 and the linear equation system L_J is given by the following

[15]Every permutation is a composition of transpositions.

Theorem 3.9 *Let $n \in \mathbb{N}$, $n \geq 2$ and $J = \{0, \ldots, n\}$ and let furthermore Δ_J^{kj} denote the determinant which is created by cancelling the k-th row and the j-th column of Δ_J; then*

- *for n odd:*

  (a) $\Delta_J^{kk} = 0$ *for* $k \in J$

  (b) $\Delta_J^{kj} = (-1)^{k+j+\frac{n+1}{2}} \cdot p_{J\setminus\{j,k\}}(t_j) \cdot \prod_{\substack{l \in J \\ l \neq j}} p_{J\setminus\{l\}}(t_l) \cdot a_J \cdot a_{J\setminus\{j,k\}}$ *for* $k, j \in J$, $k \neq j$,

  (c) $\Delta_J = (-1)^{\frac{n+1}{2}} \cdot \prod_{l \in J} p_{J\setminus\{l\}}(t_l) \cdot a_J^2$,

- *for n even:*

  $\Delta_J^{ij} = (-1)^{i+j+\frac{n}{2}} \cdot \prod_{l \in J} p_{J\setminus\{l\}}(t_l) \cdot a_{J\setminus\{i\}} \cdot a_{J\setminus\{j\}}$.

Proof. (by induction). For $n = 2$ the assertion is proved by direct computation.
Induction step for n odd:
(a) $\Delta_J^{kk} = \prod_{\substack{l \in J \\ l \neq k}} (t_l - t_k)^2 \cdot \underbrace{\Delta_{J\setminus\{k\}}}_{= 0 \text{ (theorem 3.1,(b))}} = 0.$
(b) Defining

$$\alpha(i, j; k) := \left\{ \begin{array}{ll} 0 & \text{if } k \text{ lies between } i \text{ and } j \\ 1 & \text{else} \end{array} \right\},$$

$$\beta(i, j; k) := \left\{ \begin{array}{ll} 1 & \text{if } k \text{ lies between } i \text{ and } j \\ 0 & \text{else} \end{array} \right\},$$

we have $\alpha(i, j; k) + \beta(i, j; k) = 1$.
Expanding Δ_J^{kj} by the k-th column we get

$$\Delta_J^{kj} =$$

$$\sum_{\substack{i=0 \\ i \neq k}}^{n} a_{\{i,k\}} \cdot p_{J\setminus\{i,k\}}(t_i) \cdot p_{J\setminus\{k\}}(t_k) \cdot (-1)^{i+k+\alpha(i,j;k)} \cdot \prod_{\substack{l \in J \\ l \neq k,i}} (t_l - t_k) \prod_{\substack{l \in J \\ l \neq k,j}} (t_l - t_k) \cdot \Delta_{J\setminus\{k\}}^{ij}$$

$$\overset{\text{using the induction hypothesis}}{=}$$

$$\sum_{\substack{i \in J \\ i \neq k}} a_{\{i,k\}} \cdot p_{J\setminus\{i,k\}}(t_i) \cdot p_{J\setminus\{i,k\}}(t_k) \cdot p_{J\setminus\{k\}}(t_k) \cdot p_{J\setminus\{k,j\}}(t_k) \cdot (-1)^{i+k+\alpha(i,j;k)}$$

$$\times \left[(-1)^{i+j+\beta(i,j;k)+\frac{n-1}{2}} \cdot p_{J\setminus\{j,k\}}(t_j) \prod_{\substack{l \in J \\ l \neq j,k}} p_{J\setminus\{l,k\}}(t_l) \cdot a_{J\setminus\{i,k\}} \cdot a_{J\setminus\{j,k\}} \right] =$$

$$(-1)^{k+j+\frac{n+1}{2}} \cdot p_{J\setminus\{j,k\}}(t_j) \cdot \overbrace{p_{J\setminus\{k\}}(t_k) \cdot p_{J\setminus\{j,k\}}(t_k) \cdot \prod_{\substack{l \in J \\ l \neq j,k}} p_{J\setminus\{l,k\}}(t_l)}^{= \prod_{\substack{l \in J \\ l \neq j}} p_{J\setminus\{l\}}(t_l)} \cdot a_{J\setminus\{j,k\}}$$

$$\times \underbrace{\sum_{\substack{i\in J\\ i\neq k}} p_{J\setminus\{i,k\}}(t_i)\cdot p_{J\setminus\{i,k\}}(t_k)\cdot a_{\{i,k\}}\cdot a_{J\setminus\{i,k\}}}_{=\ a_J}.$$

(c) Expanding Δ_J by the j-th column we get

$$\Delta_J \;=\; \sum_{\substack{k\in J\\ k\neq j}} a_{\{k,j\}}\cdot p_{J\setminus\{k,j\}}(t_k)\cdot p_{J\setminus\{j\}}(t_j)\cdot(-1)^{k+j}\cdot\Delta_J^{kj}$$

$$\stackrel{\text{using b)}}{=}$$

$$\sum_{\substack{k\in J\\ k\neq j}}^{n} p_{J\setminus\{k,j\}}(t_k)\cdot p_{J\setminus\{j\}}(t_j)\cdot a_{\{k,j\}}\cdot(-1)^{\frac{n+1}{2}}\cdot p_{J\setminus\{k,j\}}(t_j)\cdot\prod_{\substack{l\in J\\ l\neq j}} p_{J\setminus\{l\}}(t_l)\cdot a_J\cdot a_{J\setminus\{j,k\}} \;=$$

$$(-1)^{\frac{n+1}{2}}\cdot\prod_{l\in J} p_{J\setminus\{l\}}(t_l)\cdot a_J^2.$$

Induction step for n even:
If $i=j$ we have

$$\Delta_J^{jj} \;=\; \prod_{\substack{l\in J\\ l\neq j}}(t_l-t_j)^2\cdot\Delta_{J\setminus\{j\}}$$

$$\stackrel{\text{using the induction hypothesis}}{=}$$

$$(-1)^{\frac{n}{2}}\cdot\underbrace{\prod_{\substack{l\in J\\ l\neq j}}(t_l-t_j)^2\cdot\prod_{\substack{l\in J\\ l\neq j}} p_{J\setminus\{j,l\}}(t_l)}_{=\ \prod_{l\in J} p_{J\setminus\{l\}}(t_l)}\cdot a_{J\setminus\{j\}}^2.$$

If $i\neq j$, we are able to expand Δ_J^{ij} by the i-th column:

$$\Delta_J^{ij} \;=\; \sum_{\substack{k\in J\\ k\neq i}} p_{J\setminus\{i,k\}}(t_k)\cdot p_{J\setminus\{i\}}(t_i)\cdot a_{\{k,i\}}\cdot(-1)^{i+k+\alpha(k,j;i)}\cdot\underbrace{\prod_{\substack{l\in J\\ l\neq i,k}}(t_l-t_i)}_{=-p_{J\setminus\{i,k\}}(t_i)}\cdot\underbrace{\prod_{\substack{l\in J\\ l\neq i,j}}(t_l-t_i)}_{=-p_{J\setminus\{i,j\}}(t_i)}\cdot\Delta_{J\setminus\{i\}}^{kj}$$

$$\stackrel{\text{using the induction hypothesis}}{=}$$

$$\sum_{\substack{k\in J\\ k\neq i,j}} p_{J\setminus\{i,k\}}(t_i)\cdot p_{J\setminus\{i,k\}}(t_k)\cdot a_{\{i,k\}}\cdot(-1)^{i+k+\alpha(k,j;i)}\cdot p_{J\setminus\{i,j,k\}}(t_j)\cdot p_{J\setminus\{i\}}(t_i)\cdot p_{J\setminus\{i,j\}}(t_i)$$

$$\times\left[\prod_{\substack{l\in J\\ l\neq i,j}} p_{J\setminus\{i,l\}}(t_l)\cdot(-1)^{j+k+\beta(k,j;i)+\frac{n}{2}}\cdot a_{J\setminus\{i\}}\cdot a_{J\setminus\{i,j,k\}}\right] \;=$$

$$\overbrace{\sum_{\substack{k\in J\\k\neq i,j}} p_{J\setminus\{i,j,k\}}(t_i)\cdot p_{J\setminus\{i,j,k\}}(t_k)\cdot a_{\{i,k\}}\cdot a_{J\setminus\{i,j,k\}}}^{=\, a_{J\setminus\{j\}}}\cdot(-1)^{i+j+1+\frac{n}{2}}$$

$$\underbrace{\times\,(t_i-t_j)\cdot(t_k-t_j)\cdot p_{J\setminus\{i,j,k\}}(t_j)p_{J\setminus\{i\}}(t_i)\cdot p_{J\setminus\{i,j\}}(t_i)\cdot\prod_{\substack{l\in J\\l\neq j,i}} p_{J\setminus\{i,l\}}(t_l)}_{=\,-\prod_{l\in J} p_{J\setminus\{l\}}(t_l)}\cdot a_{J\setminus\{i\}}\quad\diamond$$

The next theorem shows that in general no solution curve exists for IP_J if n is odd.

Theorem 3.10 *If n is an odd positive integer and $a_J\neq 0$ then there exists no solution curve for IP_J.*

Proof. As $\Delta_J=(-1)^{\frac{n+1}{2}}\cdot\prod_{l\in J}p_{J\setminus\{l\}}(t_l)\cdot a_J^2\neq 0$ (theorem 3.9, (c)), the linear equation sytem L_J only has the trivial solution $w_0=0,\ldots,w_n=0$ $\diamond$

For n even we have the following

Theorem 3.11 *If n is an even positive integer and $\forall i\in J:\ a_{J\setminus\{i\}}\neq 0$ then*

(a) there is exactly one solution curve c for IP_J and $o(c)=n$.

(b) the Lagrange-representation of c is

$$c\ldots\mathbf{x}(t)\;=\;\sum_{i\in J}p_{J\setminus\{i\}}(t)\cdot p_{J\setminus\{i\}}(t_i)\cdot a_{J\setminus\{i\}}\cdot\mathbf{a}_i \tag{37}$$

Proof. Due to theorem 3.9 none of the subdeterminants Δ_J^{ij} vanishes; so rank $\mathbf{C}_J=n$. Thus the solution of L_J is a uniquely determined homogeneous $(n+1)$-tupel $(w_0,\ldots,w_n)^t$ and can for instance be computed via

$$w_0:\ldots:w_i:\ldots:w_n\;=\;(-1)^0\cdot\Delta_J^{n0}:\ldots:(-1)^i\cdot\Delta_J^{ni}:\ldots:(-1)^n\cdot\Delta_J^{nn}$$

$$\overset{\text{using theorem 3.9}}{=}$$

$$a_{J\setminus\{0\}}:\ldots:a_{J\setminus\{i\}}:\ldots:a_{J\setminus\{n\}}.$$

So the curve

$$c\ldots\mathbf{x}(t)\;=\;\sum_{i\in J}p_{J\setminus\{i\}}(t)\cdot p_{J\setminus\{i\}}(t_i)\cdot a_{J\setminus\{i\}}\cdot\mathbf{a}_i$$

is the only solution curve. Furthermore we trivially have $n\geq o(c)$ and via theorem 3.5 also $o(c)\geq$ rank $\mathbf{C}_J=n$, which in combination give $o(c)=n$ $\diamond$

Theorem 3.11 shows that generally there exists a (uniquely determined) solution curve of L_J if n is even. Moreover it also supplies us with an algorithm for the computation of the solution curve c:

Algorithm 3.1 *Let n be an even positive integer, $J = \{0, \ldots, n\}$, $t_0, \ldots, t_n$ be pairwise distinct values in $\mathbb{R}$ and $A_0, \ldots, A_n$ be points on a hyperquadric $\mathcal{Q}^{d-1} \subset \mathbb{P}^d$ with homogeneous coordinate vectors $\mathbf{a}_0, \ldots, \mathbf{a}_n$.*

1. *Compute $a_{J\setminus\{i\}}$ via the recursion given in definition 3.1.*
2. *If $\forall i \in J: \; a_{J\setminus\{i\}} \neq 0$ then*

$$c \ldots \mathbf{x}(t) \;=\; \sum_{i\in J} p_{J\setminus\{i\}}(t) \cdot p_{J\setminus\{i\}}(t_i) \cdot a_{J\setminus\{i\}} \cdot \mathbf{a}_i$$

is the (uniquely determined) univariate rational interpolant on the hyperquadric $\mathcal{Q}^{d-1}$ for the points A_i and the corresponding parameter values t_i.

If $\mathcal{Q}^{d-1}$ is an oval[16] hyperquadric the following result can be obtained:

Theorem 3.12 *Let $\mathcal{Q}^{d-1}$ be an oval hyperquadric; then the interpolation problem IP_J*

(a) has exactly one or none solution curve.

(b) has a solution curve if and only if there exists an odd integer $\overline{n}$, $1 \le \overline{n} \le n$ with

$$(1)\;\; \forall J^* := \{i_0, \ldots i_{\overline{n}}\} \subset J \;\text{ with }\; i_k \neq i_l \;\text{ for }\; k \neq l: \; a_{J^*} \neq 0.$$

$$(2)\;\; \forall J^{**} := \{i_0, \ldots i_{\overline{n}+2}\} \subset J \;\text{ with }\; i_k \neq i_l \;\text{ for }\; k \neq l: \; a_{J^{**}} = 0.$$

If (1) and (2) are fulfilled the algebraic order of the solution curve is $\overline{n} + 1$.

The proof for this theorem is given in [Gfre 1999]. In section 3.4 we will see that this result cannot be extended to arbitrary hyperquadrics.

3.3 A geometric algorithm

It is well known that in d-dimensional (real) affine space $\mathbb{A}^d$ one can construct exactly one *polynomial* interpolant (Lagrange-interpolant) for $n+1$ points $A_0, \ldots, A_n$ and corresponding parameter values $t_0, \ldots, t_n$. This interpolant has the parametrization

$$\mathbf{x}(t) \;=\; \sum_{i=0}^{n} l_{J\setminus\{i\}}(t) \cdot \mathbf{a}_i, \tag{38}$$

where $J := \{0, \ldots, n\}$ and $\mathbf{a}_i$ denotes the *affine* coordinate-vector of A_i. The functions $l_{J\setminus\{i\}}(t)$ are the Lagrange-polynomials:

$$l_{J\setminus\{i\}}(t) \;=\; \frac{p_{J\setminus\{i\}}(t)}{p_{J\setminus\{i\}}(t_i)}. \tag{39}$$

[16] A hyperquadric is of oval type if it is regular (see remark 2.1, (d)) and does not contain any real straight line. Oval hyperquadrics are projectively equivalent to the unit hypersphere: $\mathcal{S}^{d-1} \ldots \sum_{i=1}^{d} x_i^2 \; - \; x_0^2 \;=\; 0.$

Aitken's algorithm (see figure 3.1) provides a geometric construction of the Lagrange-interpolant.[17] The algorithm is based on repeated subdivision. The affine coordinate-vectors $\mathbf{a}_{i,l}(t)$ of the points $A_{i,l}(t)$ occuring in the algorithm are defined recursively by

$$\begin{aligned} \mathbf{a}_{i,0}(t) &:= \mathbf{a}_i & (40)\\ \mathbf{a}_{i,l}(t) &:= (1-\alpha(t,i,l))\cdot\mathbf{a}_{i,l-1}(t)+\alpha(t,i,l)\cdot\mathbf{a}_{i+1,l-1}(t), & (41)\\ & \quad\; l\in\{1,\ldots,n\},\;\; i\in\{0,\ldots,n-l\} & \\ \text{where }\alpha(t,i,l) &= \frac{t-t_i}{t_{i+l}-t_i}. & (42) \end{aligned}$$

Geometrically this means that

- $A_{i,l}(t)$ is on the line $[A_{i,l-1}(t), A_{i+1,l-1}(t)]$ and
- the ratios $(A_{i,l-1}(t), A_{i+1,l-1}(t); A_{i,l}(t))$ and $(t_i, t_{i+l}; t)$ are identical.

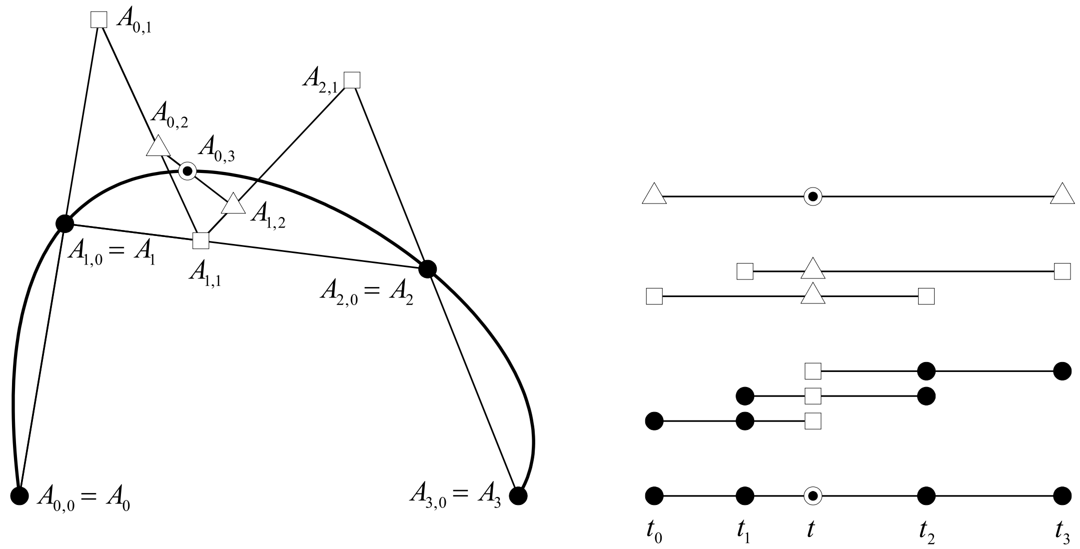

Figure 3.1 *Aitken's algorithm for constructing the polynomial interpolant in affine space.*

The aim of this section is to develop a subdivision-algorithm for *rational* interpolants on a hyperquadric $\mathcal{Q}^{d-1}$ in $d-$dimensional projective space $\mathbb{P}^d$. Obviously the concepts of Aitkens algorithm will not be useful here, as

- the line determined by two points on $\mathcal{Q}^{d-1}$ is in general not part of $\mathcal{Q}^{d-1}$ and

[17] See [Farin 1990, pages 67–70].

- the ratio of three points is an affine invariant and does not have any geometric meaning in projective space.

So, one could have the idea

- to consider triples of points on $\mathcal{Q}^{d-1}$ instead of pairs: In general three points on a hyperquadric determine a plane which intersects $\mathcal{Q}^{d-1}$ in a second-order curve c and
- to take the cross-ratio of four points on the conic section c instead of the ratio of three points on a line, as the first is a projective invariant.

Before defining and proving the algorithm we need some more properties of rational interpolants on a hyperquadric. These properties are given in the following lemmata 3.2, 3.3 and 3.4.

Lemma 3.2 *Let $\mathcal{Q}^{d-1}$ be a hyperquadric in d-dimensional projective space $\mathbb{P}^d$ and let n be an even positive integer ≥ 4; Let furthermore $J := \{j_0, \ldots, j_n\} \subset \mathbb{N}$ and let t_k be pairwise distinct values in $\mathbb{R}$ and A_k be points on $\mathcal{Q}^{d-1}$ with homogeneous coordinate vectors $\mathbf{a}_k$, $k \in J$.*
Then for any four pairwise distinct indices l_0, l_1, l_2, l_3 in J the following equation holds:

$$\begin{array}{lclcl} p_{\{l_1,l_2,l_3\}}(t_{l_0}) \cdot a_{J\setminus\{l_1,l_2,l_3\}} \cdot a_{J\setminus\{l_0\}} & + & p_{\{l_0,l_2,l_3\}}(t_{l_1}) \cdot a_{J\setminus\{l_0,l_2,l_3\}} \cdot a_{J\setminus\{l_1\}} & + & \\ p_{\{l_0,l_1,l_3\}}(t_{l_2}) \cdot a_{J\setminus\{l_0,l_1,l_3\}} \cdot a_{J\setminus\{l_2\}} & + & p_{\{l_0,l_1,l_2\}}(t_{l_3}) \cdot a_{J\setminus\{l_0,l_1,l_2\}} \cdot a_{J\setminus\{l_3\}} & = & 0. \end{array}$$

Proof. (by induction).
Initial step ($n = 4$):
Let $(l_0, l_1, l_2, l_3, l_4)$ be an arbitrary permutation of $J = \{j_0, j_1, j_2, j_3, j_4\}$; then

$$\begin{array}{lclc} p_{\{l_1,l_2,l_3\}}(t_{l_0}) \cdot a_{\{l_0,l_4\}} \cdot a_{\{l_1,l_2,l_3,l_4\}} & + & p_{\{l_0,l_2,l_3\}}(t_{l_1}) \cdot a_{\{l_1,l_4\}} \cdot a_{\{l_0,l_2,l_3,l_4\}} & + \\ p_{\{l_0,l_1,l_3\}}(t_{l_2}) \cdot a_{\{l_2,l_4\}} \cdot a_{\{l_0,l_1,l_3,l_4\}} & + & p_{\{l_0,l_1,l_2\}}(t_{l_3}) \cdot a_{\{l_3,l_4\}} \cdot a_{\{l_0,l_1,l_2,l_4\}} & \end{array}$$

$$\overset{\text{via definition 3.1}}{=}$$

$$\begin{array}{rlc} p_{\{l_1,l_2,l_3\}}(t_{l_0}) \cdot a_{\{l_0,l_4\}} \cdot [& p_{\{l_1,l_2\}}(t_{l_3}) \cdot p_{\{l_1,l_2\}}(t_{l_4}) \cdot a_{\{l_1,l_2\}} \cdot a_{\{l_3,l_4\}} & + \\ & p_{\{l_1,l_3\}}(t_{l_2}) \cdot p_{\{l_1,l_3\}}(t_{l_4}) \cdot a_{\{l_1,l_3\}} \cdot a_{\{l_2,l_4\}} & + \\ & p_{\{l_1,l_4\}}(t_{l_2}) \cdot p_{\{l_1,l_4\}}(t_{l_3}) \cdot a_{\{l_1,l_4\}} \cdot a_{\{l_2,l_3\}}] & + \\ p_{\{l_0,l_2,l_3\}}(t_{l_1}) \cdot a_{\{l_1,l_4\}} \cdot [& p_{\{l_0,l_2\}}(t_{l_3}) \cdot p_{\{l_0,l_2\}}(t_{l_4}) \cdot a_{\{l_0,l_2\}} \cdot a_{\{l_3,l_4\}} & + \\ & p_{\{l_0,l_3\}}(t_{l_2}) \cdot p_{\{l_0,l_3\}}(t_{l_4}) \cdot a_{\{l_0,l_3\}} \cdot a_{\{l_2,l_4\}} & + \\ & p_{\{l_0,l_4\}}(t_{l_2}) \cdot p_{\{l_0,l_4\}}(t_{l_3}) \cdot a_{\{l_0,l_4\}} \cdot a_{\{l_2,l_3\}}] & + \\ p_{\{l_0,l_1,l_3\}}(t_{l_2}) \cdot a_{\{l_2,l_4\}} \cdot & p_{\{l_0,l_1\}}(t_{l_3}) \cdot p_{\{l_0,l_1\}}(t_{l_4}) \cdot a_{\{l_0,l_1\}} \cdot a_{\{l_3,l_4\}} & + \\ & p_{\{l_0,l_3\}}(t_{l_1}) \cdot p_{\{l_0,l_3\}}(t_{l_4}) \cdot a_{\{l_0,l_3\}} \cdot a_{\{l_1,l_4\}} & + \\ & p_{\{l_0,l_4\}}(t_{l_1}) \cdot p_{\{l_0,l_4\}}(t_{l_3}) \cdot a_{\{l_0,l_4\}} \cdot a_{\{l_1,l_3\}}] & + \\ p_{\{l_0,l_1,l_2\}}(t_{l_3}) \cdot a_{\{l_3,l_4\}} \cdot [& p_{\{l_0,l_1\}}(t_{l_2}) \cdot p_{\{l_0,l_1\}}(t_{l_4}) \cdot a_{\{l_0,l_1\}} \cdot a_{\{l_2,l_4\}} & + \\ & p_{\{l_0,l_2\}}(t_{l_1}) \cdot p_{\{l_0,l_2\}}(t_{l_4}) \cdot a_{\{l_0,l_2\}} \cdot a_{\{l_1,l_4\}} & + \\ & p_{\{l_0,l_4\}}(t_{l_1}) \cdot p_{\{l_0,l_4\}}(t_{l_2}) \cdot a_{\{l_0,l_4\}} \cdot a_{\{l_1,l_2\}}], & \end{array}$$

which is equal to zero, as a short computation shows: e. g. the coefficient of $a_{\{l_0,l_4\}} \cdot a_{\{l_1,l_2\}} \cdot a_{\{l_3,l_4\}}$ is

$$\begin{array}{rcl} p_{\{l_1,l_2,l_3\}}(t_{l_0}) \cdot p_{\{l_1,l_2\}}(t_{l_3}) \cdot p_{\{l_1,l_2\}}(t_{l_4}) \; + \; p_{\{l_0,l_1,l_2\}}(t_{l_3}) \cdot p_{\{l_0,l_4\}}(t_{l_1}) \cdot p_{\{l_0,l_4\}}(t_{l_2}) & = & \\ (t_{l_0}-t_{l_1}) \cdot (t_{l_0}-t_{l_2}) \cdot (t_{l_0}-t_{l_3}) \cdot (t_{l_3}-t_{l_1}) \cdot (t_{l_3}-t_{l_2}) \cdot (t_{l_4}-t_{l_1}) \cdot (t_{l_4}-t_{l_2}) & + & \\ (t_{l_3}-t_{l_0}) \cdot (t_{l_3}-t_{l_1}) \cdot (t_{l_3}-t_{l_2}) \cdot (t_{l_1}-t_{l_0}) \cdot (t_{l_1}-t_{l_4}) \cdot (t_{l_2}-t_{l_0}) \cdot (t_{l_2}-t_{l_4}) & = & 0. \end{array}$$

