3R Wrist Positioning – a Classical Problem and its Geometric Background

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Abstract The *wrist centre positioning problem* (WP) peculiar to the motion produced by the first three joints of a general six revolute jointed (6R), wrist partitioned serial robot and the underlying geometry is reexamined. Conventionally a sequence of six rotational operations, alternately in terms of known geometric parameters and unknown joint angles, expresses the desired position. However the solution can be represented by four intersection points between a fourth order cyclid, and a circle. Properties of the curves of intersection of the cyclid with the absolute plane reveal why the univariate polynomial (UVP) is of fourth degree rather than eighth as indicated by the Bezout number. Simple cyclid geometry makes it convenient to investigate specific 3R positioning architectures and expose degenerate cases.

1 Introduction

The classical solution of (WP) is detailed in Angeles [1, pp. 117] employing typical "closure equations": three scalar rearrangements of the forward kinematic equation (Eq. 1) of a serial 3R chain. Another solution, with Study parameters [6] in 7-dimensional kinematic image space, of (WP) is described as intersection of a 4-dimensional object (*Segre manifold*) of algebraic degree 4 with a 3-dimensional linear subspace by Husty *et al* [3] and by Pfurner [5]). They also show that (WP) is a quartic problem. Remarkably, aside from the references cited above, the authors could find no seminal references on this topic, only textbook treatments like [2, pp.428]; essentially variants of that found in [1]. Similarly, nothing to elaborate on the origins of kinematic implication concerning Serge manifolds, used to solve (WP) *en-passant* in [3] and [5], appears in the 26 item bibliography of the survey

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article [4]. In contrast, herein (WP) is addressed explicitly and solved by intersecting common geometric 3D objects. The primary motivation of this exercise was to enhance understanding of (WP) by transforming it from a sequence of matrix operations to the visualization of three circular arcs that represent the required joint angles. One is on the circle that the wrist point traces as it is counter-rotated about the first joint axis, the second on a latitude circle of the cyclid and the third on a meridian circle of that surface.

2 Problem example, definition and formulation

The specific example illustrated in Fig. 1, b) was solved with the two-step procedure detailed just after Eq. 11 using architectural parameters and wrist position W_0 tabulated just before §**4**. Fig. 1, a), shows a 3R kinematic chain (3R robot) composed four rigid bodies $\Sigma_0, \Sigma_1, \Sigma_2, \Sigma_3$ and three revolute joints J_1, J_2, J_3: Joint J_i connects Σ_{i-1} and Σ_i. The axis of joint J_i is denoted by g_i, $i = 1, 2, 3$. Σ_0 and Σ_3 are usually called *base* (FF) and *end-effector* (EE) of the robot. In Fig. 2, the "skeleton" of the robot is revealed: N and O are the pedal points on the common perpendicular of g_1 and g_2. Analogously P and Q are the pedal points on the common perpendicular of g_2 and g_3. Robot geometry is defined by Denavit-Hartenberg parameters: a_i and α_i denote the shortest distance and the angle between the axes g_i and g_{i+1}, $i = 1, 2$ and d_2 is the distance of the two pedal points O and P on g_2. With frames $S_0 = \{N, \mathbf{e}_{0x}, \mathbf{e}_{0y}, \mathbf{e}_{0z}\}$ and $S_3 = \{Q, \mathbf{e}_{3x}, \mathbf{e}_{3y}, \mathbf{e}_{3z}\}$ on (FF) and (EE) shown in Fig. 2, the direct kinematics (DK) of (EE) motion is described by

$$\begin{bmatrix} 1 \\ \mathbf{x}_0 \end{bmatrix} = \mathbf{R}_z(u_1) \cdot \mathbf{C}_1 \cdot \mathbf{R}_z(u_2) \cdot \mathbf{C}_2 \cdot \mathbf{R}_z(u_3) \cdot \begin{bmatrix} 1 \\ \mathbf{x}_3 \end{bmatrix} \tag{1}$$

where u_i denote the joint angles, $\mathbf{x}_0 = [x_0, y_0, z_0]^\top$ and $\mathbf{x}_3 = [x_3, y_3, z_3]^\top$ are the position vectors of a point $X \in \Sigma_3$ w.r.t. the coordinate systems S_0 and S_3. For the definition of the 4×4-matrices $\mathbf{R}_z(u_i)$, $\mathbf{C}_i$ see [3].