Induction step (n even and ≥ 6):

$$\begin{array}{llll} p_{\{l_1,l_2,l_3\}}(t_{l_0}) \cdot a_{J\setminus\{l_1,l_2,l_3\}} \cdot a_{J\setminus\{l_0\}} & + & p_{\{l_0,l_2,l_3\}}(t_{l_1}) \cdot a_{J\setminus\{l_0,l_2,l_3\}} \cdot a_{J\setminus\{l_1\}} & + \\ p_{\{l_0,l_1,l_3\}}(t_{l_2}) \cdot a_{J\setminus\{l_0,l_1,l_3\}} \cdot a_{J\setminus\{l_2\}} & + & p_{\{l_0,l_1,l_2\}}(t_{l_3}) \cdot a_{J\setminus\{l_0,l_1,l_2\}} \cdot a_{J\setminus\{l_3\}} & \end{array}$$

$$\stackrel{\text{via definition 3.1}}{=}$$

$$\begin{array}{r} p_{\{l_1,l_2,l_3\}}(t_{l_0}) \cdot a_{J\setminus\{l_0\}} \\ \times \sum_{\substack{i\in J \\ i\neq l_0,l_1,l_2,l_3}} p_{J\setminus\{i,l_0,l_1,l_2,l_3\}}(t_i) \cdot p_{J\setminus\{i,l_0,l_1,l_2,l_3\}}(t_{l_0}) \cdot a_{\{i,l_0\}} \cdot a_{J\setminus\{i,l_0,l_1,l_2,l_3\}} \quad + \\ p_{\{l_0,l_2,l_3\}}(t_{l_1}) \cdot a_{J\setminus\{l_0,l_2,l_3\}} \cdot \sum_{\substack{i\in J \\ i\neq l_0,l_1}} p_{J\setminus\{i,l_0,l_1\}}(t_i) \cdot p_{J\setminus\{i,l_0,l_1\}}(t_{l_0}) \cdot a_{\{i,l_0\}} \cdot a_{J\setminus\{i,l_0,l_1\}} \quad + \\ p_{\{l_0,l_1,l_3\}}(t_{l_2}) \cdot a_{J\setminus\{l_0,l_1,l_3\}} \cdot \sum_{\substack{i\in J \\ i\neq l_0,l_2}} p_{J\setminus\{i,l_0,l_2\}}(t_i) \cdot p_{J\setminus\{i,l_0,l_2\}}(t_{l_0}) \cdot a_{\{i,l_0\}} \cdot a_{J\setminus\{i,l_0,l_2\}} \quad + \\ p_{\{l_0,l_1,l_2\}}(t_{l_3}) \cdot a_{J\setminus\{l_0,l_1,l_2\}} \cdot \sum_{\substack{i\in J \\ i\neq l_0,l_3}} p_{J\setminus\{i,l_0,l_3\}}(t_i) \cdot p_{J\setminus\{i,l_0,l_3\}}(t_{l_0}) \cdot a_{\{i,l_0\}} \cdot a_{J\setminus\{i,l_0,l_3\}}. \end{array}$$

In this expression the terms for the summation-indices l_1, l_2, l_3 are equal to zero: e. g. for $i = l_1$ we obtain

$$\begin{array}{l} a_{\{l_0,l_1\}} \cdot a_{J\setminus\{l_0,l_1,l_2\}} \cdot a_{J\setminus\{l_0,l_1,l_3\}} \cdot p_{J\setminus\{l_0,l_1,l_2,l_3\}}(t_{l_1}) \cdot p_{J\setminus\{l_0,l_1,l_2,l_3\}}(t_{l_0}) \\ \quad \times[(t_{l_2}-t_{l_0}) \cdot (t_{l_2}-t_{l_1}) \cdot (t_{l_2}-t_{l_3}) \cdot (t_{l_1}-t_{l_3}) \cdot (t_{l_0}-t_{l_3}) \quad + \\ \quad\quad (t_{l_3}-t_{l_0}) \cdot (t_{l_3}-t_{l_1}) \cdot (t_{l_3}-t_{l_2}) \cdot (t_{l_1}-t_{l_2}) \cdot (t_{l_0}-t_{l_2})] \quad = \quad 0. \end{array}$$

So what remains to be shown is that for all $i \in J \setminus \{l_0, l_1, l_2, l_3\}$ the expression

$$\begin{array}{r} p_{\{l_1,l_2,l_3\}}(t_{l_0}) \cdot p_{J\setminus\{i,l_0,l_1,l_2,l_3\}}(t_i) \cdot p_{J\setminus\{i,l_0,l_1,l_2,l_3\}}(t_i) \cdot a_{J\setminus\{l_0\}} \cdot a_{J\setminus\{i,l_0,l_1,l_2,l_3\}} \quad + \\ p_{\{l_0,l_2,l_3\}}(t_{l_1}) \cdot p_{J\setminus\{i,l_0,l_1\}}(t_{l_0}) \cdot p_{J\setminus\{i,l_0,l_1\}}(t_i) \cdot a_{J\setminus\{l_0,l_2,l_3\}} \cdot a_{J\setminus\{i,l_0,l_1\}} \quad + \\ p_{\{l_0,l_1,l_3\}}(t_{l_2}) \cdot p_{J\setminus\{i,l_0,l_2\}}(t_{l_0}) \cdot p_{J\setminus\{i,l_0,l_2\}}(t_i) \cdot a_{J\setminus\{l_0,l_1,l_3\}} \cdot a_{J\setminus\{i,l_0,l_2\}} \quad + \\ p_{\{l_0,l_1,l_2\}}(t_{l_3}) \cdot p_{J\setminus\{i,l_0,l_3\}}(t_{l_0}) \cdot p_{J\setminus\{i,l_0,l_3\}}(t_i) \cdot a_{J\setminus\{l_0,l_1,l_2\}} \cdot a_{J\setminus\{i,l_0,l_3\}} \quad \end{array} \tag{43}$$

is equal to zero. After substitution of

$$\begin{array}{rcl} p_{\{l_1,l_2,l_3\}}(t_{l_0}) \cdot p_{J\setminus\{i,l_0,l_1,l_2,l_3\}}(t_{l_0}) & = & p_{J\setminus\{i,l_0\}}(t_{l_0}), \\ p_{\{l_0,l_2,l_3\}}(t_{l_1}) \cdot p_{J\setminus\{i,l_0,l_1\}}(t_{l_0}) & = & -(t_{l_1}-t_{l_2}) \cdot (t_{l_1}-t_{l_3}) \cdot p_{J\setminus\{i,l_0\}}(t_{l_0}), \\ p_{J\setminus\{i,l_0,l_1\}}(t_i) & = & (t_i-t_{l_2}) \cdot (t_i-t_{l_3}) \cdot p_{J\setminus\{i,l_0,l_1,l_2,l_3\}}(t_i), \\ p_{\{l_0,l_1,l_3\}}(t_{l_2}) \cdot p_{J\setminus\{i,l_0,l_2\}}(t_{l_0}) & = & -(t_{l_2}-t_{l_1}) \cdot (t_{l_2}-t_{l_3}) \cdot p_{J\setminus\{i,l_0\}}(t_{l_0}), \\ p_{J\setminus\{i,l_0,l_2\}}(t_i) & = & (t_i-t_{l_1}) \cdot (t_i-t_{l_3}) \cdot p_{J\setminus\{i,l_0,l_1,l_2,l_3\}}(t_i), \\ p_{\{l_0,l_1,l_2\}}(t_{l_3}) \cdot p_{J\setminus\{i,l_0,l_3\}}(t_{l_0}) & = & -(t_{l_3}-t_{l_1}) \cdot (t_{l_3}-t_{l_2}) \cdot p_{J\setminus\{i,l_0\}}(t_{l_0}) \text{ and} \\ p_{J\setminus\{l_0,l_3,i\}}(t_i) & = & (t_i-t_{l_1}) \cdot (t_i-t_{l_2}) \cdot p_{J\setminus\{i,l_0,l_1,l_2,l_3\}}(t_i) \end{array}$$

into (43) we see that it suffices to show that for all $i \in J \setminus \{l_0, l_1, l_2, l_3\}$ the following equation is valid:

$$\begin{aligned} a_{J\setminus\{l_0\}} \cdot a_{J\setminus\{i,l_0,l_1,l_2,l_3\}} \quad = \quad & p_{\{l_2,l_3\}}(t_{l_1}) \cdot p_{\{l_2,l_3\}}(t_i) \cdot a_{J\setminus\{l_0,l_2,l_3\}} \cdot a_{J\setminus\{i,l_0,l_1\}} \quad + \\ & p_{\{l_1,l_3\}}(t_{l_2}) \cdot p_{\{l_1,l_3\}}(t_i) \cdot a_{J\setminus\{l_0,l_1,l_3\}} \cdot a_{J\setminus\{i,l_0,l_2\}} \quad + \\ & p_{\{l_1,l_2\}}(t_{l_3}) \cdot p_{\{l_1,l_2\}}(t_i) \cdot a_{J\setminus\{l_0,l_1,l_2\}} \cdot a_{J\setminus\{i,l_0,l_3\}}. \end{aligned} \tag{44}$$

We substitute[18]

$$a_{J\setminus\{l_0\}} = \sum_{\substack{k\in J\\ k\neq l_0,l_1}} p_{J\setminus\{k,l_0,l_1\}}(t_k) \cdot p_{J\setminus\{k,l_0,l_1\}}(t_{l_1}) \cdot a_{\{k,l_1\}} \cdot a_{J\setminus\{k,l_0,l_1\}},$$

$$a_{J\setminus\{l_0,l_2,l_3\}} = \sum_{\substack{k\in J\\ k\neq l_0,l_1,l_2,l_3}} p_{J\setminus\{k,l_0,l_1,l_2,l_3\}}(t_k) \cdot p_{J\setminus\{k,l_0,l_1,l_2,l_3\}}(t_{l_1}) \cdot a_{\{k,l_1\}} \cdot a_{J\setminus\{k,l_0,l_1,l_2,l_3\}},$$

$$a_{J\setminus\{i,l_0,l_2\}} = \sum_{\substack{k\in J\\ k\neq i,l_0,l_1,l_2}} p_{J\setminus\{i,k,l_0,l_1,l_2\}}(t_k) \cdot p_{J\setminus\{i,k,l_0,l_1,l_2\}}(t_{l_1}) \cdot a_{\{k,l_1\}} \cdot a_{J\setminus\{i,k,l_0,l_1,l_2\}},$$

$$a_{J\setminus\{i,l_0,l_3\}} = \sum_{\substack{k\in J\\ k\neq i,l_0,l_1,l_3}} p_{J\setminus\{i,k,l_0,l_1,l_3\}}(t_k) \cdot p_{J\setminus\{i,k,l_0,l_1,l_3\}}(t_{l_1}) \cdot a_{\{k,l_1\}} \cdot a_{J\setminus\{i,k,l_0,l_1,l_3\}}$$

into (44) and prove that the left- and right-hand sides of (44) for the summation indices $k = l_2, l_3, i$ are equal:

- $k = l_2$:

  left-hand side:

$$p_{J\setminus\{l_0,l_1,l_2\}}(t_{l_2}) \cdot p_{J\setminus\{l_0,l_1,l_2\}}(t_{l_1}) \cdot a_{\{l_1,l_2\}} \cdot a_{J\setminus\{l_0,l_1,l_2\}} \cdot a_{J\setminus\{i,l_0,l_1,l_2,l_3\}}.$$

  right-hand side:

$$\overbrace{p_{J\setminus\{i,l_0,l_1,l_2,l_3\}}(t_{l_2}) \cdot p_{J\setminus\{i,l_0,l_1,l_2,l_3\}}(t_{l_1}) \cdot p_{J\{l_1,l_2\}}(t_{l_1}) \cdot p_{J\{l_1,l_2\}}(t_i)}^{=\ p_{J\setminus\{l_0,l_1,l_2\}}(t_{l_2}) \cdot p_{J\setminus\{l_0,l_1,l_2\}}(t_{l_1})}$$
$$\times a_{\{l_1,l_2\}} \cdot a_{J\setminus\{l_0,l_1,l_2\}} \cdot a_{J\setminus\{i,l_0,l_1,l_2,l_3\}}.$$

- $k = l_3$:

  left-hand side:

$$p_{J\setminus\{l_0,l_1,l_3\}}(t_{l_3}) \cdot p_{J\setminus\{l_0,l_1,l_3\}}(t_{l_1}) \cdot a_{\{l_1,l_3\}} \cdot a_{J\setminus\{l_0,l_1,l_3\}} \cdot a_{J\setminus\{i,l_0,l_1,l_2,l_3\}}.$$

  right-hand side:

$$\overbrace{p_{J\setminus\{i,l_0,l_1,l_2,l_3\}}(t_{l_3}) \cdot p_{J\setminus\{i,l_0,l_1,l_2,l_3\}}(t_{l_1}) \cdot p_{\{l_1,l_3\}}(t_{l_2}) \cdot p_{\{l_1,l_3\}}(t_i)}^{=\ p_{J\setminus\{l_0,l_1,l_3\}}(t_{l_3}) \cdot p_{J\setminus\{l_0,l_1,l_3\}}(t_{l_1})}$$
$$\times a_{\{l_1,l_3\}} \cdot a_{J\setminus\{l_0,l_1,l_3\}} \cdot a_{J\setminus\{i,l_0,l_1,l_2,l_3\}}.$$

[18]See definition 3.1.

- $k = i$:

  left-hand side:

$$p_{J\setminus\{i,l_0,l_1\}}(t_i) \cdot p_{J\setminus\{i,l_0,l_1\}}(t_{l_1}) \cdot a_{\{i,l_1\}} \cdot a_{J\setminus\{i,l_0,l_1\}} \cdot a_{J\setminus\{i,l_0,l_1,l_2,l_3\}}.$$

  right-hand side:

$$\overbrace{p_{J\setminus\{i,l_0,l_1,l_2,l_3\}}(t_i) \cdot p_{J\setminus\{i,l_0,l_1,l_2,l_3\}}(t_{l_1}) \cdot p_{\{l_2,l_3\}}(t_i) \cdot p_{\{l_2,l_3\}}(t_{l_1})}^{=\ p_{J\setminus\{i,l_0,l_1\}}(t_i)\cdot p_{J\setminus\{i,l_0,l_1\}}(t_{l_1})}$$

$$\times a_{\{i,l_1\}} \cdot a_{J\setminus\{i,l_0,l_1\}} \cdot a_{J\setminus\{i,l_0,l_1,l_2,l_3\}}.$$

Thus (44) can also be written in the form

$$\begin{aligned}
\sum_{\substack{k\in J \\ k\neq i,l_0,l_1,l_2,l_3}} p_{J\setminus\{k,l_0,l_1\}}(t_k) \cdot p_{J\setminus\{k,l_0,l_1\}}(t_{l_1}) \cdot a_{\{k,l_1\}} \cdot a_{J\setminus\{k,l_0,l_1\}} \cdot a_{J\setminus\{i,l_0,l_1,l_2,l_3\}} &= \\
\sum_{\substack{k\in J \\ k\neq i,l_0,l_1,l_2,l_3}} [p_{\{l_2,l_3\}}(t_i) \cdot p_{\{l_2,l_3\}}(t_{l_1}) \cdot p_{\{k,l_0,l_1,l_2,l_3\}}(t_k) \cdot p_{\{k,l_0,l_1,l_2,l_3\}}(t_{l_1}) & \\
\times a_{\{k,l_1\}} \cdot a_{J\setminus\{i,l_0,l_1\}} \cdot a_{J\setminus\{k,l_0,l_1,l_2,l_3\}} &+ \\
p_{\{l_1,l_3\}}(t_i) \cdot p_{\{l_1,l_3\}}(t_{l_2}) \cdot p_{\{i,k,l_0,l_1,l_2\}}(t_k) \cdot p_{\{i,k,l_0,l_1,l_2\}}(t_{l_1}) & \\
\times a_{\{k,l_1\}} \cdot a_{J\setminus\{l_0,l_1,l_3\}} \cdot a_{J\setminus\{i,k,l_0,l_1,l_2\}} &+ \\
p_{\{l_1,l_2\}}(t_i) \cdot p_{\{l_1,l_2\}}(t_{l_3}) \cdot p_{\{i,k,l_0,l_1,l_3\}}(t_k) \cdot p_{\{i,k,l_0,l_1,l_3\}}(t_{l_1}) & \\
\times a_{\{k,l_1\}} \cdot a_{J\setminus\{l_0,l_1,l_2\}} \cdot a_{J\setminus\{i,k,l_0,l_1,l_3\}}]. &
\end{aligned}$$

or simplified

$$\sum_{\substack{k\in J \\ k\neq i,l_0,l_1,l_2,l_3}} p_{\{i,l_2,l_3\}}(t_k) \cdot p_{J\setminus\{i,k,l_0,l_1,l_2,l_3\}}(t_k) \cdot p_{J\setminus\{i,k,l_0,l_1,l_2,l_3\}}(t_{l_1}) \cdot e \quad = \quad 0,$$

with

$$\begin{aligned}
e \;:=\; & \\
& p_{\{i,l_2,l_3\}}(t_k) \cdot a_{J\setminus\{i,l_0,l_1,l_2,l_3\}} \cdot a_{J\setminus\{k,l_0,l_1\}} \;+\; p_{\{k,l_2,l_3\}}(t_i) \cdot a_{J\setminus\{k,l_0,l_1,l_2,l_3\}} \cdot a_{J\setminus\{i,l_0,l_1\}} \;+ \\
& p_{\{i,k,l_2\}}(t_{l_3}) \cdot a_{J\setminus\{i,k,l_0,l_1,l_2\}} \cdot a_{J\setminus\{l_0,l_1,l_3\}} \;+\; p_{\{i,k,l_3\}}(t_{l_2}) \cdot a_{J\setminus\{i,k,l_0,l_1,l_3\}} \cdot a_{J\setminus\{l_0,l_1,l_2\}}.
\end{aligned}$$

Applying the induction hypothesis on the expression e for $J^* := J \setminus \{l_0, l_1\}$ and the four indices $k, i, l_2, l_3 \in J^*$ makes clear that $e = 0$, which completes the proof $\diamond$

Lemma 3.3 *Let n be an even positive integer, $J := \{j_0, \ldots, j_n\}$, $J^* := \{j_0, \ldots, j_{l-1}, j_l^*, j_{l+1}, \ldots, j_n\}$ with $j_0, \ldots, j_n, j_l^* \in \mathbb{N}$. Let furthermore $\mathcal{Q}^{d-1}$ be a hyperquadric in d-dimensional projective space $\mathbb{P}^d$ and $A_{j_0}, \ldots, A_{j_n}, A_{j_l^*}$ be points on $\mathcal{Q}^{d-1}$ with homogeneous coordinate vectors $\mathbf{a}_{j_0}, \ldots, \mathbf{a}_{j_n}, \mathbf{a}_{j_l^*}$. Let $t_{j_0}, \ldots, t_{j_n}$ be pairwise distinct values in $\mathbb{R}$ and $t_{j_l^*} \in \mathbb{R}$ with $t_{j_l^*} \neq t_k$ for $k \in J \setminus \{j_l\}$.*

Then for the two parametric representations

$$\begin{aligned}\mathbf{x}(t) &= \sum_{i\in J} p_{J\setminus\{i\}}(t)\cdot p_{J\setminus\{i\}}(t_i)\cdot a_{J\setminus\{i\}}\cdot \mathbf{a}_i\\ \mathbf{x}^*(t) &= \sum_{i\in J^*} p_{J^*\setminus\{i\}}(t)\cdot p_{J^*\setminus\{i\}}(t_i)\cdot a_{J^*\setminus\{i\}}\cdot \mathbf{a}_i\end{aligned}$$

the following equation holds:

$$\langle \mathbf{x}(t),\mathbf{x}^*(t)\rangle = p^2_{J\setminus\{j_l\}}(t)\cdot a_{J\setminus\{j_l\}}\cdot a_{J\cup\{j_l^*\}}.$$

Proof. Evaluating the polynomial $f(t) := \langle \mathbf{x}(t),\mathbf{x}^*(t)\rangle$ for $t = t_k$, $k\in J\setminus\{j_l\}$ yields

$$\begin{aligned} f(t_k) &= \langle \mathbf{x}(t_k),\mathbf{x}^*(t_k)\rangle = \\ \langle p^2_{J\setminus\{k\}}(t_k)\cdot a_{J\setminus\{k\}}\cdot \mathbf{a}_k\ ,\ p^2_{J^*\setminus\{k\}}(t_k)\cdot a_{J^*\setminus\{k\}}\cdot \mathbf{a}_k\rangle &= \\ p^2_{J\setminus\{k\}}(t_k)\cdot p^2_{J^*\setminus\{k\}}(t_k)\cdot a_{J\setminus\{k\}}\cdot a_{J^*\setminus\{k\}}\cdot \underbrace{\langle \mathbf{a}_k,\mathbf{a}_k\rangle}_{=0} &= 0.\end{aligned} \tag{45}$$

Evaluating the first derivative of $f(t)$ for $t = t_k$, $k\in J\setminus\{j_l\}$ gives us

$$\begin{aligned}\tfrac{d}{dt}f(t_k) &= \langle \tfrac{d}{dt}\mathbf{x}(t_k),\mathbf{x}^*(t_k)\rangle + \langle \tfrac{d}{dt}\mathbf{x}^*(t_k),\mathbf{x}(t_k)\rangle = \\ p^2_{J\setminus\{k\}}(t_k)\cdot a_{J\setminus\{k\}}\cdot\langle \mathbf{a}_k,\tfrac{d}{dt}\mathbf{x}^*(t_k)\rangle &+ p^2_{J^*\setminus\{k\}}(t_k)\cdot a_{J^*\setminus\{k\}}\cdot\langle \mathbf{a}_k,\tfrac{d}{dt}\mathbf{x}(t_k)\rangle.\end{aligned} \tag{46}$$

- If A_k is a regular point on $\mathcal{Q}^{d-1}$ then $\frac{d}{dt}\mathbf{x}^*(t_k)$ represents a point in the hyperplane tangent to $\mathcal{Q}^{d-1}$ in A_k, as the curve represented by the parametrization $\mathbf{x}^*(t)$ is part of $\mathcal{Q}^{d-1}$. This yields $\langle \mathbf{a}_k,\frac{d}{dt}\mathbf{x}^*(t_k)\rangle = 0$.
- If A_k is a singular point on $\mathcal{Q}^{d-1}$ we trivially[19] have $\langle \mathbf{a}_k,\frac{d}{dt}\mathbf{x}^*(t_k)\rangle = 0$.

Analogously $\langle \mathbf{a}_k,\frac{d}{dt}\mathbf{x}(t_k)\rangle = 0$. This implies

$$\frac{d}{dt}f(t_k) = 0. \tag{47}$$

From (45) and (47) we obtain that $(t-t_k)^2$ is a factor of the polynomial $f(t)$ for all $k\in J\setminus\{j_l\}$. As furthermore $\deg f\le 2\cdot n$ we have

$$f(t) = \prod_{\substack{k\in J\\ k\ne j_l}}(t-t_k)^2\cdot c = p^2_{J\setminus\{j_l\}}(t)\cdot c, \tag{48}$$

where c is a constant factor. To determine c, we evaluate the polynomial $f(t)$ for $t = t_{j_l}$: On the one hand

$$\begin{aligned} f(t_{j_l}) = \langle \mathbf{x}(t_{j_l}),\mathbf{x}^*(t_{j_l})\rangle &= p^2_{J\setminus\{j_l\}}(t_{j_l})\cdot a_{J\setminus\{j_l\}} \\ \times\quad & \underbrace{\langle \mathbf{a}_{j_l}\ ,\ \sum_{i\in J^*} p_{J^*\setminus\{i\}}(t_{j_l})\cdot p_{J^*\setminus\{i\}}(t_i)\cdot a_{J^*\setminus\{i\}}\cdot \mathbf{a}_i\rangle}_{=\sum_{i\in J^*} p_{J^*\setminus\{i\}}(t_i)\cdot p_{J^*\setminus\{i\}}(t_{j_l})\cdot a_{\{i,j_l\}}\cdot a_{J^*\setminus\{i\}} = a_{J\cup\{j_l^*\}}\ \text{(see definition (3.1))}}.\end{aligned} \tag{49}$$

[19]Compare with remark 2.1, (d).

On the other hand we have due to (48)

$$f(t_{j_l}) \ = \ p^2_{J\setminus\{j_l\}}(t_{j_l}) \cdot c. \tag{50}$$

Comparing (49) and (50) we get

$$c \ = \ a_{J\setminus\{j_l\}} \cdot a_{J\cup\{j_l^*\}}, \tag{51}$$

which completes the proof $\diamond$

Lemma 3.4 *Let n be an even positive integer ≥ 4, $J := \{j_0, \ldots, j_n\} \subset \mathbb{N}$; let furthermore $\mathcal{Q}^{d-1}$ be a hyperquadric in d-dimensional projective space $\mathbb{P}^d$ and A_i be points on $\mathcal{Q}^{d-1}$ with homogeneous coordinate vectors $\mathbf{a}_i$ and corresponding pairwise distinct parameter values $t_i \in \mathbb{R}$, $i \in J$.*
Let l_0, l_1, l_2 be three pairwise distinct indices in J and

$$\begin{aligned}
\mathbf{x}_{J\setminus\{l_1,l_2\}}(t) \ &:= \ \sum_{\substack{i\in J\\ i\neq l_1,l_2}} p_{J\setminus\{i,l_1,l_2\}}(t) \cdot p_{J\setminus\{i,l_1,l_2\}}(t_i) \cdot a_{J\setminus\{i,l_1,l_2\}} \cdot \mathbf{a}_i,\\
\mathbf{x}_{J\setminus\{l_0,l_2\}}(t) \ &:= \ \sum_{\substack{i\in J\\ i\neq l_0,l_2}} p_{J\setminus\{i,l_0,l_2\}}(t) \cdot p_{J\setminus\{i,l_0,l_2\}}(t_i) \cdot a_{J\setminus\{i,l_0,l_2\}} \cdot \mathbf{a}_i,\\
\mathbf{x}_{J\setminus\{l_0,l_1\}}(t) \ &:= \ \sum_{\substack{i\in J\\ i\neq l_0,l_1}} p_{J\setminus\{i,l_0,l_1\}}(t) \cdot p_{J\setminus\{i,l_0,l_1\}}(t_i) \cdot a_{J\setminus\{i,l_0,l_1\}} \cdot \mathbf{a}_i,\\
\mathbf{x}_J(t) \ &:= \ \sum_{i\in J} p_{J\setminus\{i\}}(t) \cdot p_{J\setminus\{i\}}(t_i) \cdot a_{J\setminus\{i\}} \cdot \mathbf{a}_i;
\end{aligned}$$

then

$$\begin{aligned}
p_{\{l_1,l_2\}}(t) \cdot p_{\{l_1,l_2\}}(t_{l_0}) \cdot \langle \mathbf{x}_{J\setminus\{l_0,l_2\}}(t), \mathbf{x}_{J\setminus\{l_0,l_1\}}(t)\rangle \cdot \mathbf{x}_{J\setminus\{l_1,l_2\}}(t) \ &+\\
p_{\{l_0,l_2\}}(t) \cdot p_{\{l_0,l_2\}}(t_{l_1}) \cdot \langle \mathbf{x}_{J\setminus\{l_1,l_2\}}(t), \mathbf{x}_{J\setminus\{l_0,l_1\}}(t)\rangle \cdot \mathbf{x}_{J\setminus\{l_0,l_2\}}(t) \ &+\\
p_{\{l_0,l_1\}}(t) \cdot p_{\{l_0,l_1\}}(t_{l_2}) \cdot \langle \mathbf{x}_{J\setminus\{l_1,l_2\}}(t), \mathbf{x}_{J\setminus\{l_0,l_2\}}(t)\rangle \cdot \mathbf{x}_{J\setminus\{l_0,l_1\}}(t) \ &=\\
p^2_{J\setminus\{l_0,l_1,l_2\}}(t) \cdot a^2_{J\setminus\{l_0,l_1,l_2\}} \cdot \mathbf{x}_J(t).
\end{aligned} \tag{52}$$

Proof. With the help of lemma 3.3 we get

$$\begin{aligned}
\langle \mathbf{x}_{J\setminus\{l_0,l_2\}}(t), \mathbf{x}_{J\setminus\{l_0,l_1\}}(t)\rangle \ &= \ p^2_{J\setminus\{l_0,l_1,l_2\}}(t) \cdot a_{J\setminus\{l_0,l_1,l_2\}} \cdot a_{J\setminus\{l_0\}},\\
\langle \mathbf{x}_{J\setminus\{l_1,l_2\}}(t), \mathbf{x}_{J\setminus\{l_0,l_1\}}(t)\rangle \ &= \ p^2_{J\setminus\{l_0,l_1,l_2\}}(t) \cdot a_{J\setminus\{l_0,l_1,l_2\}} \cdot a_{J\setminus\{l_1\}},\\
\langle \mathbf{x}_{J\setminus\{l_1,l_2\}}(t), \mathbf{x}_{J\setminus\{l_0,l_2\}}(t)\rangle \ &= \ p^2_{J\setminus\{l_0,l_1,l_2\}}(t) \cdot a_{J\setminus\{l_0,l_1,l_2\}} \cdot a_{J\setminus\{l_2\}}.
\end{aligned}$$

Thus the left-hand side of (52) can be written in the form

$$p^2_{J\setminus\{l_0,l_1,l_2\}}(t)\cdot a_{J\setminus\{l_0,l_1,l_2\}}$$

$$\times[p_{\{l_1,l_2\}}(t)p_{\{l_1,l_2\}}(t_{l_0})\cdot a_{J\setminus\{l_0\}}\cdot\sum_{\substack{i\in J\\ i\neq l_1,l_2}} p_{J\setminus\{i,l_1,l_2\}}(t)\cdot p_{J\setminus\{i,l_1,l_2\}}(t_i)\cdot a_{J\setminus\{i,l_1,l_2\}}\cdot \mathbf{a}_i \quad +$$

$$p_{\{l_0,l_2\}}(t)p_{\{l_0,l_2\}}(t_{l_1})\cdot a_{J\setminus\{l_1\}}\cdot\sum_{\substack{i\in J\\ i\neq l_0,l_2}} p_{J\setminus\{i,l_0,l_2\}}(t)\cdot p_{J\setminus\{i,l_0,l_2\}}(t_i)\cdot a_{J\setminus\{i,l_0,l_2\}}\cdot \mathbf{a}_i \quad +$$

$$p_{\{l_0,l_1\}}(t)p_{\{l_0,l_1\}}(t_{l_2})\cdot a_{J\setminus\{l_2\}}\cdot\sum_{\substack{i\in J\\ i\neq l_0,l_1}} p_{J\setminus\{i,l_0,l_1\}}(t)\cdot p_{J\setminus\{i,l_0,l_1\}}(t_i)\cdot a_{J\setminus\{i,l_0,l_1\}}\cdot \mathbf{a}_i].$$

We compute the coefficient of $\mathbf{a}_i$ in this expression:

- coefficient of $\mathbf{a}_{l_j}$ for $j\in\{0,1,2\}$:

  $$p^2_{J\setminus\{l_0,l_1,l_2\}}(t)\cdot a^2_{J\setminus\{l_0,l_1,l_2\}}\cdot p_{J\setminus\{l_j\}}(t)\cdot p_{J\setminus\{l_j\}}(t_{l_j})\cdot a_{J\setminus\{l_j\}}.$$

- coefficient of $\mathbf{a}_i$ for $i\in J\setminus\{l_0,l_1,l_2\}$:

  $$p^2_{J\setminus\{l_0,l_1,l_2\}}(t)\cdot a_{J\setminus\{l_0,l_1,l_2\}}\cdot p_{J\setminus\{i\}}(t)\cdot p_{J\setminus\{i,l_0,l_1,l_2\}}(t_i)\cdot c,$$

  with

  $$\begin{aligned} c \;:=\; & -\; p_{\{i,l_1,l_2\}}(t_{l_0})\cdot a_{J\setminus\{i,l_1,l_2\}}\cdot a_{J\setminus\{l_0\}}\\ & -\; p_{\{i,l_0,l_2\}}(t_{l_1})\cdot a_{J\setminus\{i,l_0,l_2\}}\cdot a_{J\setminus\{l_1\}}\\ & -\; p_{\{i,l_0,l_1\}}(t_{l_2})\cdot a_{J\setminus\{i,l_0,l_1\}}\cdot a_{J\setminus\{l_2\}}.\end{aligned}$$

  Due to lemma 3.2 we have

  $$c \;=\; p_{\{l_0,l_1,l_2\}}(t_i)\cdot a_{J\setminus\{l_0,l_1,l_2\}}\cdot a_{J\setminus\{i\}}.$$

  So the coefficient of $\mathbf{a}_i$ for $i\in J\setminus\{l_0,l_1,l_2\}$ is

  $$p^2_{J\setminus\{l_0,l_1,l_2\}}(t)\cdot a^2_{J\setminus\{l_0,l_1,l_2\}}\cdot p_{J\setminus\{i\}}(t)\cdot p_{J\setminus\{i\}}(t_i)\cdot a_{J\setminus\{i\}}.$$

  This completes the proof $\diamond$

Now we are well-prepared to define and prove a geometric subdivision-algorithm to construct the rational interpolant for a given set of points on a hyperquadric.

Theorem 3.13 *Let n be an even positive integer, $J:=\{0,\ldots,n\}$; let furthermore $\mathcal{Q}^{d-1}$ be a hyperquadric in d-dimensional projective space $\mathbb{P}^d$ and A_i be points on $\mathcal{Q}^{d-1}$ with homogeneous coordinate vectors $\mathbf{a}_i$ and corresponding pairwise distinct parameter values $t_i\in\mathbb{R}$, $i\in J$. If the vectors $\mathbf{a}_{i,l}(t)$ are defined via*

$$\mathbf{a}_{i,0}(t) \;:=\; \mathbf{a}_i \text{ for } i\in J, \tag{53}$$

$$\begin{aligned}\mathbf{a}_{i,l}(t) \;:=\; & p_{\{i,n-l+1\}}(t)\cdot p_{\{i,n-l+1\}}(t_{l-1})\cdot\langle \mathbf{a}_{i,l-1}(t),\mathbf{a}_{n-l+1,l-1}(t)\rangle\cdot\mathbf{a}_{l-1,l-1}(t)\\ +\; & p_{\{l-1,n-l+1\}}(t)\cdot p_{\{l-1,n-l+1\}}(t_i)\cdot\langle \mathbf{a}_{l-1,l-1}(t),\mathbf{a}_{n-l+1,l-1}(t)\rangle\cdot\mathbf{a}_{i,l-1}(t)\\ +\; & p_{\{l-1,i\}}(t)\cdot p_{\{l-1,i\}}(t_{n-l+1})\cdot\langle \mathbf{a}_{l-1,l-1}(t),\mathbf{a}_{i,l-1}(t)\rangle\cdot\mathbf{a}_{n-l+1,l-1}(t)\\ & \text{for } l\in\{1,\ldots,\frac{n}{2}\} \text{ and } i\in\{l,\ldots,n-l\}.\end{aligned} \tag{54}$$

then

$$\mathbf{a}_{i,1}(t) = \mathbf{x}_{\{0,i,n\}}(t) \tag{55}$$

and

$$\begin{aligned}\mathbf{a}_{i,l}(t) &= \prod_{k=1}^{l-1}\left[\left((t-t_{k-1})\cdot(t-t_{n-k+1})\right)^{3^{l-k}-1}\cdot a^{2\cdot 3^{l-k-1}}_{\{0,\ldots,k-1,n-k+1,\ldots,n\}}\right] \\ &\times \mathbf{x}_{\{0,\ldots,l-1,i,n-l+1,\ldots,n\}}(t)\end{aligned} \tag{56}$$

for $l \in \{2,\ldots,\frac{n}{2}\}$ and $i \in \{l,\ldots,n-l\}$.

Proof. For $l = 1$ we have

$$\begin{aligned}\mathbf{a}_{i,1}(t) &= p_{\{i,n\}}(t)\cdot p_{\{i,n\}}(t_0)\cdot a_{\{i,n\}}\cdot \mathbf{a}_0 \\ &+ p_{\{0,n\}}(t)\cdot p_{\{0,n\}}(t_i)\cdot a_{\{0,n\}}\cdot \mathbf{a}_i \\ &+ p_{\{0,i\}}(t)\cdot p_{\{0,i\}}(t_n)\cdot a_{\{0,i\}}\cdot \mathbf{a}_n \\ &= \mathbf{x}_{\{0,i,n\}}(t).\end{aligned}$$

The proof for $l \geq 2$ is given by induction.
Initial step ($l = 2$): Using (55) we obtain

$$\begin{aligned}\mathbf{a}_{i,2}(t) &= p_{\{i,n-1\}}(t)\cdot p_{\{i,n-1\}}(t_1)\cdot\langle \mathbf{x}_{\{0,i,n\}}(t), \mathbf{x}_{\{0,n-1,n\}}(t)\rangle\cdot \mathbf{x}_{\{0,1,n\}}(t) \\ &+ p_{\{1,n-1\}}(t)\cdot p_{\{1,n-1\}}(t_i)\cdot\langle \mathbf{x}_{\{0,1,n\}}(t), \mathbf{x}_{\{0,n-1,n\}}(t)\rangle\cdot \mathbf{x}_{\{0,i,n\}}(t) \\ &+ p_{\{1,i\}}(t)\cdot p_{\{1,i\}}(t_{n-1})\cdot\langle \mathbf{x}_{\{0,1,n\}}(t), \mathbf{x}_{\{0,i,n\}}(t)\rangle\cdot \mathbf{x}_{\{0,n-1,n\}}(t) \\ &\overset{\text{via lemma 3.4}}{=} (t-t_0)^2\cdot(t-t_n)^2\cdot a^2_{\{0,n\}}\cdot \mathbf{x}_{\{0,1,i,n-1,n\}}(t).\end{aligned}$$

Induction step: With the help of the induction hypothesis we can assume

$$\begin{aligned}\mathbf{a}_{i,l-1}(t) &= \prod_{k=1}^{l-2}\left[\left((t-t_{k-1})\cdot(t-t_{n-k+1})\right)^{3^{l-1-k}-1}\cdot a^{2\cdot 3^{l-k-2}}_{\{0,\ldots,k-1,n-k+1,\ldots,n\}}\right] \\ &\times \mathbf{x}_{\{0,\ldots,l-2,i,n-l+2,\ldots,n\}}(t),\end{aligned} \tag{57}$$

$$\begin{aligned}\mathbf{a}_{n-l+1,l-1}(t) &= \prod_{k=1}^{l-2}\left[\left((t-t_{k-1})\cdot(t-t_{n-k+1})\right)^{3^{l-1-k}-1}\cdot a^{2\cdot 3^{l-k-2}}_{\{0,\ldots,k-1,n-k+1,\ldots,n\}}\right] \\ &\times \mathbf{x}_{\{0,\ldots,l-2,n-l+1,n-l+2,\ldots,n\}}(t),\end{aligned} \tag{58}$$

$$\mathbf{a}_{l-1,l-1}(t) = \prod_{k=1}^{l-2}\left[((t-t_{k-1})\cdot(t-t_{n-k+1}))^{3^{l-1-k}-1}\cdot a^{2\cdot 3^{l-k-2}}_{\{0,\ldots,k-1,n-k+1,\ldots,n\}}\right] \times \mathbf{x}_{\{0,\ldots,l-2,l-1,n-l+2,\ldots,n\}}(t). \tag{59}$$

Thus, using the definition (54) of the $\mathbf{a}_{i,l}(t)$ and (57), (58), (59) we get

$$\begin{aligned}\mathbf{a}_{i,l}(t) = & \prod_{k=1}^{l-2}\left[((t-t_{k-1})\cdot(t-t_{n-k+1}))^{3^{l-k}-3}\cdot a^{2\cdot 3^{l-k-1}}_{\{0,\ldots,k-1,n-k+1,\ldots,n\}}\right] \\ \times & \big[p_{\{i,n-l+1\}}(t)\cdot p_{\{i,n-l+1\}}(t_{l-1}) \\ & \times\langle \mathbf{x}_{\{0,\ldots,l-2,i,n-l+2,\ldots,n\}}(t), \mathbf{x}_{\{0,\ldots,l-2,n-l+1,n-l+2,\ldots,n\}}(t)\rangle \\ & \times\mathbf{x}_{\{0,\ldots,l-2,l-1,n-l+2,\ldots,n\}}(t) + \\ & p_{\{l-1,n-l+1\}}(t)\cdot p_{\{l-1,n-l+1\}}(t_i) \\ & \times\langle \mathbf{x}_{\{0,\ldots,l-2,l-1,n-l+2,\ldots,n\}}(t), \mathbf{x}_{\{0,\ldots,l-2,n-l+1,n-l+2,\ldots,n\}}(t)\rangle \\ & \times\mathbf{x}_{\{0,\ldots,l-2,i,n-l+2,\ldots,n\}}(t) + \\ & p_{\{l-1,i\}}(t)\cdot p_{\{l-1,i\}}(t_{n-l+1}) \\ & \times\langle \mathbf{x}_{\{0,\ldots,l-2,l-1,n-l+2,\ldots,n\}}(t), \mathbf{x}_{\{0,\ldots,l-2,i,n-l+2,\ldots,n\}}(t)\rangle \\ & \times\mathbf{x}_{\{0,\ldots,l-2,n-l+1,n-l+2,\ldots,n\}}(t)\big]\end{aligned}$$

$$\overset{\text{using lemma 3.4}}{=}$$

$$\prod_{k=1}^{l-1}\left[((t-t_{k-1})\cdot(t-t_{n-k+1}))^{3^{l-k}-1}\cdot a^{2\cdot 3^{l-k-1}}_{\{0,\ldots,k-1,n-k+1,\ldots,n\}}\right]\cdot \mathbf{x}_{\{0,\ldots,l-1,i,n-l+1,\ldots,n\}}(t)$$

$\diamond$

Now we want to investigate the geometric meaning of the given recursion formulas (53), (54). Let the assumptions of theorem 3.13 be fulfilled and let furthermore $\forall i \in J = \{0,\ldots,n\} : a_{J\setminus\{i\}} \neq 0$ (general case); this condition gives us the guarantee, that there exists exactly one solution curve c (compare with theorem 3.11). Furthermore the solution curve has the parametrization

$$\mathbf{x}_J(t) := \sum_{i\in J} p_{J\setminus\{i\}}(t)\cdot p_{J\setminus\{i\}}(t_i)\cdot a_{J\setminus\{i\}}\cdot \mathbf{a}_i.$$

Then vector $\mathbf{a}_{\frac{n}{2},\frac{n}{2}}(t)$ computed by the recursion (53), (54) has the form

$$\mathbf{a}_{\frac{n}{2},\frac{n}{2}}(t) = \prod_{k=1}^{\frac{n}{2}-1}\left[((t-t_{k-1})\cdot(t-t_{n-k+1}))^{3^{\frac{n}{2}-k}-1}\cdot a^{2\cdot 3^{\frac{n}{2}-k-1}}_{\{0,\ldots,k-1,n-k+1,\ldots,n\}}\right] \times \mathbf{x}_J(t) \tag{60}$$

due to theorem 3.13.
Thus we see that $\mathbf{a}_{\frac{n}{2},\frac{n}{2}}(t)$ either

(a) is the zero-vector

or

(b) represents the point on the solution curce c belonging to the parameter value t.

Case (a) only occurs if

(a1) one of the values $a_{\{0,n\}}, a_{\{0,1,n-1,n\}}, \ldots, a_{\{0,\ldots,\frac{n}{2}-2,\frac{n}{2}-2,\ldots,n\}}$ is zero (then $\mathbf{a}_{\frac{n}{2},\frac{n}{2}}(t)$ is the zero-vector *for all* $t \in \mathbb{R}$)

or

(a2) if t is equal to one of the values $t_0, \ldots, t_{\frac{n}{2}-2}, t_{\frac{n}{2}+2}, \ldots, t_n$.

We end up at the following

Algorithm 3.2 *Let the assumptions of theorem 3.13 be fulfilled and let furthermore*

- $\forall i \in J = \{0, \ldots, n\} : a_{J \setminus \{i\}} \neq 0,$
- $\forall k \in \{1, \ldots, \frac{n}{2} - 1\} : a_{\{0,\ldots,k-1,n-k+1,\ldots,n\}} \neq 0,$

then for any $t \in \mathbb{R} \setminus \{t_0, \ldots, t_{\frac{n}{2}-2}, t_{\frac{n}{2}+2}, \ldots, t_n\}$ *the vector* $\mathbf{a}_{\frac{n}{2},\frac{n}{2}}(t)$ *computed via the recursion formulas (53), (54) represents the point belonging to* t *on the (uniquely determined) solution-curve* c *of the interpolation-problem* IP_J.

The diagram below illustrates the generation of the vectors $\mathbf{a}_{i,l}(t)$. A storage-optimized implementation of the algorithm only needs *one* array of vectors as $\mathbf{a}_{i,l}(t)$ only depends on $\mathbf{a}_{l-1,l-1}(t)$, $\mathbf{a}_{i,l-1}(t)$, $\mathbf{a}_{n-l+1,l-1}(t)$ and thus can overwrite $\mathbf{a}_{i,l-1}(t)$.

$$
\begin{array}{cccccc}
\mathbf{a}_{0,0}(t) & & & & & \\
\mathbf{a}_{1,0}(t) & \mathbf{a}_{1,1}(t) & & & & \\
* & \mathbf{a}_{2,1}(t) & & & & \\
* & * & * & & & \\
* & * & * & * & & \\
* & * & * & * & \mathbf{a}_{\frac{n}{2}-1,\frac{n}{2}-1}(t) & \\
* & * & * & * & \mathbf{a}_{\frac{n}{2},\frac{n}{2}-1}(t) & \mathbf{a}_{\frac{n}{2},\frac{n}{2}}(t) \sim \mathbf{x}_J(t) \\
* & * & * & * & \mathbf{a}_{\frac{n}{2}+1,\frac{n}{2}-1}(t) & \\
* & * & * & * & & \\
* & * & * & & & \\
* & \mathbf{a}_{n-2,1}(t) & & & & \\
\mathbf{a}_{n-1,0}(t) & \mathbf{a}_{n-1,1}(t) & & & & \\
\mathbf{a}_{n,0}(t) & & & & &
\end{array}
$$

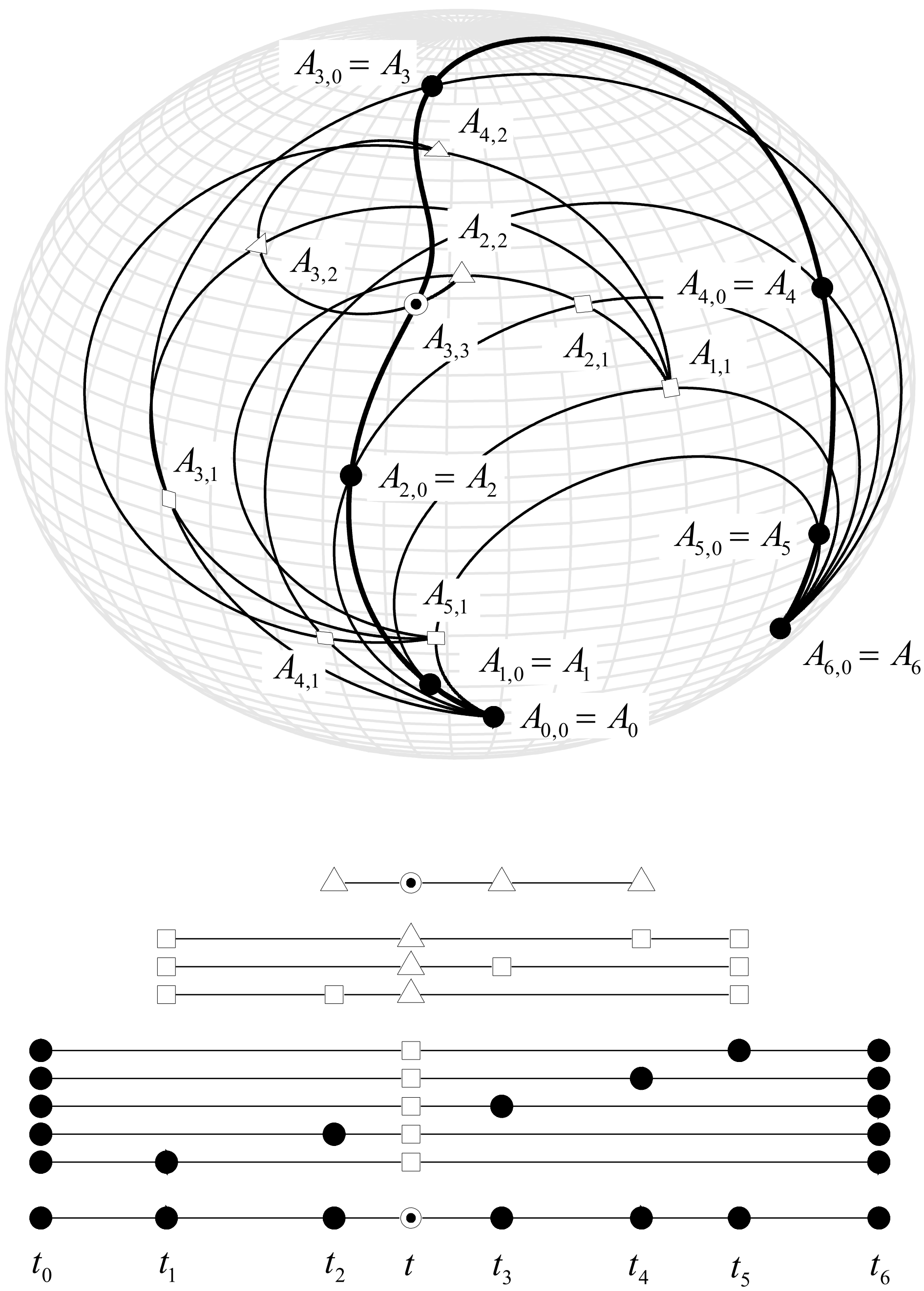

Figure 3.2 *The geometric algorithm for constructing the rational interpolant on a hyperquadric ($n = 6$).*
Input points: $A_0, \ldots, A_6$ (black circles).
First generation of points: $A_{1,1}, \ldots, A_{5,1}$ (quadrangles).
Second generation of points: $A_{2,2}, A_{3,2}, A_{4,2}$ (triangles).
Resulting point on the curve: $A_{3,3}$ (black dotted circle).

Remark 3.1 *(a) In general the vectors* $\mathbf{a}_{l-1,l-1}(t)$, $\mathbf{a}_{i,l-1}(t)$, $\mathbf{a}_{n-l+1,l-1}(t)$ *represent points* $A_{l-1,l-1}(t), A_{i,l-1}(t), A_{n-l+1,l-1}(t)$ *on the hyperquadric which span a plane* $[A_{l-1,l-1}(t), A_{i,l-1}(t), A_{n-l+1,l-1}(t)]_p$, *intersecting the hyperquadric in a conic section. Then*

- $A_{i,l}(t)$ *is on this conic section and*
- *the cross-ratios* $(A_{l-1,l-1}(t)\ A_{i,l-1}(t)\ A_{n-l+1,l-1}(t)\ A_{i,l}(t))$ *and* $(t_{l-1}\ t_i\ t_{n-l+1}\ t)$ *are identical.*

(b) As the exponents of $(t-t_{k-1})\cdot(t-t_{n-k+1})$ *occuring in (56) are very large for n large, the implementation of the given algorithm requires some care to guarantee numerical stability.*

3.4 The case $n = 3$

We have seen that in general there exists exactly one solution curve of the interpolation problem IP_J if n is even and none if n is odd. But also a pencil of solution curves is possible as the discussion of our interpolation problem in the case $n = 3$ will show.

Lemma 3.5 *Let* $\mathcal{Q}^2$ *be a quadric in 3-dimensional projective space* $\mathbb{P}^3$ *and* A_0, A_1, A_2, A_3 *be four real points on it with* $\dim[A_0, A_1, A_2, A_3]_p = 3$. *Let furthermore* $c := [A_0, A_1, A_2]_p \cap \mathcal{Q}^2$ *be a regular second order curve and at most one of the lines* $[A_i, A_3]_p$, $i \in \{0, 1, 2\}$ *be contained by* $\mathcal{Q}^2$. *Then at least one of the planes* $[A_0, A_1, A_3]_p$, $[A_0, A_2, A_3]_p$, $[A_1, A_2, A_3]_p$ *intersects* $\mathcal{Q}^2$ *in a regular second-order curve.*

Proof. $\mathcal{Q}^2$ must either be of oval type or a real quadratic cone or an annular quadric, as $[A_0, A_1, A_2]_p$ intersects the quadric in a regular second-order curve. If $\mathcal{Q}^2$ is an oval quadric, a real quadratic cone or an annular quadric then 0, 1 or 2 real generators pass through A_3, respectively. So at least two of the lines $[A_i, A_3]_p$ - let's say $[A_0, A_3]_p$ and $[A_1, A_3]_p$ are not part of $\mathcal{Q}^2$. Thus $[A_0, A_1, A_3]_p \cap \mathcal{Q}^2$ has to be a regular second-order curve $\diamond$

Let now $\mathcal{Q}^{d-1}$ be a hyperquadric in d-dimensional projective space $\mathbb{P}^d$ and A_0, A_1, A_2, A_3 be points on $\mathcal{Q}^{d-1}$ represented by homogeneous coordinate vectors $\mathbf{a}_0$, $\mathbf{a}_1$, $\mathbf{a}_2$, $\mathbf{a}_3$. Let furthermore four corresponding pairwise distinct parameter values t_0, t_1, t_2, t_3 be given. The weights of any solution curve of $IP_{\{0,1,2,3\}}$ have to satisfy the homogeneous linear equation system $L_{\{0,1,2,3\}}$ with the coefficient matrix $\mathbf{C}_{\{0,1,2,3\}}$ (theorem 3.3). According to theorem 3.1 rank $C_{\{0,1,2,3\}} \in \{0, 2, 4\}$.
If rank $\mathbf{C}_{\{0,1,2,3\}} = 4$, no solution curve can exist.
If rank $\mathbf{C}_{\{0,1,2,3\}} = 0$ then $[A_0, A_1, A_2, A_3]_p \subset \mathcal{Q}^{d-1}$ (see theorem 3.6). But in this case *any* choice of the weights $w_0, \ldots, w_3$ with $w_i \neq 0$ yields a solution curve; so we either have

- exactly one *"solution curve"* if $A_0 = A_1 = A_2 = A_3$ - the solution curve is this point - or

- exactly one solution curve if $[A_0, A_1, A_2, A_3]_p$ is a line - the solution curve is this line - or
- a $\overline{d}$-parametric set of solution curves if $\overline{d} := \dim[A_0, A_1, A_2, A_3]_p \in \{2, 3\}$.

Now let rank $\mathbf{C}_{\{0,1,2,3\}} = 2$. If $d > 3$ we intersect $\mathcal{Q}^{d-1}$ with a 3-space $\mathbb{P}^3$ containing $A_0, \dots, A_3$ which yields a two-dimensional quadric $\mathcal{Q}^2$ as $[A_0, A_1, A_2, A_3]_p \not\subset \mathcal{Q}^{d-1}$.
So without loss of generality we can assume

$$a_{\{0,1,2,3\}} = 0, \ \text{rank}\ \mathbf{C}_{\{0,1,2,3\}} = 2, \ d = 3 \tag{61}$$

for our further investigations. Moreover $\overline{d} = \dim[A_0, A_1, A_2, A_3]_p$ cannot be zero in this case.[20] In the following we will discuss the remaining cases $\overline{d} = 1, 2, 3$.

Case 1: $\overline{d} = 1$: This means that $[A_0, A_1, A_2, A_3]_p$ is a line, which cannot be contained by $\mathcal{Q}^2$ as rank $\mathbf{C}_{\{0,1,2,3\}} = 2$. As a conclusion three of the points must be identical.[21] For example $A_1 = A_2 = A_3$, $A_0 \neq A_1$. But in this case the coefficient matrix has the shape

$$\mathbf{C}_{\{0,1,2,3\}} = \begin{pmatrix} 0 & * & * & * \\ * & 0 & 0 & 0 \\ * & 0 & 0 & 0 \\ * & 0 & 0 & 0 \end{pmatrix}, \tag{62}$$

which implies $w_1 = w_2 = w_3 = 0$. Thus no solution curve can exist.

Case 2: $\overline{d} = 2$: $[A_0, A_1, A_2, A_3]_p$ is a plane, again not being part of $\mathcal{Q}^2$. So this plane intersects $\mathcal{Q}^2$ in a second order curve c.
If c is singular, it must consist of two distinct lines, as $\overline{d} = 2$. Obviously no solution curve exists in this case. Now let c be a regular second-order curve. Because of $\overline{d} = 2$ three of the points, lets say A_0, A_1, A_2, have to be pairwise distinct. Furthermore none of the lines $[A_0, A_1]_p, [A_0, A_2]_p, [A_1, A_2]_p$ can be part of $\mathcal{Q}^2$. As a conclusion $a_{\{0,1\}} \neq 0$, $a_{\{0,2\}} \neq 0$, $a_{\{1,2\}} \neq 0$. So according to theorem 3.2 the curve c with the parametrization $\mathbf{x}(t)$ (see 28) is the unique solution of the interpolation problem $IP_{\{0,1,2\}}$. As additionally

$$\langle \mathbf{a}_3, \mathbf{x}(t_3) \rangle \ = \ a_{\{0,1,2,3\}} \ = \ 0$$

holds, the point A_3 is represented by the vector $\mathbf{x}(t_3)$. Thus c also is (the uniquely determined) solution curve of $IP_{\{0,1,2,3\}}$. According to section 2.2 the cross ratio of the four points A_0, A_1, A_2, A_3 on c is equal to that one of the four corresponding parameter values:

$$(A_0 \ A_1 \ A_2 \ A_3) = (t_0 \ t_1 \ t_2 \ t_3).$$

Case 3: $\overline{d} = 3$: $[A_0, A_1, A_2, A_3]_p = \mathcal{P}^3$. Two cases can occur:

[20] $\overline{d} = 0$ would imply rank $\mathbf{C}_{\{0,1,2,3\}} = 0$ in contradiction to rank $\mathbf{C}_{\{0,1,2,3\}} = 2$.

[21] A hyperquadric $\mathcal{Q}^{d-1}$ and a line l intersect in exactly two points if l is neither tangent to nor contained by $\mathcal{Q}^{d-1}$.

(a) There exists a triple i_0, i_1, i_2 in $\{0,1,2,3\}$ with $a_{\{i_0,i_1\}} \neq 0$, $a_{\{i_0,i_2\}} \neq 0$, $a_{\{i_1,i_2\}} \neq 0$.
(b) There exists no such triple.

Case 3a: Without loss of generality let $i_0 = 0$, $i_1 = 1$, $i_2 = 2$. Then due to theorem 3.2 the interpolation problem $IP_{\{0,1,2\}}$ has a uniquely determined solution curve $c_{\{0,1,2\}}$ with the Lagrange-representation.

$$\begin{aligned} \mathbf{x}_{\{0,1,2\}}(t) &= p_{\{1,2\}}(t) \cdot p_{\{1,2\}}(t_0) \cdot a_{\{1,2\}} \cdot \mathbf{a}_0 \;+\; p_{\{0,2\}}(t) \cdot p_{\{0,2\}}(t_1) \cdot a_{\{0,2\}} \cdot \mathbf{a}_1 \\ &+ \; p_{\{0,1\}}(t) \cdot p_{\{0,1\}}(t_2) \cdot a_{\{0,1\}} \cdot \mathbf{a}_2. \end{aligned} \tag{63}$$

Furthermore we have

$$\langle \mathbf{a}_3, \mathbf{x}_{\{0,1,2\}}(t_3) \rangle \;=\; a_{\{0,1,2,3\}} \;=\; 0.$$

So, if $X_{\{0,1,2\}}(t_3)$ denotes the point represented by $\mathbf{x}_{\{0,1,2\}}(t_3)$ the line $[A_3, X_{\{0,1,2\}}(t_3)]_p$ must be contained[22] by $\mathcal{Q}^2$. Thus this quadric has a real generator and therefore must either be a real quadratic cone or an annular quadric.[23] Two cases are possible:

- $\mathcal{Q}^2$ is an annular quadric or a real quadratic cone with vertex different from A_3. Here the assumptions of lemma 3.5 are fulfilled; thus at least one of the planes $[A_0, A_1, A_3]_p$, $[A_0, A_2, A_3]_p$, $[A_1, A_2, A_3]_p$, let's say $[A_0, A_1, A_3]_p$ intersects $\mathcal{Q}^2$ in a regular second-order curve $c_{\{0,1,3\}}$. This curve together with its Lagrange representation

$$\begin{aligned} \mathbf{x}_{\{0,1,3\}}(t) &= p_{\{1,3\}}(t) \cdot p_{\{1,3\}}(t_0) \cdot a_{\{1,3\}} \cdot \mathbf{a}_0 \;+\; p_{\{0,3\}}(t) \cdot p_{\{0,3\}}(t_1) \cdot a_{\{0,3\}} \cdot \mathbf{a}_1 \\ &+ \; p_{\{0,1\}}(t) \cdot p_{\{0,1\}}(t_3) \cdot a_{\{0,1\}} \cdot \mathbf{a}_3. \end{aligned} \tag{64}$$

  is the uniquely determined solution curve of $IP_{\{0,1,3\}}$. We consider the one-parametric set of cubics

$$c_{\lambda:\mu} \ldots \mathbf{x}_{\lambda:\mu}(t) \;=\; \lambda \cdot (t - t_3) \cdot \mathbf{x}_{\{0,1,2\}}(t) \;+\; \mu \cdot (t - t_2) \cdot \mathbf{x}_{\{0,1,3\}}(t) \tag{65}$$

  with $\lambda : \mu \in \mathbb{R}$, $\lambda \neq 0$, $\mu \neq 0$. Any of them clearly interpolates the points A_0, A_1, A_2, A_3 for the parameter values t_0, t_1, t_2, t_3. Moreover we have

$$\begin{aligned} \langle \mathbf{x}_{\lambda:\mu}(t), \mathbf{x}_{\lambda:\mu}(t) \rangle &= \lambda^2 \cdot (t - t_3)^2 \cdot \langle \mathbf{x}_{\{0,1,2\}}(t), \mathbf{x}_{\{0,1,2\}}(t) \rangle \\ &+ \; \mu^2 \cdot (t - t_2)^2 \cdot \langle \mathbf{x}_{\{0,1,3\}}(t), \mathbf{x}_{\{0,1,3\}}(t) \rangle \\ &+ \; 2 \cdot \lambda \cdot \mu \cdot (t - t_2) \cdot (t - t_3) \cdot \langle \mathbf{x}_{\{0,1,2\}}(t), \mathbf{x}_{\{0,1,3\}}(t) \rangle. \end{aligned} \tag{66}$$

  Trivially $\langle \mathbf{x}_{\{0,1,2\}}(t), \mathbf{x}_{\{0,1,2\}}(t) \rangle$ and $\langle \mathbf{x}_{\{0,1,3\}}(t), \mathbf{x}_{\{0,1,3\}}(t) \rangle$ are zero. Due to lemma 3.3 we furthermore have

$$\langle \mathbf{x}_{\{0,1,2\}}(t), \mathbf{x}_{\{0,1,3\}}(t) \rangle \;=\; (t - t_0)^2 \cdot (t - t_1)^2 \cdot a_{\{0,1\}} \cdot \underbrace{a_{\{0,1,2,3\}}}_{=\,0},$$

[22] Compare with remark 2.1, (c).

[23] Compare with the possible types of quadrics in $\mathbb{P}^3$ listed in section 2.1: $\mathcal{Q}^2$ cannot be a pair of planes or a doubly-counted plane, as $a_{\{i_0,i_1\}} \neq 0$, $a_{\{i_0,i_2\}} \neq 0$, $a_{\{i_1,i_2\}} \neq 0$. It also cannot be a quadratic null cone or an oval quadric, as these types do not contain real straight lines.

which shows us that the third summand of the right-hand side of (66) vanishes too.[24] So we have

$$\langle \mathbf{x}_{\lambda:\mu}(t), \mathbf{x}_{\lambda:\mu}(t) \rangle \;\; = \;\; 0,$$

which shows that any of the curves $c_{\lambda:\mu}$ is on $\mathcal{Q}^2$ and thus is a solution of $IP_{\{0,1,2,3\}}$.

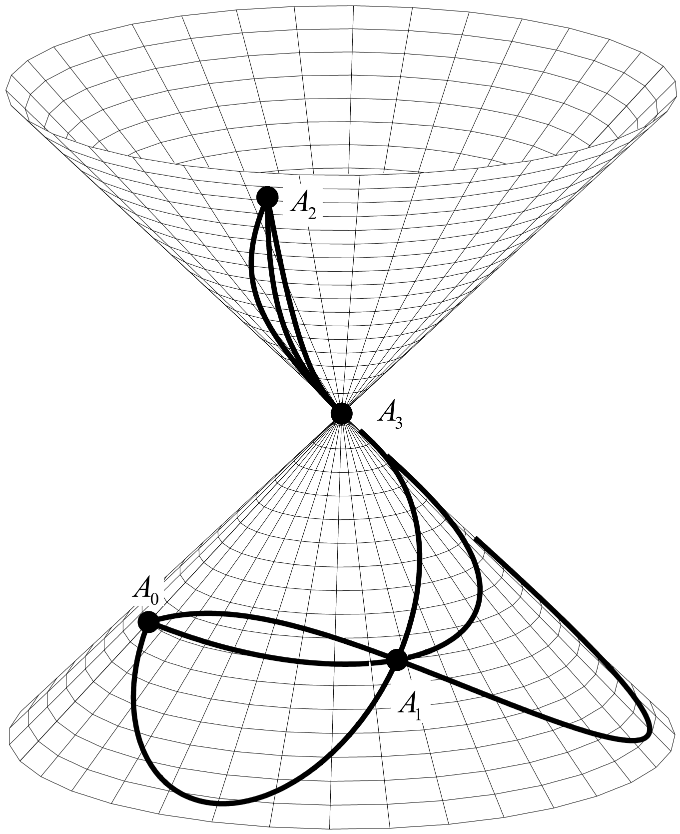

Figure 3.3 *Different solution curves in the case $n = 3$ on a quadratic cone, one of the points to be interpolated being the vertice.*

- $\mathcal{Q}^2$ is a real quadratic cone with vertex A_3. Then *any* of the one-parametric set of cubics

$$c_{\lambda:\mu} \ldots \mathbf{x}_{\lambda:\mu}(t) \;\; = \;\; \lambda \cdot (t - t_3) \cdot \mathbf{x}_{\{0,1,2\}}(t) \;\; + \;\; \mu \cdot \prod_{i=0}^{2}(t - t_i) \cdot \mathbf{a}_3 \qquad (67)$$

[24]This means that the two points $X_{\{0,1,2\}}(t) \in c_{\{0,1,2\}}$ and $X_{\{0,1,3\}}(t) \in c_{\{0,1,3\}}$ belonging to the same parameter-value t always lie on a common generator of $\mathcal{Q}^2$.

with $\lambda : \mu \in \mathbb{R}$, $\lambda \neq 0$, $\mu \neq 0$ is a solution of $IP_{\{0,1,2,3\}}$ because

$$\mathbf{x}_{\lambda:\mu}(t_i) \stackrel{\wedge}{=} A_i \text{ for } i \in \{0,1,2,3\}$$

and

$$\begin{aligned}
&\langle \mathbf{x}_{\lambda:\mu}(t), \mathbf{x}_{\lambda:\mu}(t) \rangle = \\
&\lambda^2 \cdot (t-t_3)^2 \cdot \underbrace{\langle \mathbf{x}_{\{0,1,2\}}(t), \mathbf{x}_{\{0,1,2\}}(t) \rangle}_{=\ 0} + \mu^2 \cdot \prod_{i \in \{0,1,2\}} (t-t_i)^2 \cdot \underbrace{\langle \mathbf{a}_3, \mathbf{a}_3 \rangle}_{=\ 0} + \\
&2 \cdot \lambda \cdot \mu \cdot \prod_{i \in \{0,1,2,3\}} (t-t_i) \cdot \underbrace{\langle \mathbf{x}_{\{0,1,2\}}(t), \mathbf{a}_3 \rangle}_{=\ 0, \text{ as } A_3 \text{ is the vertex.}} .
\end{aligned}$$

Figure 3.3 gives an impression of this particular case.

The existence of a one-parametric set of solution cubics is obvious, if one considers the problem in a more geometrical way: Take a solution cubic c; then under *any* of the collineations belonging to the one-parametric set of perspective collineations with center A_3 and fixed plane $[A_0, A_1, A_2]_p$ the cone and the four points A_i are fixed, whereas the cubic is mapped into another one.

Case 3b: There exists *no* triple $i_0, i_1, i_2 \in \{0,1,2,3\}$ with $a_{\{i_0,i_1\}} \neq 0$, $a_{\{i_0,i_2\}} \neq 0$, $a_{\{i_1,i_2\}} \neq 0$. This implies either

- $\exists i_0, i_1, i_2 \in \{0,1,2,3\}$, i_0, i_1, i_2 pairwise distinct and $a_{\{i_0,i_1\}}, a_{\{i_0,i_2\}}, a_{\{i_1,i_2\}}$ are zero

or

- $\exists i_0, i_1, i_2, i_3 \in \{0,1,2,3\}$, i_0, i_1, i_2, i_3 pairwise distinct and $a_{\{i_0,i_1\}}, a_{\{i_2,i_3\}}$ are zero.

In the first case the plane $[A_{i_0}, A_{i_1}, A_{i_2}]_p$ is part of $\mathcal{Q}^2$ which must therefore consist of two planes, $[A_{i_0}, A_{i_1}, A_{i_2}]_p$ being one of them and A_{i_3} lying in the other one due to $\dim[A_0, A_1, A_2, A_3]_p = 3$.

In the second case the two skew lines $l_{\{i_0,i_1\}} = [A_{i_0}, A_{i_1}]_p$ and $l_{\{i_2,i_3\}} = [A_{i_2}, A_{i_3}]_p$ belong to $\mathcal{Q}^2$ which implies that $\mathcal{Q}^2$ is either a pair of planes or an annular quadric. Obviously no solution curve can exist if $\mathcal{Q}^2$ is a pair of planes. In case that $\mathcal{Q}^2$ is an annular quadric, the generators $l_{\{i_0,i_1\}}$ and $l_{\{i_2,i_3\}}$ belong to the same regulus $\mathcal{R}$ on it (see figure 3.4).
They can be parametrized by

$$\begin{aligned} l_{\{i_0,i_1\}} \ldots \mathbf{x}_{\{i_0,i_1\}}(t) &= (t-t_{i_1}) \cdot (t_{i_0} - t_{i_2}) \cdot a_{\{i_1,i_2\}} \cdot \mathbf{a}_{i_0} \\ &+ (t-t_{i_0}) \cdot (t_{i_2} - t_{i_1}) \cdot a_{\{i_0,i_2\}} \cdot \mathbf{a}_{i_1}, \end{aligned} \tag{68}$$

$$\begin{aligned} l_{\{i_2,i_3\}} \ldots \mathbf{x}_{\{i_2,i_3\}}(t) &= (t-t_{i_3}) \cdot (t_{i_2} - t_{i_0}) \cdot a_{\{i_0,i_3\}} \cdot \mathbf{a}_{i_2} \\ &+ (t-t_{i_2}) \cdot (t_{i_0} - t_{i_3}) \cdot a_{\{i_0,i_2\}} \cdot \mathbf{a}_{i_3}. \end{aligned} \tag{69}$$

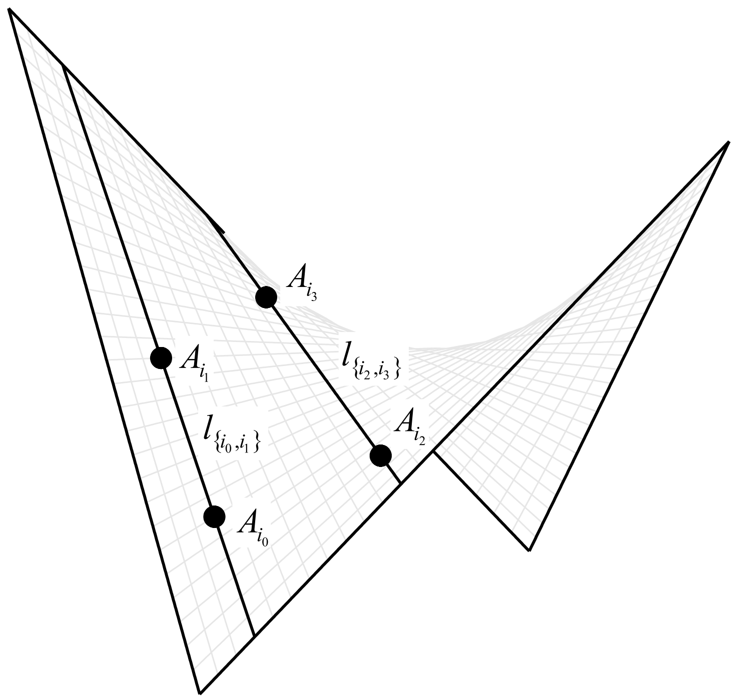

Figure 3.4 *The second case of 3b.*

We consider the one-parametric set of cubics

$$\begin{aligned} c_{\lambda:\mu} \ldots \mathbf{x}_{\lambda:\mu}(t) &= \lambda \cdot (t - t_{i_2}) \cdot (t - t_{i_3}) \cdot \mathbf{x}_{\{i_0,i_1\}}(t) \\ &+ \mu \cdot (t - t_{i_0}) \cdot (t - t_{i_1}) \cdot \mathbf{x}_{\{i_2,i_3\}}(t) \end{aligned} \tag{70}$$

with $\lambda : \mu \in \mathbb{R}$, $\lambda \neq 0$, $\mu \neq 0$. Any of them clearly interpolates the points A_0, A_1, A_2, A_3 for the parameter values t_0, t_1, t_2, t_3, respectively. Moreover we have

$$\begin{aligned} \langle \mathbf{x}_{\lambda:\mu}(t), \mathbf{x}_{\lambda:\mu}(t) \rangle &= \lambda^2 \cdot (t - t_{i_2})^2 \cdot (t - t_{i_3})^2 \cdot \langle \mathbf{x}_{\{i_0,i_1\}}(t), \mathbf{x}_{\{i_0,i_1\}}(t) \rangle \\ &+ \mu^2 \cdot (t - t_{i_0})^2 \cdot (t - t_{i_1})^2 \cdot \langle \mathbf{x}_{\{i_2,i_3\}}(t), \mathbf{x}_{\{i_2,i_3\}}(t) \rangle \\ &+ 2 \cdot \lambda \cdot \mu \cdot \prod_{i=0}^{3} (t - t_i) \cdot \langle \mathbf{x}_{\{i_0,i_1\}}(t), \mathbf{x}_{\{i_2,i_3\}}(t) \rangle. \end{aligned} \tag{71}$$

Trivially $\langle \mathbf{x}_{\{i_0,i_1\}}(t), \mathbf{x}_{\{i_0,i_1\}}(t) \rangle$ and $\langle \mathbf{x}_{\{i_2,i_3\}}(t), \mathbf{x}_{\{i_2,i_3\}}(t) \rangle$ are zero.
Because $a_{\{i_0,i_1\}} = a_{\{i_2,i_3\}} = 0$ we have

$$\begin{aligned} 0 = a_{\{0,1,2,3\}} &= a_{\{i_0,i_2\}} \cdot a_{\{i_1,i_3\}} \cdot g_{\{i_0,i_2\}}(t_{i_1}) \cdot g_{\{i_0,i_2\}}(t_{i_3}) \\ &+ a_{\{i_0,i_3\}} \cdot a_{\{i_1,i_2\}} \cdot g_{\{i_0,i_3\}}(t_{i_1}) \cdot g_{\{i_0,i_3\}}(t_{i_2}), \end{aligned}$$

which yields

$$\frac{a_{\{i_0,i_2\}} \cdot a_{\{i_1,i_3\}}}{a_{\{i_0,i_3\}} \cdot a_{\{i_1,i_2\}}} = \frac{(t_{i_0} - t_{i_2}) \cdot (t_{i_1} - t_{i_3})}{(t_{i_0} - t_{i_3}) \cdot (t_{i_1} - t_{i_2})} = (t_{i_0}\ t_{i_1}\ t_{i_2}\ t_{i_3}). \tag{72}$$

With the help of (72) we get

$$\langle \mathbf{x}_{\{i_0,i_1\}}(t), \mathbf{x}_{\{i_2,i_3\}}(t) \rangle \;=\; 0. \tag{73}$$

So, the third summand of the right-hand side of (71) is zero too and as a conclusion any of the curves $c_{\lambda:\mu}$ is on $\mathcal{Q}^2$. Thus any of these curves is a solution of $IP_{\{0,1,2,3\}}$.
Now we want to investigate the geometric meaning of the conditions given in case 3b:
Let $X_{\{i_0,i_1\}}(t)$ denote the point on $l_{\{i_0,i_1\}}$ represented by $\mathbf{x}_{\{i_0,i_1\}}(t)$ and $X_{\{i_2,i_3\}}(t)$ the one on $l_{\{i_2,i_3\}}$ represented by $\mathbf{x}_{\{i_2,i_3\}}(t)$. Then due to (73) $\bar{l}(t) := [X_{\{i_0,i_1\}}(t), X_{\{i_2,i_3\}}(t)]_p$ is a generator of $\mathcal{Q}^2$ belonging to the complementary regulus $\overline{\mathfrak{R}}$ for any $t \in \mathbb{R}$.
Let furthermore $\bar{l}_i$ denote the generator passing through A_i and belonging to $\overline{\mathfrak{R}}$. Then we have[25]

$$\begin{aligned} \tfrac{a_{\{i_0,i_2\}}\cdot a_{\{i_1,i_3\}}}{a_{\{i_0,i_3\}}\cdot a_{\{i_1,i_1\}}} &= (l_{\{i_0,i_1\}}\; l_{\{i_0,i_1\}}\; l_{\{i_2,i_3\}}\; l_{\{i_2,i_3\}}) \cdot (\bar{l}_{i_0}\; \bar{l}_{i_1}\; \bar{l}_{i_2}\; \bar{l}_{i_3}) \\ &= (A_{i_0}\; A_{i_1}\; A_{i_2}\; A_{i_3}) \cdot \overline{(A_{i_0}\; A_{i_1}\; A_{i_2}\; A_{i_3})} \end{aligned} \tag{74}$$

and because of

$$(l_{\{i_0,i_1\}}\; l_{\{i_0,i_1\}}\; l_{\{i_2,i_3\}}\; l_{\{i_2,i_3\}}) \;=\; (A_{i_0}\; A_{i_1}\; A_{i_2}\; A_{i_3}) \;=\; 1$$

we get

$$\tfrac{a_{\{i_0,i_2\}}\cdot a_{\{i_1,i_3\}}}{a_{\{i_0,i_3\}}\cdot a_{\{i_1,i_2\}}} \;=\; (A_{i_0}\; A_{i_1}\; A_{i_2}\; A_{i_3}). \tag{75}$$

[25] Compare with section 2.2, (7).

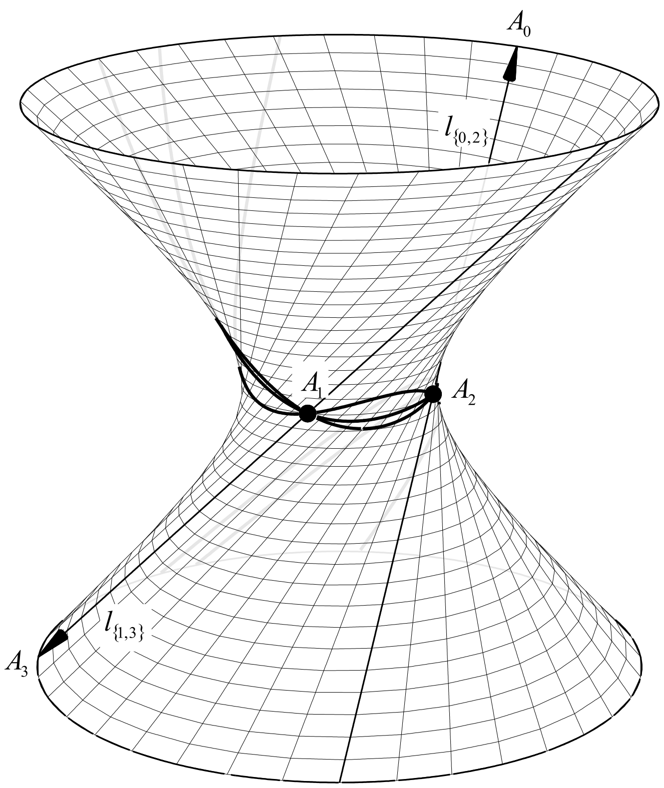

Figure 3.5 *Different solution curves in the case $n = 3$ on an annular quadric.*

Comparing the equations (72) and (75) gives us

$$\overline{(A_{i_0}\ A_{i_1}\ A_{i_2}\ A_{i_3})} \quad = \quad (t_{i_0}\ t_{i_1}\ t_{i_2}\ t_{i_3}). \tag{76}$$

Figure 3.5 demonstrates an example for the situation: The four points A_0, A_1, A_2, A_3 were chosen on an annular quadric, A_0 and A_2 lying on a common generator $l_{\{0,2\}}$, A_1 and A_3 lying on a common generator $l_{\{1,3\}}$; the points A_0 and A_3 were chosen to be the points at infinity of $l_{\{0,2\}}$ and $l_{\{1,3\}}$, respectively. Furthermore the corresponding parameter values $t_0, \ldots, t_3$ were chosen in a way that (76) holds. Three exemplares of the one-parametric set of solution cubics are shown.
Summarizing we get the following

Theorem 3.14 *Let $\mathcal{Q}^{d-1}$ be a hyperquadric in d-dimensional projective space $\mathbb{P}^d$ and let four points A_0, A_1, A_2, A_3 on $\mathcal{Q}^{d-1}$ and four corresponding pairwise distinct parameter values t_0, t_1, t_2, t_3 be given; then the following can be said about the solutions of the interpolation problem $IP_{\{0,1,2,3\}}$:*

1. *If $a_{\{0,1,2,3\}} \neq 0$ then there is no solution curve for $IP_{\{0,1,2,3\}}$ (general case).*

2. *If* $\forall i, j \in \{0,1,2,3\} : a_{\{i,j\}} = 0$ *- which of course implies* $a_{\{0,1,2,3\}} = 0$ *- then we either have*

    (a) *exactly one "solution curve" if* $A_0 = A_1 = A_2 = A_3$ *- the solution curve is this point - or*

    (b) *exactly one solution curve if* $[A_0, A_1, A_2, A_3]_p$ *is a line - the solution curve is this line - or*

    (c) *a* $\overline{d}$*-parametric set of solution curves where* $\overline{d} := \dim[A_0, A_1, A_2, A_3]_p \in \{2, 3\}$.

3. *If* $a_{\{0,1,2,3\}} = 0$ *but not all of the values* $a_{\{i,j\}}$ *are zero, a solution only exists if either*

    (a) $[A_0, A_1, A_2, A_3]_p$ *is a plane which intersects* $\mathcal{Q}^{d-1}$ *in a regular second-order curve or*

    (b) $[A_0, A_1, A_2, A_3]_p$ *is a 3-space, intersecting* $\mathcal{Q}^{d-1}$ *in a real quadratic cone, its vertex being one of these points or*

    (c) $[A_0, A_1, A_2, A_3]_p$ *is a 3-space, intersecting* $\mathcal{Q}^{d-1}$ *in a real quadratic cone or in an annular quadric* $\mathcal{Q}^2$ *and* $(A_0\ A_1\ A_2\ A_3) = (t_0\ t_1\ t_2\ t_3)$*, where the left-hand side of this equation denotes (one of) the cross-ratio(s) on* $\mathcal{Q}^2$.

In case 3 (a) we have exactly one solution curve, in case 3 (b) and 3 (c) a one-parametric set of solution curves all of them being cubics.

4 QB-curves

In affine space we construct Lagrange-interpolants with the help of Aitken's algorithm (see section 3.3). This subdivision-algorithm uses the ratio $\alpha(t, i, l) = \frac{t-t_i}{t_{i+l}-t_i}$ (see (42)). It is well-known that by simply replacing $\alpha(t, i, l)$ by the (i, l)-constant ratio

$$\beta(t) := \frac{t - t_0}{t_1 - t_0} \tag{77}$$

one gets a Bézier curve instead of the Lagrange-interpolant (de Casteljau algorithm, see figure 4.1).

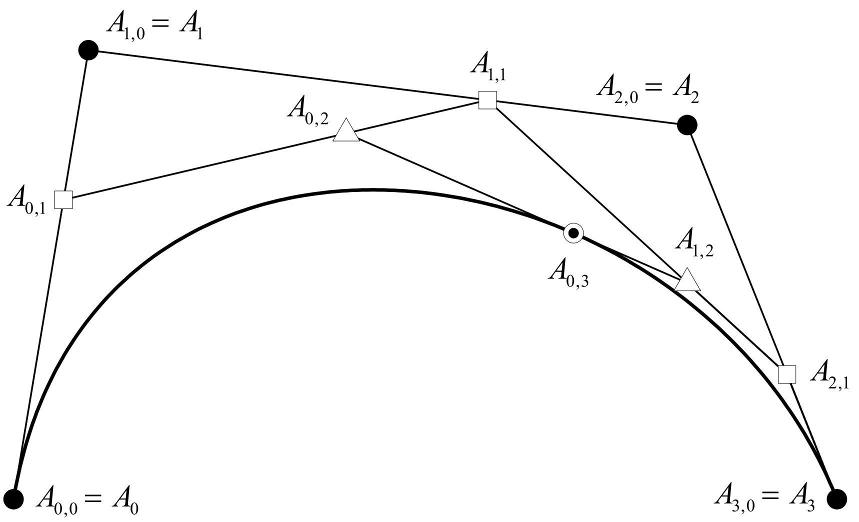

Figure 4.1 *Algorithm of de Casteljau for the construction of a Bézier curve.*

Bézier curves are represented by the parametrization

$$\mathbf{x}(t) = \sum_{i=0}^{n} b_i^n(t) \cdot \mathbf{a}_i, \tag{78}$$

where the functions $b_i^n(t)$ are the Bernstein-polynomials

$$b_i^n(t) = \frac{1}{(t_1 - t_0)^n} \cdot \binom{n}{i} \cdot (t - t_0)^i \cdot (t_1 - t)^{n-i} \tag{79}$$

on the intervall $[t_0, t_1]$ and $\mathbf{a}_i$ are again the affine coordinate-vectors of given points A_i, $i \in \{0, \ldots, n\}$.
Bézier curves do not interpolate the given points but they are connected with their control-structure in a nice geometric way:[26]

Theorem 4.1 *(a)* $\mathbf{x}(t_0) = \mathbf{a}_0$*,* $\mathbf{x}(t_1) = \mathbf{a}_n$*; for the parameter values* t_0 *and* t_1 *the first and the last points of the control-structure are interpolated, respectively.*

(b) The k-th derivative $\mathbf{x}^{(k)}(t)$ *for* $t = t_0$ *only depends on the points* $A_0, \ldots, A_k$*,* $k \in \{0, \ldots, n\}$*. Analogously: The k-th derivative* $\mathbf{x}^{(k)}(t)$ *for* $t = t_1$ *only depends on the points* $A_n, \ldots, A_{n-k}$*,* $k \in \{0, \ldots, n\}$*.*

(c) The curve is invariantly combined with its control-structure $A_0, \ldots, A_n$ *with respect to affine transformations.*

[26] For the proofs see e. g. [Hosch 1992, pages 115–128] or [Farin 1990, pages 30–32 and 40–46].

(d) A change of the starting- and end-parameter values: $t_0, t_1 \longrightarrow s_0, s_1$ only effects the parametrization of the curve (and not the curve itself).

For example property (b) yields that the first edge $[A_0, A_1]_p$ and the last edge $[A_{n-1}, A_n]_p$ of the control-structure are tangent to the curve in A_0 and A_1, respectively.
We naturally ask the following question: What kind of curves on a hyperquadric do we get if replacing the cross-ratio $(t_{l-1}\ t_i\ t_{i+1}\ t)$ used in formulas (53), (54) of section 3.3 by the (i,l)-constant crossratio $(t_0\ t_1\ t_2\ t)$? The definition and investigation of these curves will be the contents of this chapter.

4.1 Definition of QB-curves

For the further considerations we will use the notations:

$$\begin{array}{rcl} f_0(t) & := & (t_0 - t_1)\cdot(t_0 - t_2)\cdot(t - t_1)\cdot(t - t_2), \\ f_1(t) & := & (t_1 - t_0)\cdot(t_1 - t_2)\cdot(t - t_0)\cdot(t - t_2), \\ f_2(t) & := & (t_2 - t_0)\cdot(t_2 - t_1)\cdot(t - t_0)\cdot(t - t_1). \end{array} \tag{80}$$

For the polynomials f_i the following identity holds:

$$f_0(t)\cdot f_1(t) + f_1(t)\cdot f_2(t) + f_0(t)\cdot f_2(t) \quad \equiv \quad 0. \tag{81}$$

Furthermore, if

$$J \quad := \quad (j_0, \ldots, j_n) \tag{82}$$

is a (finite) sequence[27] of numbers we will denote the subsequence which one gets by taking away the numbers $j_{i_0}, \ldots, j_{i_k}$ from J with $k < n$ and $0 \le i_0 < \ldots < i_k \le n$ by $J_{j_{i_0},\ldots,j_{i_k}}$.
We now define a new set of polynomials:

Definition 4.1 *Let n be an even positive integer and $J := (j_0, \ldots, j_n)$ a sequence of positive integers; let furthermore $\mathcal{Q}^{d-1}$ be a hyperquadric in d-dimensional projective space $\mathbb{P}^d$ and A_{j_i} be points on $\mathcal{Q}^{d-1}$ with homogeneous coordinate vectors $\mathbf{a}_{j_i}$ for $i \in \{0, \ldots, n\}$ and let $t_0, t_1, t_2 \in \mathbb{R}$, pairwise distinct. Then we define polynomials $q_{J,i}(t)$ via the following recursion:*

For $n = 2$ $(J = (j_0, j_1, j_2))$:

$$\begin{array}{rcl} q_{J,0}(t) & := & f_0(t)\cdot a_{\{j_1,j_2\}} \\ q_{J,1}(t) & := & f_1(t)\cdot a_{\{j_0,j_2\}} \\ q_{J,2}(t) & := & f_2(t)\cdot a_{\{j_0,j_1\}} \end{array} \tag{83}$$

[27] The order of appearance of the numbers j_i is important; this is different to just regarding the set $\{j_0, \ldots, j_n\}$.

For n even, $n \geq 4$:

$$
\begin{array}{llll}
i \in \{0,\dots,\frac{n}{2}-2\}: & q_{J,i}(t) & := & a_{\{j_{\frac{n}{2}},j_{\frac{n}{2}+1}\}} \cdot f_0(t) \cdot q_{J_{j_{\frac{n}{2}},j_{\frac{n}{2}+1}},i}(t) \\
 & & + & a_{\{j_{\frac{n}{2}-1},j_{\frac{n}{2}+1}\}} \cdot f_1(t) \cdot q_{J_{j_{\frac{n}{2}-1},j_{\frac{n}{2}+1}},i}(t) \\
 & & + & a_{\{j_{\frac{n}{2}-1},j_{\frac{n}{2}}\}} \cdot f_2(t) \cdot q_{J_{j_{\frac{n}{2}-1},j_{\frac{n}{2}}},i}(t) \\
i = \frac{n}{2}-1: & q_{J,\frac{n}{2}-1}(t) & := & a_{\{j_{\frac{n}{2}},j_{\frac{n}{2}+1}\}} \cdot f_0(t) \cdot q_{J_{j_{\frac{n}{2}},j_{\frac{n}{2}+1}},\frac{n}{2}-1}(t) \\
i = \frac{n}{2}: & q_{J,\frac{n}{2}}(t) & := & a_{\{j_{\frac{n}{2}-1},j_{\frac{n}{2}+1}\}} \cdot f_1(t) \cdot q_{J_{j_{\frac{n}{2}-1},j_{\frac{n}{2}+1}},\frac{n}{2}-1}(t) \\
i = \frac{n}{2}+1: & q_{J,\frac{n}{2}+1}(t) & := & a_{\{j_{\frac{n}{2}-1},j_{\frac{n}{2}}\}} \cdot f_2(t) \cdot q_{J_{j_{\frac{n}{2}-1},j_{\frac{n}{2}}},\frac{n}{2}-1}(t) \\
i = \{\frac{n}{2}+2,\dots,n\}: & q_{J,i}(t) & := & a_{\{j_{\frac{n}{2}},j_{\frac{n}{2}+1}\}} \cdot f_0(t) \cdot q_{J_{j_{\frac{n}{2}},j_{\frac{n}{2}+1}},i-2}(t) \\
 & & + & a_{\{j_{\frac{n}{2}-1},j_{\frac{n}{2}+1}\}} \cdot f_1(t) \cdot q_{J_{j_{\frac{n}{2}-1},j_{\frac{n}{2}+1}},i-2}(t) \\
 & & + & a_{\{j_{\frac{n}{2}-1},j_{\frac{n}{2}}\}} \cdot f_2(t) \cdot q_{J_{j_{\frac{n}{2}-1},j_{\frac{n}{2}}},i-2}(t)
\end{array}
\tag{84}
$$

where $a_{i,j} := \langle \mathbf{a}_i, \mathbf{a}_j \rangle$ and $\langle .,. \rangle$ denotes the bilinear form belonging to $\mathcal{Q}^{d-1}$.

For example, if $n = 4$ we get

$$
\begin{aligned}
q_{J,0}(t) &= f_0(t) \cdot (a_{\{j_1,j_4\}} \cdot a_{\{j_2,j_3\}} \cdot f_0(t) + a_{\{j_1,j_3\}} \cdot a_{\{j_2,j_4\}} \cdot f_1(t) + a_{\{j_1,j_2\}} \cdot a_{\{j_3,j_4\}} \cdot f_2(t)) \\
q_{J,1}(t) &= a_{\{j_0,j_4\}} \cdot a_{\{j_2,j_3\}} \cdot f_0(t) \cdot f_1(t) \\
q_{J,2}(t) &= a_{\{j_0,j_4\}} \cdot a_{\{j_1,j_3\}} \cdot f_1^2(t) \\
q_{J,3}(t) &= a_{\{j_0,j_4\}} \cdot a_{\{j_1,j_2\}} \cdot f_1(t) \cdot f_2(t) \\
q_{J,4}(t) &= f_2(t) \cdot (a_{\{j_0,j_1\}} \cdot a_{\{j_2,j_3\}} \cdot f_0(t) + a_{\{j_0,j_2\}} \cdot a_{\{j_1,j_3\}} \cdot f_1(t) + a_{\{j_0,j_3\}} \cdot a_{\{j_1,j_2\}} \cdot f_2(t))
\end{aligned}
\tag{85}
$$

Using the polynomials $q_{J,i}(t)$ we now define a curve via its parametrization $\mathbf{y}_J(t)$:

Definition 4.2 *Let n, the sequence J, the hyperquadric $\mathcal{Q}^{d-1}$, the points A_{j_i} on $\mathcal{Q}^{d-1}$ and the values t_0, t_1, t_2 be given like in definition 4.1; then we call the curve*

$$
\mathbf{y}_J(t) \ := \ \sum_{i=0}^{n} q_{J,i}(t) \cdot \mathbf{a}_{j_i} \tag{86}
$$

a QB-curve.[28]

[28] The "Q" stands for quadric and the "B" for Bézier.

Lemma 4.1 *Let again n, the sequence J, the hyperquadric $\mathcal{Q}^{d-1}$, the points A_{j_i} and the values t_0, t_1, t_2 be given like in definition 4.1 and let furthermore $A_{j^*_{\frac{n}{2}}} \in \mathcal{Q}^{d-1}$,*

$$J^* \quad := \quad (j_0^* = j_0, \ldots, j^*_{\frac{n}{2}-1} = j_{\frac{n}{2}-1}, j^*_{\frac{n}{2}}, j^*_{\frac{n}{2}+1} = j_{\frac{n}{2}+1}, \ldots, j_n^* = j_n),$$

and

$$\mathbf{y}_J^*(t) \quad := \quad \sum_{i=0}^{n} q_{J^*,i}(t) \cdot \mathbf{a}_{j_i^*}$$

(This means that we get the QB-curve $\mathbf{y}_J^*(t)$ by simply replacing $A_{j_{\frac{n}{2}}}$ by $A_{j^*_{\frac{n}{2}}}$.) Then

(a)
$$\begin{aligned} \mathbf{y}_J(t) &= a_{\{j_{\frac{n}{2}}, j_{\frac{n}{2}+1}\}} \cdot f_0(t) \cdot \mathbf{y}_{J_{j_{\frac{n}{2}}, j_{\frac{n}{2}+1}}}(t) \\ &+ a_{\{j_{\frac{n}{2}-1}, j_{\frac{n}{2}+1}\}} \cdot f_1(t) \cdot \mathbf{y}_{J_{j_{\frac{n}{2}-1}, j_{\frac{n}{2}+1}}}(t) \\ &+ a_{\{j_{\frac{n}{2}-1}, j_{\frac{n}{2}}\}} \cdot f_2(t) \cdot \mathbf{y}_{J_{j_{\frac{n}{2}-1}, j_{\frac{n}{2}}}}(t), \end{aligned}$$

(b) $\langle \mathbf{y}_J(t), \mathbf{y}_J(t) \rangle \equiv 0$,

(c) $\langle \mathbf{y}_J(t), \mathbf{y}_{J^*}(t) \rangle = \left[\prod_{k=0}^{\frac{n}{2}-1} a^2_{\{j_k, j_{n-k}\}} \right] \cdot \langle \mathbf{a}_{j_{\frac{n}{2}}}, \mathbf{a}_{j^*_{\frac{n}{2}}} \rangle \cdot f_1^n(t)$.

Proof. (a) follows by direct computation after having substituted definition 4.1. For (b) and (c) we will use induction over n:

Initial step ($n = 2$).

(b) Here $\mathbf{y}_J(t)$ is identical with the parametrization (28) given in section 3.1, which shows that (b) is true for $n = 2$.

(c) First we compute $a_{\{j_0, j_1, j_2, j_1^*\}}$ (see definition 3.1):

$$\begin{aligned} a_{\{j_0,j_1,j_2,j_1^*\}} &\overset{\text{via definition 3.1}}{=} [p_{\{0,1\}}(t_2) \cdot p_{\{0,1\}}(t_1) \cdot a_{\{j_0,j_1\}} \cdot a_{\{j_2,j_1^*\}} \\ &\quad + p_{\{0,2\}}(t_1) \cdot p_{\{0,2\}}(t_1) \cdot a_{\{j_0,j_2\}} \cdot a_{\{j_1,j_1^*\}} \\ &\quad + p_{\{0,1\}}(t_2) \cdot p_{\{0,1\}}(t_1) \cdot a_{\{j_0,j_1^*\}} \cdot a_{\{j_1,j_2\}}] \\ &\overset{\text{as } p_{\{0,1\}}(t_1)=0}{=} p^2_{\{0,2\}}(t_1) \cdot a_{\{j_0,j_2\}} \cdot a_{\{j_1,j_1^*\}} \\ &= p^2_{\{0,2\}}(t_1) \cdot a_{\{j_0,j_2\}} \cdot \langle \mathbf{a}_{j_1}, \mathbf{a}_{j_1^*} \rangle. \end{aligned}$$

With the help of this identity and by making use of lemma 3.3 we then get

$$\langle \mathbf{y}_{(j_0,j_1,j_2)}(t), \mathbf{y}_{(j_0,j_1^*,j_2)}(t) \rangle \quad = \quad p^2_{\{j_0,j_2\}}(t) \cdot a_{\{j_0,j_2\}} \cdot a_{\{j_0,j_1,j_2,j_1^*\}}$$

$$= \quad (t_{j_1} - t_{j_0})^2 \cdot (t_{j_1} - t_{j_2})^2 \cdot (t - t_{j_0})^2 \cdot (t - t_{j_0})^2 \cdot a^2_{\{j_0,j_2\}} \cdot \langle \mathbf{a}_{j_1}, \mathbf{a}_{j_1^*} \rangle$$

$$= \quad a^2_{\{j_0,j_2\}} \cdot \langle \mathbf{a}_{j_1}, \mathbf{a}_{j_1^*} \rangle \cdot f_1^2(t).$$

Induction step (n even and ≥ 4).

(b) Substituting (a) and using definition 4.1 of the polynomials $q_{J,i}(t)$ and the induction hypothesis for (b) and (c) we obtain

$$\langle \mathbf{y}_J(t), \mathbf{y}_J(t)\rangle = $$

$$2 \cdot \underbrace{(f_0(t)\cdot f_1(t) + f_0(t)\cdot f_2(t) + f_1(t)\cdot f_2(t))}_{=\ 0 \text{ by using } (81).}$$

$$\times \left[\prod_{k=0}^{\frac{n}{2}-2} a^2_{\{j_k, j_{n-k}\}}\right] \cdot a_{\{j_{\frac{n}{2}}, j_{\frac{n}{2}+1}\}} \cdot a_{\{j_{\frac{n}{2}-1}, j_{\frac{n}{2}+1}\}} \cdot a_{\{j_{\frac{n}{2}-1}, j_{\frac{n}{2}}\}} \cdot f_1^{n-2}(t).$$

(c) Again we substitute (a) and make use of definition 4.1 and the induction hypothesis for (b) and (c); then

$$\langle \mathbf{y}_J(t), \mathbf{y}_{J^*}(t)\rangle = \left[\prod_{k=0}^{\frac{n}{2}-1} a^2_{\{j_k, j_{n-k}\}}\right] \cdot \langle \mathbf{a}_{j_{\frac{n}{2}}}, \mathbf{a}_{j^*_{\frac{n}{2}}}\rangle \cdot f_1^n(t)$$

$$+ \underbrace{(f_0(t)\cdot f_1(t) + f_0(t)\cdot f_2(t) + f_1(t)\cdot f_2(t))}_{=\ 0 \text{ by using } (81).} \cdot \left[\prod_{k=0}^{\frac{n}{2}-2} a^2_{\{j_k, j_{n-k}\}}\right] \cdot a_{\{j_{\frac{n}{2}-1}, j_{\frac{n}{2}+1}\}}$$

$$\times \left[a_{\{j_{\frac{n}{2}}, j_{\frac{n}{2}+1}\}} \cdot \langle \mathbf{p}_{j_{\frac{n}{2}-1}}, \mathbf{p}_{j^*_{\frac{n}{2}}}\rangle + a_{\{j_{\frac{n}{2}-1}, j_{\frac{n}{2}}\}} \cdot \langle \mathbf{p}_{j_{\frac{n}{2}+1}}, \mathbf{p}_{j^*_{\frac{n}{2}}}\rangle\right].$$

◇

Remark 4.1 *Due to lemma 4.1, (b) a QB-curve is part of the hyperquadric $\mathcal{Q}^{d-1}$ belonging to it.*

4.2 Some properties of QB-curves

We now intend to study QB-curves and the defining polynomials $q_{J,i}(t)$ in a more-detailed way. Some properties of the polynomials $q_{J,i}(t)$ are listed in

Theorem 4.2 *(a)*

$$q_{J,0}(t_0) = \left[\prod_{k=1}^{\frac{n}{2}} a_{\{j_k, j_{n-k+1}\}}\right] \cdot f_0^{\frac{n}{2}}(t_0),$$

$$q_{J,\frac{n}{2}}(t_1) = \left[\prod_{k=1}^{\frac{n}{2}} a_{\{j_{k-1}, j_{n-k+1}\}}\right] \cdot f_1^{\frac{n}{2}}(t_1),$$

$$q_{J,n}(t_2) = \left[\prod_{k=1}^{\frac{n}{2}} a_{\{j_{k-1}, j_{n-k}\}}\right] \cdot f_2^{\frac{n}{2}}(t_2).$$

(b)

$$\begin{aligned}\forall i\in\{1,\dots,n\}:\ q_{J,i}(t_0) &= 0,\\ \forall i\in\{0,\dots,\tfrac{n}{2}-1,\tfrac{n}{2}+1,\dots,n\}:\ q_{J,i}(t_1) &= 0,\\ \forall i\in\{0,\dots,n-1\}:\ q_{J,i}(t_2) &= 0.\end{aligned}$$

(c) Let n be even and ≥ 4; then

$$\forall i\in\{1,\dots,n-1\}:\ q_{J,i}(t)=a_{\{j_0,j_n\}}\cdot f_1(t)\cdot q_{J_{j_0,j_n},i-1}(t),$$

(d)

$$\forall k\in\{0,\dots,\frac{n}{2}-1\}:$$

$$\left\{\begin{array}{ccccccc}\frac{d^k}{(dt)^k}q_{J,k+1}(t_0) &=& \dots &=& \frac{d^k}{(dt)^k}q_{J,n-k}(t_0) &=& 0,\\ \frac{d^k}{(dt)^k}q_{J,k}(t_2) &=& \dots &=& \frac{d^k}{(dt)^k}q_{J,n-k-1}(t_2) &=& 0.\end{array}\right\}$$

(e) If $a_{\{i,j\}}\neq 0$ for all $i,j\in J$ with $i\neq j$ then the polynomials $q_{J,i}(t)$ form a basis of the vector-space of all polynomials of degree $\leq n$.

Proof.

(a) and (b) follow directly from definition 4.1 by using $f_1(t_0)=f_2(t_0)=f_0(t_1)=f_2(t_1)=f_0(t_2)=f_1(t_2)=0$.

(c) (Induction over n)

Initial step ($n=4$). The statement follows directly by checking the formulas (85).

Induction step (n even and ≥ 6). We have to give single proofs for

(1) $i\in\{1,\dots,\frac{n}{2}-2\}$,
(2) $i=\frac{n}{2}-1$,
(3) $i=\frac{n}{2}$,
(4) $i=\frac{n}{2}+1$ and
(5) $i\in\{\frac{n}{2}+2,\dots,n-1\}$.

Due to the recursive definition of the polynomials $q_{J,i}$ the proofs for (1) and (5) and also that ones of (2), (3) and (4) have the same architecture. So we exemplarily prove the assertions in the cases (1) and (2):

(1) $i \in \{1, \ldots, \frac{n}{2} - 2\}$:

$$
\begin{aligned}
q_{J,i}(t) \quad & \overset{\text{using definition 4.1}}{=} \quad a_{\{j_{\frac{n}{2}}, j_{\frac{n}{2}+1}\}} \cdot f_0(t) \cdot q_{J_{j_{\frac{n}{2}}, j_{\frac{n}{2}+1}}, i}(t) \\
& \quad + \quad a_{\{j_{\frac{n}{2}-1}, j_{\frac{n}{2}+1}\}} \cdot f_1(t) \cdot q_{J_{j_{\frac{n}{2}-1}, j_{\frac{n}{2}+1}}, i}(t) \\
& \quad + \quad a_{\{j_{\frac{n}{2}-1}, j_{\frac{n}{2}}\}} \cdot f_2(t) \cdot q_{J_{j_{\frac{n}{2}-1}, j_{\frac{n}{2}}}, i}(t) \\
& \overset{\text{by the induction hypothesis}}{=} \quad a_{\{j_0, j_n\}} \cdot f_1(t) \\
& \quad \times \quad [a_{\{j_{\frac{n}{2}}, j_{\frac{n}{2}+1}\}} \cdot f_0(t) \cdot q_{J_{j_0, j_{\frac{n}{2}}, j_{\frac{n}{2}+1}, j_n}, i}(t) \\
& \quad + \quad a_{\{j_{\frac{n}{2}-1}, j_{\frac{n}{2}+1}\}} \cdot f_1(t) \cdot q_{J_{j_0, j_{\frac{n}{2}-1}, j_{\frac{n}{2}+1}, j_n}, i}(t) \\
& \quad + \quad a_{\{j_{\frac{n}{2}-1}, j_{\frac{n}{2}}\}} \cdot f_2(t) \cdot q_{J_{j_0, j_{\frac{n}{2}-1}, j_{\frac{n}{2}}, j_n}, i}(t)] \\
& \overset{\text{using definition 4.1}}{=} \quad a_{\{j_0, j_n\}} \cdot f_1(t) \cdot q_{J_{j_0, j_n}, i-1}(t).
\end{aligned}
$$

(2) $i = \frac{n}{2} - 1$:

$$
\begin{aligned}
q_{J,\frac{n}{2}-1}(t) \quad & \overset{\text{using definition 4.1}}{=} \quad a_{\{j_{\frac{n}{2}}, j_{\frac{n}{2}+1}\}} \cdot f_0(t) \cdot q_{J_{j_{\frac{n}{2}}, j_{\frac{n}{2}+1}}, \frac{n}{2}-1}(t) \\
& \overset{\text{by the induction hypothesis}}{=} \quad a_{\{j_0, j_n\}} \cdot f_1(t) \\
& \quad \times \quad [a_{\{j_{\frac{n}{2}}, j_{\frac{n}{2}+1}\}} \cdot f_0(t) \cdot q_{J_{j_0, j_{\frac{n}{2}}, j_{\frac{n}{2}+1}, j_n}, \frac{n}{2}-2}(t) \\
& \overset{\text{using definition 4.1}}{=} \quad a_{\{j_0, j_n\}} \cdot f_1(t) \cdot q_{J_{j_0, j_n}, \frac{n}{2}-2}(t).
\end{aligned}
$$

(d) We only will prove the first line of the statement as the proof of the second is completely analogous. We have to show that $(t - t_0)^{k+1}$ is a factor of the polynomial $q_{J,i}(t)$ for all $k \in \{0, \ldots, \frac{n}{2} - 1\}$ and $i \in \{k + 1, \ldots, n - k\}$. We use induction over k:

Initial step ($k = 0$). In this case $i \in \{1, \ldots n\}$; so the statement is true due to (b).

Induction step ($k \in \{1, \ldots, \frac{n}{2} - 1\}$). As $i \in \{2, \ldots, n - 1\}$ we have

$$q_{J,i}(t) \quad = \quad a_{\{j_0, j_n\}} \cdot f_1(t) \cdot q_{J_{j_0, j_n}, i-1}(t)$$

due to (c). As $(t - t_0)$ is a factor of $f_1(t)$ and due to the induction hypothesis $(t - t_0)^k$ is a factor of $q_{J_{0,n}, i-1}(t)$ we obtain the desired result.

(e) (Induction over n.)

Initial step ($n = 2$). In this case the assertion is true as for any triple of pairwise distinct real numbers t_0, t_1, t_2 the polynomials $(t - t_0) \cdot (t - t_1)$, $(t - t_0) \cdot (t - t_2)$ and $(t - t_1) \cdot (t - t_2)$ form a basis of the vector-space of all polynomials of degree ≤ 2.

Induction step (n even and ≤ 4*).* It suffices to show that for any identically vanishing linear combination

$$\sum_{i=0}^{n} \lambda_i \cdot q_{J,i}(t) \overset{(t)}{\equiv} 0, \quad \lambda_i \in \mathbb{R}$$

all λ_i have to be zero. For $t = t_0$ this equation reads with the help of (a) and (b) like follows:

$$\lambda_0 \cdot \left[\prod_{k=1}^{\frac{n}{2}} a_{\{j_k, j_{n-k+1}\}} \right] \cdot f_0^{\frac{n}{2}}(t_0) \quad = \quad 0.$$

As moreover $a_{\{i,j\}} \neq 0$ for all $i, j \in J$, $i \neq j$ this means

$$\lambda_0 \quad = \quad 0.$$

Analogously we get

$$\lambda_n \quad = \quad 0.$$

With the help of (c) we have

$$\sum_{i=1}^{n-1} \lambda_i \cdot q_{J,i}(t) \quad = \quad a_{\{j_0,j_n\}} \cdot f_0(t) \cdot \sum_{i=1}^{n-1} \lambda_i \cdot q_{J_{j_0,j_n},i-1}(t).$$

But then using the induction hypothesis and the assumption $a_{\{j_0,j_n\}} \neq 0$ we obtain

$$\lambda_1 \quad = \quad \ldots \quad = \quad \lambda_{n-1} \quad = \quad 0.$$

$\diamond$

Using the recursion given in lemma 4.1, (a), we get the following formulas for the derivatives of the parametrization $\mathbf{y}_J(t)$ of a QB-curve:

$$\begin{aligned} \left(\tfrac{d}{dt}\mathbf{y}_J\right)(t) \quad = \quad & a_{\{j_{\frac{n}{2}}, j_{\frac{n}{2}+1}\}} \cdot \left(\tfrac{d}{dt}f_0\right)(t) \cdot \mathbf{y}_{J_{j_{\frac{n}{2}}, j_{\frac{n}{2}+1}}}(t) \\ + \quad & a_{\{j_{\frac{n}{2}-1}, j_{\frac{n}{2}+1}\}} \cdot \left(\tfrac{d}{dt}f_1\right)(t) \cdot \mathbf{y}_{J_{j_{\frac{n}{2}-1}, j_{\frac{n}{2}+1}}}(t) \\ + \quad & a_{\{j_{\frac{n}{2}-1}, j_{\frac{n}{2}}\}} \cdot \left(\tfrac{d}{dt}f_2\right)(t) \cdot \mathbf{y}_{J_{j_{\frac{n}{2}-1}, j_{\frac{n}{2}}}}(t) \\ + \quad & a_{\{j_{\frac{n}{2}}, j_{\frac{n}{2}+1}\}} \cdot f_0(t) \cdot \left(\tfrac{d}{dt}\mathbf{y}_{J_{j_{\frac{n}{2}}, j_{\frac{n}{2}+1}}}\right)(t) \\ + \quad & a_{\{j_{\frac{n}{2}-1}, j_{\frac{n}{2}+1}\}} \cdot f_1(t) \cdot \left(\tfrac{d}{dt}\mathbf{y}_{J_{j_{\frac{n}{2}-1}, j_{\frac{n}{2}+1}}}\right)(t) \\ + \quad & a_{\{j_{\frac{n}{2}-1}, j_{\frac{n}{2}}\}} \cdot f_2(t) \cdot \left(\tfrac{d}{dt}\mathbf{y}_{J_{j_{\frac{n}{2}-1}, j_{\frac{n}{2}}}}\right)(t) \end{aligned} \tag{87}$$

and for $k \geq 2$:

$$
\begin{aligned}
\left(\tfrac{d^k}{(dt)^k}\mathbf{y}_J\right)(t) &= \tbinom{k}{2}\cdot\Big[a_{\{j_{\frac{n}{2}},j_{\frac{n}{2}+1}\}}\cdot\left(\tfrac{d^2}{(dt)^2}f_0\right)(t)\cdot\left(\tfrac{d^{k-2}}{(dt)^{k-2}}\mathbf{y}_{J_{j_{\frac{n}{2}},j_{\frac{n}{2}+1}}}\right)(t)\\
&+ a_{\{j_{\frac{n}{2}-1},j_{\frac{n}{2}+1}\}}\cdot\left(\tfrac{d^2}{(dt)^2}f_1\right)(t)\cdot\left(\tfrac{d^{k-2}}{(dt)^{k-2}}\mathbf{y}_{J_{j_{\frac{n}{2}-1},j_{\frac{n}{2}+1}}}\right)(t)\\
&+ a_{\{j_{\frac{n}{2}-1},j_{\frac{n}{2}}\}}\cdot\left(\tfrac{d^2}{(dt)^2}f_2\right)(t)\cdot\left(\tfrac{d^{k-2}}{(dt)^{k-2}}\mathbf{y}_{J_{j_{\frac{n}{2}-1},j_{\frac{n}{2}}}}\right)(t)\Big]\\
&+ k\cdot\Big[a_{\{j_{\frac{n}{2}},j_{\frac{n}{2}+1}\}}\cdot\left(\tfrac{d}{dt}f_0\right)(t)\cdot\left(\tfrac{d^{k-1}}{(dt)^{k-1}}\mathbf{y}_{J_{j_{\frac{n}{2}},j_{\frac{n}{2}+1}}}\right)(t)\\
&+ a_{\{j_{\frac{n}{2}-1},j_{\frac{n}{2}+1}\}}\cdot\left(\tfrac{d}{dt}f_1\right)(t)\cdot\left(\tfrac{d^{k-1}}{(dt)^{k-1}}\mathbf{y}_{J_{j_{\frac{n}{2}-1},j_{\frac{n}{2}+1}}}\right)(t)\\
&+ a_{\{j_{\frac{n}{2}-1},j_{\frac{n}{2}}\}}\cdot\left(\tfrac{d}{dt}f_2\right)(t)\cdot\left(\tfrac{d^{k-1}}{(dt)^{k-1}}\mathbf{y}_{J_{j_{\frac{n}{2}-1},j_{\frac{n}{2}}}}\right)(t)\Big]\\
&+ \Big[a_{\{j_{\frac{n}{2}},j_{\frac{n}{2}+1}\}}\cdot f_0(t)\cdot\left(\tfrac{d^{k}}{(dt)^{k}}\mathbf{y}_{J_{j_{\frac{n}{2}},j_{\frac{n}{2}+1}}}\right)(t)\\
&+ a_{\{j_{\frac{n}{2}-1},j_{\frac{n}{2}+1}\}}\cdot f_1(t)\cdot\left(\tfrac{d^{k}}{(dt)^{k}}\mathbf{y}_{J_{j_{\frac{n}{2}-1},j_{\frac{n}{2}+1}}}\right)(t)\\
&+ a_{\{j_{\frac{n}{2}-1},j_{\frac{n}{2}}\}}\cdot f_2(t)\cdot\left(\tfrac{d^{k}}{(dt)^{k}}\mathbf{y}_{J_{j_{\frac{n}{2}-1},j_{\frac{n}{2}}}}\right)(t)\Big]
\end{aligned}
\tag{88}
$$

Our aim now is to derive properties of QB-curves similar to the ones of Bézier curves listed in theorem 4.1. For this we need some basic concepts of projective differential geometry:

Definition 4.3 *(a) Let $\mathbf{x} = \mathbf{x}(t) = (x_0(t), \ldots, x_d(t))^t$ be the parametrization of a curve c in d-dimensional projective space $\mathbb{P}^d$.*

If $g(t) \neq 0$ is an arbitrary function of t then

$$\overline{\mathbf{x}}(t) \; := \; g(t)\cdot\mathbf{x}(t)$$

is another parametrization of the curve c. This is called renormalization[29] *and $g(t)$ is called normalization function.*

If $t = h(s)$ is an arbitrary function with $\forall s : \frac{dh}{ds}(s) \neq 0$ then we also get a new parametrization of c via

$$\mathbf{x}^*(s) \; := \; \mathbf{x}(h(s)) \; .$$

The function $h(s)$ is called (regular) parameter transformation.

(b) Let now $\mathbf{x}_1(t)$ and $\mathbf{x}_2(s)$ be the parametrizations of two different curves c_1 and c_2 in d-dimensional projective space $\mathbb{P}^d$. Furthermore let the vector functions $\mathbf{x}_1(t)$ and $\mathbf{x}_2(s)$ be differentiable of class C^k. If $\mathbf{x}_1(t_0)$ and $\mathbf{x}_2(s_0)$ represent the same point A_0, then we say[30] *c_1 and c_2 have contact of order k in A_0 if there exist a parameter transformation $s = h(t)$ with $h(t_0) = s_0$ and a normalization function $g(t)$ so that for $\overline{\mathbf{y}}^*(t) := g(t)\cdot\mathbf{y}(h(t))$*

$$\forall i \in \{0,\ldots,k\} : \; \tfrac{d^i}{(dt)^i}\mathbf{x}(t_0) \; = \; \tfrac{d^i}{(dt)^i}\overline{\mathbf{y}}^*(t_0).$$

[29]In German language: *Umnormung*; compare with [Bol 1950, page 3].

[30]See [Bol 1950, page 136].

Theorem 4.3 *(a) To have contact of order k in a point is an equivalence relation on the set of curves which have C^k-differentiable parametrizations.*

(b) If c_1 and c_2 have contact of order k in A_0 then for all $i \in \{1,\dots,k\}$:

$$\left[\tfrac{d^0}{(dt)^0}\mathbf{x}_1(t_0),\dots,\tfrac{d^i}{(dt)^i}\mathbf{x}_1(t_0)\right] \;=\; \left[\tfrac{d^0}{(ds)^0}\mathbf{x}_2(s_0),\dots,\tfrac{d^i}{(ds)^i}\mathbf{x}_2(s_0)\right].$$

(c) If $\dim\left[\mathbf{x}_1(t_0),\tfrac{d}{dt}\mathbf{x}_1(t_0)\right]=2$ *then the following statement is true:*

c_1 and c_2 have contact of order 1 in A_0 if and only if they have the same tangent in this point.

(d) Now let $c\dots\mathbf{x}(t)$ be contained by a hypersurface $\mathcal{S}^{d-1}\subset\mathbb{P}^d$ and let

$$\dim\left[\mathbf{x}(t_0),\tfrac{d}{dt}\mathbf{x}(t_0),\tfrac{d^2}{(dt)^2}\mathbf{x}(t_0)\right] \;=\; 3.$$

Let furthermore $A_0^{(1)}$ and $A_0^{(2)}$ denote the points represented by the homogeneous coordinate vectors $\tfrac{d}{dt}\mathbf{x}(t_0)$ and $\tfrac{d^2}{(dt)^2}\mathbf{x}(t_0)$, respectively. Then the following statement is true:

If the plane[31] $[A_0,A_0^{(1)},A_0^{(2)}]_p$ is not part of the tangential-space of $\mathcal{S}^{d-1}$ in A_0 then the intersection curve $\overline{c}$ of $[A_0,A_0^{(1)},A_0^{(2)}]_p$ with $\mathcal{S}^{d-1}$ and c have contact of order 2.

Definition 4.4 *An equivalence class of curves determined by the relation described in theorem 4.3, (a) is called a line element of order k.*

Remark 4.2 *In general the converse statement of theorem 4.3, (b) is true only if $k=1$.*

The next four theorems will demonstrate that QB-curves and ordinary Bézier curves have similar properties.[32]

Theorem 4.4 *(Starting-middle-end-point interpolation, see figure 4.2.)*
Let n be an even positive integer, $J:=\{0,\dots,n\}$, $\mathcal{Q}^{d-1}$ be a hyperquadric in d-dimensional projective space $\mathbb{P}^d$, A_i points on $\mathcal{Q}^{d-1}$ with homogeneous coordinate vectors $\mathbf{a}_i$ for $i\in J$. Let furthermore t_0,t_1,t_2 be pairwise distinct values in $\mathbb{R}$ and let again

$$c \;\dots\; \mathbf{y}_J(t) \;=\; \sum_{i=0}^{n} q_{J,i}(t)\cdot\mathbf{a}_i$$

denote the QB-curve on $\mathcal{Q}^{d-1}$ belonging to this data. Then,

[31] $[A_0,A_0^{(1)},A_0^{(2)}]_p$ is the osculating plane of c in A_0.

[32] Compare the properties given in these four theorems with the properties (a), (b), (c), (d) of ordinary Bézier curves given in theorem 4.1.

- *if* $a_{\{1,n\}} \neq 0, a_{\{2,n-1\}} \neq 0, \ldots, a_{\{\frac{n}{2},\frac{n}{2}+1\}} \neq 0$ *then* $\mathbf{y}_J(t_0) \overset{\wedge}{=} A_0$,
- *if* $a_{\{0,n\}} \neq 0, a_{\{1,n-2\}} \neq 0, \ldots, a_{\{\frac{n}{2}-1,\frac{n}{2}+1\}} \neq 0$ *then* $\mathbf{y}_J(t_1) \overset{\wedge}{=} A_{\frac{n}{2}}$,
- *if* $a_{\{0,n-1\}} \neq 0, a_{\{1,n-2\}} \neq 0, \ldots, a_{\{\frac{n}{2}-1,\frac{n}{2}\}} \neq 0$ *then* $\mathbf{y}_J(t_2) \overset{\wedge}{=} A_n$.

This means that the curve interpolates the points A_0, $A_{\frac{n}{2}}$ *and* A_n *for the parameter values* t_0, t_1 *and* t_2, *respectively.*

Proof. The statements are direct consequences of theorem 4.2, (a) $\diamond$

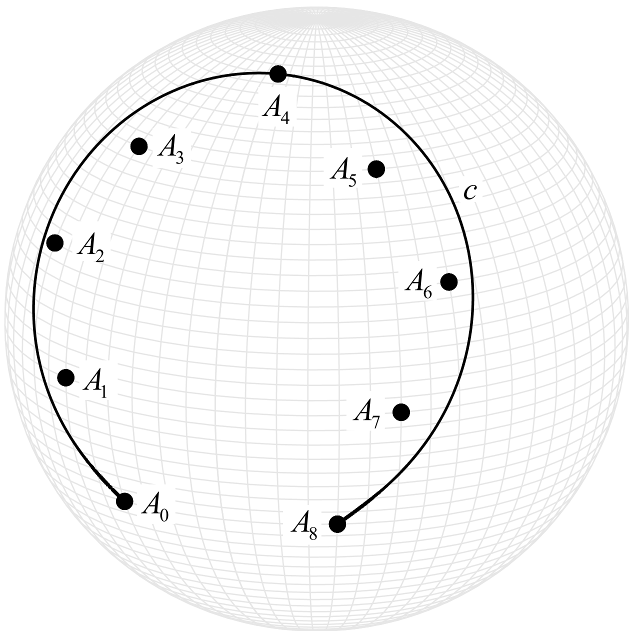

Figure 4.2 *A QB-curve c interpolates the starting point* A_0, *the midpoint* $A_{\frac{n}{2}}$ *and the endpoint* A_n *of the control points. In this example* $n = 8$ *and the quadric was chosen as sphere.*

Lemma 4.2 *Let* n, J, $\mathcal{Q}^{d-1}$, *the points* A_i, *the real numbers* t_0, t_1, t_2 *and the QB-curve* $c \ldots \mathbf{y}_J(t)$ *be given like in theorem 4.4 and let furthermore* $a_{\{i,j\}} = \langle \mathbf{a}_i, \mathbf{a}_j \rangle \neq 0 \ \forall i, j \in J;\ i \neq j$. *Then*

(a) $\mathbf{y}_J(t_0), \frac{d}{dt}\mathbf{y}_J(t_0), \frac{d^2}{(dt)^2}\mathbf{y}_J(t_0)$ *are linearly independent vectors.*

(b) $\mathbf{y}_J(t_2), \frac{d}{dt}\mathbf{y}_J(t_2), \frac{d^2}{(dt)^2}\mathbf{y}_J(t_2)$ *are linearly independent vectors.*

Proof. We only prove (a); then (b) follows due to the symmetry in the definition of the parametrization $\mathbf{y}_J(t)$. We use induction over n.

Initial step ($n = 2$). Here the statement trivially holds as $\mathbf{y}_J(t)$ is a quadratic parametrization of the conic section c.

Induction step (n even and ≥ 4). Due to theorem 4.2, (d) we have

$$\begin{aligned} \mathbf{y}_J(t_0) &= q_{J,0}(t_0) \cdot \mathbf{a}_0 \\ \tfrac{d}{dt}\mathbf{y}_J(t_0) &= \tfrac{d}{dt}q_{J,0}(t_0) \cdot \mathbf{a}_0 + \tfrac{d}{dt}q_{J,1}(t_0) \cdot \mathbf{a}_1 + \tfrac{d}{dt}q_{J,n}(t_0) \cdot \mathbf{a}_n. \end{aligned}$$

As moreover $\mathbf{a}_0, \mathbf{a}_1, \mathbf{a}_n$ are linearly independent vectors due to[33] $\langle \mathbf{a}_i, \mathbf{a}_j \rangle \neq 0$ and by using theorem 4.2, (a) and (c) we get

$$\begin{aligned} q_{J,0}(t_0) &\neq 0 \text{ and} \\ \tfrac{d}{dt}q_{J,1}(t_0) &\neq 0 \end{aligned}$$

So we have proved that $\mathbf{y}_J(t_0)$, $\frac{d}{dt}\mathbf{y}_J(t_0)$ are linearly independent vectors.

Putting $k = 2$ in (88) and evaluating this equality for $t = t_0$ we get

$$\begin{aligned} \tfrac{d^2}{(dt)^2}\mathbf{y}_J(t_0) &= a_{\{j_{\frac{n}{2}}, j_{\frac{n}{2}+1}\}} \cdot \left(\tfrac{d^2}{(dt)^2} f_0\right)(t_0) \cdot \mathbf{y}_{J_{j_{\frac{n}{2}}, j_{\frac{n}{2}+1}}}(t_0) \\ &+ a_{\{j_{\frac{n}{2}-1}, j_{\frac{n}{2}+1}\}} \cdot \left(\tfrac{d^2}{(dt)^2} f_1\right)(t_0) \cdot \mathbf{y}_{J_{j_{\frac{n}{2}-1}, j_{\frac{n}{2}+1}}}(t_0) \\ &+ a_{\{j_{\frac{n}{2}-1}, j_{\frac{n}{2}}\}} \cdot \left(\tfrac{d^2}{(dt)^2} f_2\right)(t_0) \cdot \mathbf{y}_{J_{j_{\frac{n}{2}-1}, j_{\frac{n}{2}}}}(t_0) \\ &+ 2 \cdot \Big[a_{\{j_{\frac{n}{2}}, j_{\frac{n}{2}+1}\}} \cdot \left(\tfrac{d}{dt} f_0\right)(t_0) \cdot \left(\tfrac{d}{dt}\mathbf{y}_{J_{j_{\frac{n}{2}}, j_{\frac{n}{2}+1}}}\right)(t_0) \\ &+ a_{\{j_{\frac{n}{2}-1}, j_{\frac{n}{2}+1}\}} \cdot \left(\tfrac{d}{dt} f_1\right)(t_0) \cdot \left(\tfrac{d}{dt}\mathbf{y}_{J_{j_{\frac{n}{2}-1}, j_{\frac{n}{2}+1}}}\right)(t_0) \\ &+ a_{\{j_{\frac{n}{2}-1}, j_{\frac{n}{2}}\}} \cdot \left(\tfrac{d}{dt} f_2\right)(t_0) \cdot \left(\tfrac{d}{dt}\mathbf{y}_{J_{j_{\frac{n}{2}-1}, j_{\frac{n}{2}}}}\right)(t_0)\Big] \\ &+ a_{\{j_{\frac{n}{2}}, j_{\frac{n}{2}+1}\}} \cdot f_0(t_0) \cdot \left(\tfrac{d^2}{(dt)^2}\mathbf{y}_{J_{j_{\frac{n}{2}}, j_{\frac{n}{2}+1}}}\right)(t_0). \end{aligned} \tag{89}$$

Additionally we have due to theorem 4.4

$$\begin{aligned} \mathbf{y}_{J_{j_{\frac{n}{2}}, j_{\frac{n}{2}+1}}}(t_0) &= \lambda_0 \cdot \mathbf{a}_0, \ \ \lambda_0 \neq 0, \\ \mathbf{y}_{J_{j_{\frac{n}{2}-1}, j_{\frac{n}{2}+1}}}(t_0) &= \lambda_1 \cdot \mathbf{a}_0, \ \ \lambda_1 \neq 0, \\ \mathbf{y}_{J_{j_{\frac{n}{2}-1}, j_{\frac{n}{2}}}}(t_0) &= \lambda_2 \cdot \mathbf{a}_0, \ \ \lambda_2 \neq 0. \end{aligned} \tag{90}$$

[33]Compare with section 3.1, discussion of the case $n = 2$.

Furthermore

$$\frac{d}{dt}\mathbf{y}_{J_{j_{\frac{n}{2}-1},j_{\frac{n}{2}+1}}}(t_0), \frac{d}{dt}\mathbf{y}_{J_{j_{\frac{n}{2}-1},j_{\frac{n}{2}}}}(t_0) \in \left[\mathbf{y}_{J_{j_{\frac{n}{2}},j_{\frac{n}{2}+1}}}(t_0),\ \frac{d}{dt}\mathbf{y}_{J_{j_{\frac{n}{2}},j_{\frac{n}{2}+1}}}(t_0)\right]. \tag{91}$$

Using the induction hypothesis for $\mathbf{y}_{J_{j_{\frac{n}{2}},j_{\frac{n}{2}+1}}}(t)$, the linear independence of the vectors $\mathbf{y}_J(t_0), \frac{d}{dt}\mathbf{y}_J(t_0)$ and (89), (90), (91) we get the desired result $\diamond$

As a conclusion we have

Remark 4.3 *Under the assumptions of lemma 4.2 the starting-point A_0 and the end-point A_n cannot be inflection points on the QB-curve c.*

Theorem 4.5 *(Differential geometry properties in the starting- and the end-point.) Let n, J, $\mathcal{Q}^{d-1}$, the points A_i, the real numbers t_0, t_1, t_2 and the QB-curve $c \ldots \mathbf{y}_J(t)$ be given like in theorem 4.4. Let moreover*

$$c^* \ldots \mathbf{y}^*_{J^*}(t) = \sum_{i=0}^{n^*} q^*_{J^*,i}(t) \cdot \mathbf{a}^*_i$$

denote another QB-curve belonging to n^*; $J^* := \{0, \ldots, n^*\}$ and $A^*_i \in \mathcal{Q}^{d-1}$; and let furthermore $a_{\{i,j\}} = \langle \mathbf{a}_i, \mathbf{a}_j \rangle \neq 0\ \forall i,j \in J;\ i \neq j$ and analogously $a^*_{\{i,j\}} = \langle \mathbf{a}^*_i, \mathbf{a}^*_j \rangle \neq 0\ \forall i,j \in J^*;\ i \neq j$; then the following statements are true:

(a) If $A^*_0 = A_0, A^*_1 = A_1, A^*_n = A_n$ (equality of the first two and the last points) then c and c^* have contact of order 1 in A_0.

(b) If $A^*_0 = A_0, A^*_{n^*-1} = A_{n-1}, A_{n^*} = A_n$ (equality of the first and the last two points) then c and c^* have contact of order 1 in A_n.

(c) If $A^*_0 = A_0, A^*_1 = A_1, A^*_2 = A_2, A^*_{n^*-1} = A_{n-1}, A^*_n = A_n$ (equality of the first three and the last two points) then c and c^* have contact of order 2 in A_0.

(d) If $A^*_0 = A_0, A^*_1 = A_1, A^*_{n^*-2} = A_{n-2}, A^*_{n^*-1} = A_{n-1}, A_{n^*} = A_n$ (equality of the first two and the last three points) then c and c^* have contact of order 2 in A_n.

Proof. It suffices to prove (a) and (c). Then the statements (b) and (d) follow due to the symmetry in the definition of the parametrization $\mathbf{y}_J(t)$.

(a) As $\langle \mathbf{a}_0, \mathbf{a}_1 \rangle \neq 0$, $\langle \mathbf{a}_0, \mathbf{a}_n \rangle \neq 0$, $\langle \mathbf{a}_1, \mathbf{a}_n \rangle \neq 0$ the three points A_0, A_1, A_n span up a plane $[A_0, A_1, A_n]_p$ which intersects $\mathcal{Q}^{d-1}$ in a regular second-order curve (conic section).[34] Let $A_0^{(1)}$, $A_0^{(1)*}$ denote the points represented by the vectors $\frac{d}{dt}\mathbf{y}_J(t_0)$, $\frac{d}{dt}\mathbf{y}^*_{J^*}(t_0)$, respectively . Due to theorem 4.2, (d) we have $A_0^{(1)} \in [A_0, A_1, A_n]_p$. On the other hand $A_0^{(1)}$ has to lie in the hyperplane tangent[35] to $\mathcal{Q}^{d-1}$ in A_0. So, $A_0^{(1)}$ is on the intersection line of this hyperplane and $[A_0, A_1, A_n]_p$. As the same is true for $A_0^{(1)*}$ the two curves have the same tangent in A_0. So they have contact of order one in this point (theorem 4.3, (c)).

[34] See section 3.1, discussion of the case $n = 2$.
[35] A_0 is a regular point on $\mathcal{Q}^{d-1}$.

(c) Let $\overline{c}$ be the QB-curve defined by the points $A_0, A_1, A_2, A_{n-1}, A_n$ and let $\overline{\mathbf{y}}(t)$ denote its parametrization. An examination of (89) shows that

$$\frac{d^2}{(dt)^2}\mathbf{y}_J(t_0) \in [\overline{\mathbf{y}}(t_0), \frac{d}{dt}\overline{\mathbf{y}}(t_0), \frac{d^2}{(dt)^2}\overline{\mathbf{y}}(t_0)]$$

(this follows by induction). Moreover

$$[\mathbf{y}_J(t_0), \frac{d}{dt}\mathbf{y}_J(t_0), \frac{d^2}{(dt)^2}\mathbf{y}_J(t_0)] = [\overline{\mathbf{y}}(t_0), \frac{d}{dt}\overline{\mathbf{y}}(t_0), \frac{d^2}{(dt)^2}\overline{\mathbf{y}}(t_0)]$$

and

$$\dim\,[\mathbf{y}_J(t_0), \frac{d}{dt}\mathbf{y}_J(t_0), \frac{d^2}{(dt)^2}\mathbf{y}_J(t_0)] = 3$$

(Lemma 4.2). So we see that c and $\overline{c}$ have the same osculating plane in A_0. Analogously the osculating planes of c^* and $\overline{c}$ are identical.

Moreover this plane contains the common (see (a)) tangent of c and c^* in A_0 and thus has to intersect the hyperquadric $\mathcal{Q}^{d-1}$ in a regular[36] second-order curve $\tilde{c}$. So, due to theorem 4.3, (d) both curves have contact of order 2 in A_0 with this conic section $\tilde{c}$. Thus c and c^* also have contact of order 2 in A_0 with each other (theorem 4.3, (a)) $\diamond$

[36] Any plane through a line tangent to but not contained by a hyperquadric $\mathcal{Q}^{d-1}$ intersects $\mathcal{Q}^{d-1}$ in a regular second-order curve.

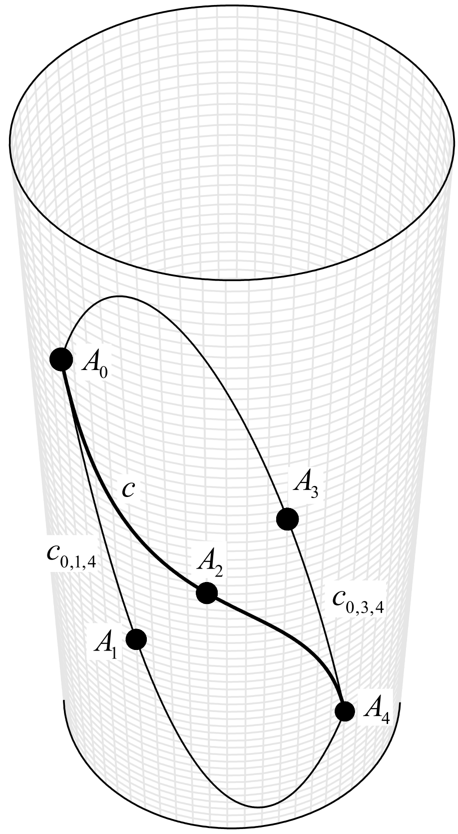

Figure 4.3 *The conic section $c_{0,1,n}$ determined by the points A_0, A_1, A_n is tangent to the QB-curve c in A_0. Analogously: The conic section $c_{0,n-1,n}$ determined by the points A_0, A_{n-1}, A_n is tangent to c in A_n. In this example $n = 4$ and the quadric was chosen as cylinder.*

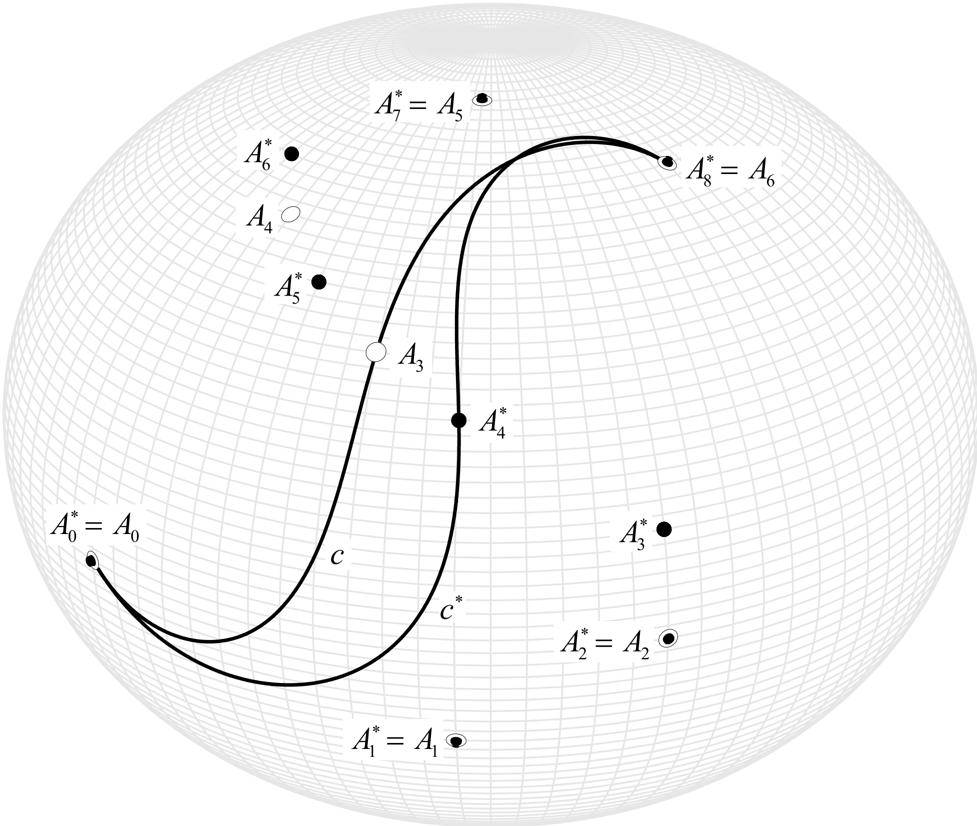

Figure 4.4 *The two QB-curves c (control points A_i, large empty circles) and c^* (control points A_i^*, small black-filled circles) are of algebraic order* 6 *and* 8*, respectively. As $A_0^* = A_0$, $A_1^* = A_1$, $A_2^* = A_2$, $A_7^* = A_5$, $A_8^* = A_6$ they have contact of order two in $A_0^* = A_0$ and contact of order one $A_8^* = A_6$. In this example the quadric was chosen as ellipsoid.*

The previous theorem implies that the line element of order one determined by the QB-curve c and

- the point A_0 only depends on the points A_0, A_1, A_n,
- the point A_n only depends on the points A_0, A_{n-1}, A_n.

Analogously, the line element of order two determined by the QB-curve c and

- the point A_0 only depends on the points $A_0, A_1, A_2, A_{n-1}, A_n$,
- the point A_n only depends on the points $A_0, A_1, A_{n-2}, A_{n-1}, A_n$.

Theorem 4.6 *(Projective-invariant connection of a QB-curve with its control structure $A_0, \ldots, A_n$.)*

Let n, J, $\mathcal{Q}^{d-1}$, the points A_i, the real numbers t_0, t_1, t_2 and the QB-curve $c \ldots \mathbf{y}_J(t)$ be given like in theorem 4.4. If κ is an autocollineation of $\mathbb{P}^d$ and $\tilde{\mathcal{Q}}^{d-1} := \kappa(\mathcal{Q}^{d-1})$, $\tilde{A}_i := \kappa(A_i)$ and $\tilde{q}_{J,i}(t)$ denote the polynomials belonging to the points $\tilde{A}_0, \ldots, \tilde{A}_n$ according to definition 4.1 then $\tilde{c} := \kappa(c)$ is the QB-curve represented by

$$\tilde{\mathbf{y}}_J(t) \;=\; \sum_{i=0}^{n} \tilde{q}_{J,i}(t) \cdot \tilde{\mathbf{a}}_i$$

on $\tilde{\mathcal{Q}}^{d-1}$.
This means that if for a given control structure $A_0, \ldots, A_n$ one constructs a QB-curve and then maps it via a collineation he gets the same result as if first mapping the control structure and then constructing the QB-curve.

Proof. The transformation equation of the collineation κ has the form

$$\tilde{\mathbf{x}} \;=\; \mathbf{K} \cdot \mathbf{x}, \tag{92}$$

where $\mathbf{K}$ is a regular $(d+1) \times (d+1)$-matrix and $\mathbf{x}, \tilde{\mathbf{x}}$ are the homogeneous coordinate vectors of a point X and its image $\tilde{X} = \kappa(X)$, respectively. So, $\tilde{c}$ has the parametrization

$$\mathbf{K} \cdot \mathbf{y}_J(t) \;=\; \sum_{i=0}^{n} q_{J,i}(t) \cdot \tilde{\mathbf{a}}_i \tag{93}$$

If

$$\langle \mathbf{x}, \mathbf{y} \rangle \;=\; \mathbf{x}^t \cdot \mathbf{M} \cdot \mathbf{y}$$

denotes the bilinear form belonging to $\mathcal{Q}^{d-1}$ then

$$\langle \tilde{\mathbf{x}}, \tilde{\mathbf{y}} \rangle_{\sim} \;=\; \tilde{\mathbf{x}}^t \cdot (\mathbf{K}^{-1})^t \cdot \mathbf{M} \cdot \mathbf{K}^{-1} \cdot \tilde{\mathbf{y}}$$

is that one of $\tilde{\mathcal{Q}}^{d-1}$. Thus the inner products $\langle \mathbf{a}_i, \mathbf{a}_j \rangle$, are not effected by κ. As the polynomials $q_{J,i}(t)$ only depend on these inner products, we also have

$$q_{J,i}(t) \;= \tilde{q}_{J,i}(t), \text{ for } i \in \{0, \ldots, n\}.$$

This together with (93) proves the statement $\diamond$

Theorem 4.7 *(Parameter-invariance of QB-curves.)*
Let n, J, $\mathcal{Q}^{d-1}$, the points A_i, the real numbers t_0, t_1, t_2 and the QB-curve $c \ldots \mathbf{y}_J(t)$ be given like in theorem 4.4. If we replace the real numbers t_0, t_1, t_2 by pairwise distinct numbers s_0, s_1, s_2, we get the parametrization of the QB-curve belonging to the old points A_i and the new triple s_0, s_1, s_2 via

$$\mathbf{y}^*_J(s) \;=\; \sum_{i=0}^{n} q^*_{J,i}(s) \cdot \tilde{\mathbf{a}}_i \;=\; g(s)^{\frac{n}{2}} \cdot \mathbf{y}_J(h(s)),$$

with the renormalization function

$$g(s) \; := \; \frac{(c \cdot s_0 + d)^2 \cdot (c \cdot s_1 + d)^2 \cdot (c \cdot s_2 + d)^2}{(a \cdot d - b \cdot c)^4} \cdot (c \cdot s + d)^2 \tag{94}$$

and the fractional linear parameter transformation[37]

$$t \; = \; h(s) \; := \; \frac{a \cdot s + b}{c \cdot s + d}, \tag{95}$$

where $a : b : c : d$ are uniquely determined by the three equations

$$t_i \; = \; \frac{a \cdot s_i + b}{c \cdot s_i + d}, \;\; i \in \{0, 1, 2\}. \tag{96}$$

This means that replacing t_0, t_1, t_2 by s_0, s_1, s_2 only effects the parametrization of the curve and not the curve itself.

Proof. For the polynomials $f_0(t), f_1(t), f_2(t)$ (see (80)) and the polynomials

$$\begin{array}{rcl} f_0^*(s) & := & (s_0 - s_1) \cdot (s_0 - s_2) \cdot (s - s_1) \cdot (s - s_2), \\ f_1^*(s) & := & (s_1 - s_0) \cdot (s_1 - s_2) \cdot (s - s_0) \cdot (s - s_2), \\ f_2^*(s) & := & (s_2 - s_0) \cdot (s_2 - s_1) \cdot (s - s_0) \cdot (s - s_1). \end{array} \tag{97}$$

we by direct computation verify the identity

$$f_i^*(s) \; := \; g(s) \cdot f_i(h(s)), \;\; \text{for} \;\; i \in \{0, 1, 2\}. \tag{98}$$

But then due to the recursive definition of the polynomials $q_{J,i}(t)$ we also have

$$q_{J,i}^*(s) \; := \; g(s)^{\frac{n}{2}} \cdot q_{J,i}(h(s)), \;\; \text{for} \;\; i \in \{0, \ldots, n\}, \tag{99}$$

which proves the statement $\diamond$

4.3 Geometric construction of QB-curves

In this section we will see that replacing the cross-ratio $(t_{l-1}\ t_i\ t_{i+1}\ t)$ used in the algorithm for computing the rational interpolant[38] by the (i, l)-constant crossratio $(t_0\ t_1\ t_2\ t)$ yields a geometric algorithm for the computation of QB-curves.

[37] In terms of projective geometry this describes a collineation (projectivity) on the line of real numbers considering it as a line in projective space.

[38] See section 3.3

Theorem 4.8 *Let n be an even positive integer and $J := (0,\dots,n)$; let furthermore $\mathcal{Q}^{d-1}$ be a hyperquadric in d-dimensional projective space $\mathbb{P}^d$ and A_i be points on $\mathcal{Q}^{d-1}$ with homogeneous coordinate vectors $\mathbf{a}_i$ for $i \in J$ and let $t_0, t_1, t_2 \in \mathbb{R}$, pairwise distinct. If the vectors $\mathbf{b}_{i,l}(t)$ are defined via*

$$\begin{aligned}
\mathbf{b}_{i,0}(t) &:= \mathbf{a}_i \text{ for } i \in J, && (100)\\
\mathbf{b}_{i,l}(t) &:= f_0(t)\cdot\langle \mathbf{b}_{i,l-1}(t), \mathbf{b}_{n-l+1,l-1}(t)\rangle \cdot \mathbf{b}_{l-1,l-1}(t)\\
&+ f_1(t)\cdot\langle \mathbf{b}_{l-1,l-1}(t), \mathbf{b}_{n-l+1,l-1}(t)\rangle \cdot \mathbf{b}_{i,l-1}(t)\\
&+ f_2(t)\cdot\langle \mathbf{b}_{l-1,l-1}(t), \mathbf{b}_{i,l-1}(t)\rangle \cdot \mathbf{b}_{n-l+1,l-1}(t)\\
&\text{for } l \in \{1,\dots,\frac{n}{2}\} \text{ and } i \in \{l,\dots,n-l\}. && (101)
\end{aligned}$$

then

$$\mathbf{b}_{i,1}(t) = \mathbf{y}_{(0,i,n)}(t), \qquad (102)$$

and for $l \in \{2,\dots,\frac{n}{2}\}$ and $i \in \{l,\dots,n-l\}$:

$$\mathbf{b}_{i,l}(t) = \left[\prod_{k=0}^{l-2} a_{\{k,n-k\}}^{e_{k,l}}\right] \cdot f_1^{\overline{e}_l}(t) \cdot \mathbf{y}_{(0,\dots,l-1,i,n-l+1,\dots,n)}(t), \qquad (103)$$

where

$$e_{k,l} = 2 \cdot \sum_{m=0}^{l-k-2} 3^m \qquad (104)$$

and

$$\overline{e}_l = 2 \cdot \sum_{m=0}^{l-2} (l-m-1)\cdot 3^m. \qquad (105)$$

Proof. For $l = 1$ we have

$$\begin{aligned}
\mathbf{b}_{i,1}(t) &= f_0(t)\cdot a_{\{i,n\}}\cdot \mathbf{b}_0 + f_1(t)\cdot a_{\{0,n\}}\cdot \mathbf{b}_i + f_2(t)\cdot a_{\{0,i\}}\cdot \mathbf{b}_n\\
&= \mathbf{y}_{(0,i,n)}(t).
\end{aligned}$$

The proof for $l \geq 2$ is given by induction. For the exponents $e_{k,l}$ and $\overline{e}_l$ the recursion formulas

$$e_{k,l} = 3\cdot e_{k,l-1} + 2, \qquad (106)$$

$$\overline{e}_l = 3\cdot \overline{e}_l + 2\cdot(l-1) \qquad (107)$$

can be verified by direct computation.

Initial step ($l = 2$): Using (102) we obtain

$$\begin{aligned}\mathbf{b}_{i,2}(t) &= f_0(t) \cdot \langle \mathbf{y}_{(0,i,n)}(t), \mathbf{y}_{(0,n-1,n)}(t) \rangle \cdot \mathbf{y}_{(0,1,n)}(t) \\ &+ f_1(t) \cdot \langle \mathbf{y}_{(0,1,n)}(t), \mathbf{y}_{(0,n-1,n)}(t) \rangle \cdot \mathbf{y}_{(0,i,n)}(t) \\ &+ f_2(t) \cdot \langle \mathbf{y}_{(0,1,n)}(t), \mathbf{y}_{(0,i,n)}(t) \rangle \cdot \mathbf{y}_{(0,n-1,n)}(t) \\ &\overset{\text{via lemma 4.1, (c),(a)}}{=} \\ & f_1^2(t) \cdot a_{\{0,n\}}^2 \cdot \mathbf{y}_{(0,1,i,n-1,n)}(t).\end{aligned}$$

This proves the statement for $l = 2$, as $e_{0,2} = \overline{e}_2 = 2$.

Induction step ($l \geq 3$): With the help of the induction hypothesis we can assume

$$\mathbf{b}_{i,l-1}(t) = \left[\prod_{k=0}^{l-3} a_{\{k,n-k\}}^{e_{k,l-1}}\right] \cdot f_1^{\overline{e}_{l-1}}(t) \cdot \mathbf{y}_{(0,\dots,l-2,i,n-l+2,\dots,n)}(t), \tag{108}$$

$$\mathbf{b}_{n-l+1,l-1}(t) = \left[\prod_{k=0}^{l-3} a_{\{k,n-k\}}^{e_{k,l-1}}\right] \cdot f_1^{\overline{e}_{l-1}}(t) \cdot \mathbf{y}_{(0,\dots,l-2,n-l+1,n-l+2,\dots,n)}(t), \tag{109}$$

$$\mathbf{b}_{n-l+1,l-1}(t) = \left[\prod_{k=0}^{l-3} a_{\{k,n-k\}}^{e_{k,l-1}}\right] \cdot f_1^{\overline{e}_{l-1}}(t) \cdot \mathbf{y}_{(0,\dots,l-2,l-1,n-l+2,\dots,n)}(t). \tag{110}$$

Thus, using (108), (109), (110) and the definition (101) of the $\mathbf{b}_{i,l}(t)$ we get

$$\begin{aligned}\mathbf{b}_{i,l}(t) &= \left[\prod_{k=0}^{l-3} a_{\{k,n-k\}}^{3\cdot e_{k,l-1}}\right] \cdot f_1^{3\cdot\overline{e}_{l-1}}(t) \\ &\times \big[f_0(t) \cdot \langle \mathbf{y}_{(0,\dots,l-2,i,n-l+2,\dots,n)}(t), \mathbf{y}_{(0,\dots,l-2,n-l+1,n-l+2,\dots,n)}(t) \rangle \\ &\qquad \times \mathbf{y}_{(0,\dots,l-2,l-1,n-l+2,\dots,n)}(t) \\ &+ f_1(t) \cdot \langle \mathbf{y}_{(0,\dots,l-2,l-1,n-l+2,\dots,n)}(t), \mathbf{y}_{(0,\dots,l-2,n-l+1,n-l+2,\dots,n)}(t) \rangle \\ &\qquad \times \mathbf{y}_{(0,\dots,l-2,i,n-l+2,\dots,n)}(t) \\ &+ f_2(t) \cdot \langle \mathbf{y}_{(0,\dots,l-2,l-1,n-l+2,\dots,n)}(t), \mathbf{y}_{(0,\dots,l-2,i,n-l+2,\dots,n)}(t) \rangle \\ &\qquad \times \mathbf{y}_{(0,\dots,l-2,n-l+1,n-l+2,\dots,n)}(t)\big] \\ &\overset{\text{using lemma 4.1, (c)}}{=} \\ &\left[\prod_{k=0}^{l-3} a_{\{k,n-k\}}^{3\cdot e_{k,l-1}}\right] \cdot \left[\prod_{k=0}^{l-2} a_{\{k,n-k\}}^{2}\right] \cdot f_1^{3\cdot\overline{e}_{l-1}}(t) \cdot f_1^{2\cdot(l-1)} \cdot \mathbf{y}_{(0,\dots,l-1,i,n-l+1,\dots,n)}(t) \ .\end{aligned}$$

With the help of (106) and (107) we obtain

$$\left[\prod_{k=0}^{l-3} a_{\{k,n-k\}}^{3\cdot e_{k,l-1}}\right]\cdot\left[\prod_{k=0}^{l-2} a_{\{k,n-k\}}^{2}\right] = \left[\prod_{k=0}^{l-2} a_{\{k,n-k\}}^{e_{k,l}}\right],$$
$$f_1^{3\cdot\overline{e}_{l-1}}(t)\cdot f_1^{2\cdot(l-1)} = f_1^{\overline{e}_l}(t),$$

which completes the proof $\diamond$

As

$$\mathbf{b}_{\frac{n}{2},\frac{n}{2}}(t) = \left[\prod_{k=0}^{\frac{n}{2}-2} a_{\{k,n-k\}}^{e_{k,\frac{n}{2}}}\right]\cdot f_1^{\overline{e}_{\frac{n}{2}}}(t)\cdot\mathbf{y}_J(t) \tag{111}$$

we end up at the following

Algorithm 4.1 *Let the assumptions of theorem 4.8 be fulfilled and let furthermore*

$$\forall k\in\{0,\ldots,\frac{n}{2}-2\}: a_{\{k,n-k\}}\neq 0,$$

then for any $t\in\mathbb{R}\setminus\{t_0,t_2\}$ the vector $\mathbf{b}_{\frac{n}{2},\frac{n}{2}}(t)$ computed via the recursion formulas (100), (101) represents the point belonging to t on the QB-curve with the control-structure $A_0,\ldots,A_n$.

Remark 4.4 *(a) If $a_{\{k,n-k\}}=0$ for any $k\in\{0,\ldots,\frac{n}{2}-2\}$ then $\mathbf{b}_{\frac{n}{2},\frac{n}{2}}(t)$ is the zero-vector for all t.*

(b) $\mathbf{b}_{\frac{n}{2},\frac{n}{2}}(t_0) = \mathbf{b}_{\frac{n}{2},\frac{n}{2}}(t_2) = (0,\ldots,0)^t$.

(c) In general the vectors $\mathbf{b}_{l-1,l-1}(t)$, $\mathbf{b}_{i,l-1}(t)$, $\mathbf{b}_{n-l+1,l-1}(t)$ represent points $B_{l-1,l-1}(t), B_{i,l-1}(t), B_{n-l+1,l-1}(t)$ on the hyperquadric which span a plane $[B_{l-1,l-1}(t), B_{i,l-1}(t), B_{n-l+1,l-1}(t)]_p$, intersecting the hyperquadric in a conic section. Then

- *$B_{i,l}(t)$ is on this conic section and*
- *the crossratios $(B_{l-1,l-1}(t)\ B_{i,l-1}(t)\ B_{n-l+1,l-1}(t)\ B_{i,l}(t))$ and $(t_0\ t_1\ t_2\ t)$ are identical.*

(d) As the exponents $e_{k,l}$ and $\overline{e}_l$ occuring in (103) are very large for n large, the implementation of the given algorithm requires some care to guarantee numerical stability.

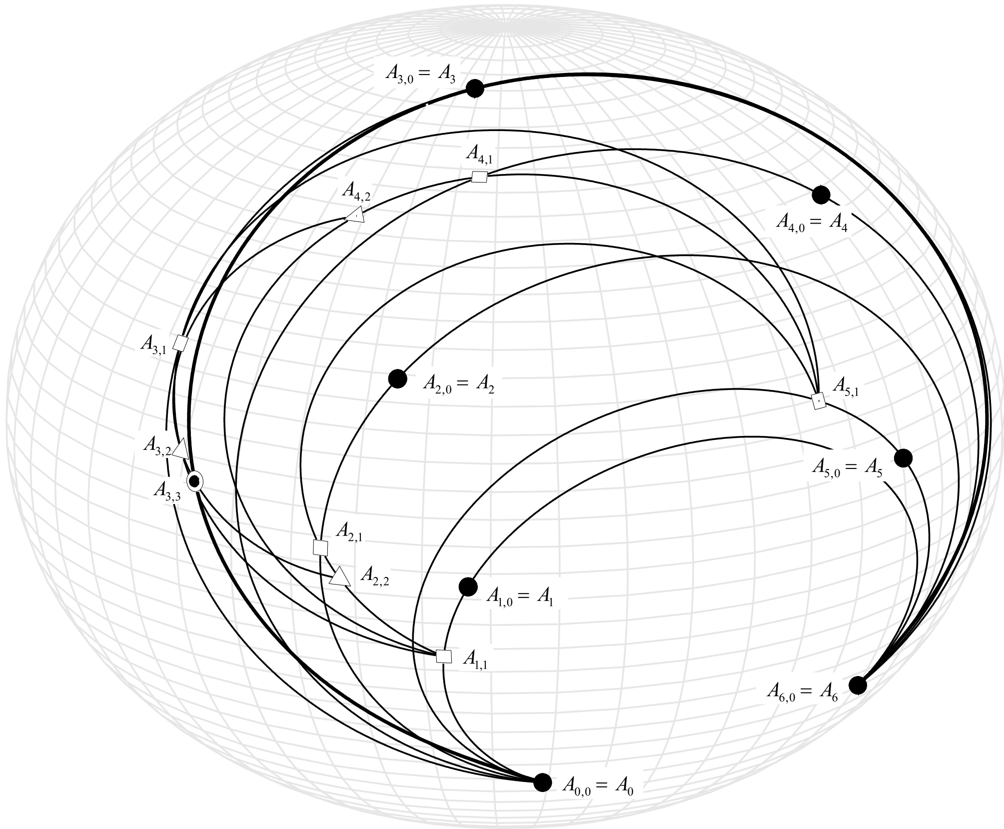

Figure 4.5 *The geometric algorithm for constructing a QB-curve on a hyperquadric (here $n = 6$).*
Input points: $A_0, \ldots, A_6$ (black circles).
First generation of points: $A_{1,1}, \ldots, A_{5,1}$ (quadrangles).
Second generation of points: $A_{2,2}, A_{3,2}, A_{4,2}$ (triangles).
Resulting point on the curve: $A_{3,3}$ (black dotted circle).

References

[Barn 1974] Barnhill, R. E., Riesenfeld, R. F. (ed.) (1974), *Computer Aided Geometric Design*, Academic Press, New York–San Francisco–London.

[Bert 1924] Bertini, E., Duschek, A. (1924), *Einführung in die Projektive Geometrie Mehrdimensionaler Räume*, Verlag Seidel & Sohn, Wien.

[Blasch 1960] Blaschke, W. (1960), *Kinematik und Quaternionen*, Deutscher Verlag der Wissenschaften, Berlin.

[Böhm 1984] Böhm, W., Farin, G., Kahmann, J. (1984), *A survey of curve and surface methods in CAGD*, Computer Aided Geometric Design 1, 1–60.

[Böhm 1992] Böhm, W., Hansford, D. (1992) *Parametric representation of quadric surfaces*, Math. Modeling, and Numerical Analysis 26, No. 1, 191–200.

[Böhm 1993] Böhm, W. (1993), *Some remarks on quadrics*, Computer Aided Geometric Design 10, 231–236.

[Böhm 1999] Böhm, W., Müller, A. (1999), *On de Casteljau's algorithm*, Computer Aided Geometric Design 16, 587-605.

[Bol 1950] Bol, G. (1950), *Projektive Differentialgeometrie 1. Teil*, Vandenhoeck & Ruprecht, Göttingen.

[Dietz 1993] Dietz, R., Hoschek, J. Jüttler, B. (1993), *An algebraic approach to curves and surfaces on the sphere and other quadrics*, Computer Aided Geometric Design 10, 211-229.

[Dietz 1995] Dietz, R., Hoschek, J., Jüttler, B. (1995), *Rational patches on quadric surfaces*, Computer-Aided Design 27, 27–40.

[Farin 1983] Farin, G. (1983), *Algorithms for rational Bézier curves*, Computer Aided Design 15, 73–77.

[Farin 1990] Farin, G. (1990), *Curves and Surfaces for Computer Aided Geometric Design*, 2^{nd} ed., Academic Press, Boston–New York–London.

[Geise 1991] Geise, G., Langbecker, U. (1991), *Finite quadric segments with four conic boundary curves*, Computer Aided Geometric Design 7, 141–150.

[Gfre 1999] Gfrerrer, A. (1999), *Rational interpolation on a hypersphere*, Computer Aided Geometric Design 16, 21–37.

[Gier 1982] Giering, O. (1982), *Vorlesungen über Höhere Geometrie*, Vieweg & Sohn, Braunschweig–Wiesbaden.

[Greub 1967] Greub, W. H. (1967), *Linear Algebra*, 3^{rd} edition, Springer–Verlag, New York.

[Hodge 1952] Hodge, W. V. D., Pedoe, D. (1952), *Methods of Algebraic Geometry Vol. II*, Cambridge University Press.

[Hosch 1992] Hoschek, J., Lasser, D. (1992), *Grundlagen der geometrischen Datenverarbeitung*, 2. Auflage, B. G. Teubner, Stuttgart.

[Jütt 1993] Jüttler, B. (1993), *Zur Konstruktion rationaler Kurven und Flächen auf Quadriken*, Journal of Geometry 47, 53–64.

[Kleij 1991] Kleij, R. van, Roussou, R. (1991), *A CSG scanline algorithm for quadrics*, Modeling in computer graphics (Kunii, T. L., ed.), Springer, Tokyo, 247–263.

[Levin 1976] Levin, J. (1976), *A parametric algorithm for drawing pictures of solid objects composed of quadric surfaces*, Communications of the ACM 19, No. 10, 555–563.

[Mill 1988] Miller, J. R. (1988), *Analysis of quadric-surface-based solid models*, IEEE computer graphics & applications, 28–42.

[Piegl 1986] Piegl, L. (1986), *A geometric investigation of the rational Bézier scheme of computer aided design*, Computers in industry 7, 401–410.

[Piegl 1986 a] Piegl, L. (1986), *The sphere as a rational Bézier surface*, Computer Aided Geometric Design 3, 45–52.

[Piegl 1987] Piegl, L. (1987), *Interactive data interpolation by rational Bézier curves*, IEEE Computer graphics and applications 7, 45–58.

[Piegl 1988] Piegl, L. (1988), *Hermite- and Coons-like interpolants using rational Bézier approximation form with infinite control points*, Computer Aided Design 20, 2–10.

[Rösch 1991] Röschel, O. (1991), *Kinematic rational Bézier patches I*, RAD HAZU 10, 95–108.

[Rösch 1991 a] Röschel, O. (1991), *Kinematic rational Bézier patches II*, RAD HAZU 10, 131–138.

[Sabin 1976] Sabin, M. A. (1976), *Method for displaying the intersection curve of two quadric surfaces*, Computer Journal 19, 336–338.

[Study 1903] Study, E. (1903), *Geometrie der Dynamen*, B. G. Teubner Verlag, Berlin.

[Weiss 1935] Weiss, E. A. (1935), *Einführung in die Liniengeometrie und Kinematik*, B. G. Teubner, Berlin.

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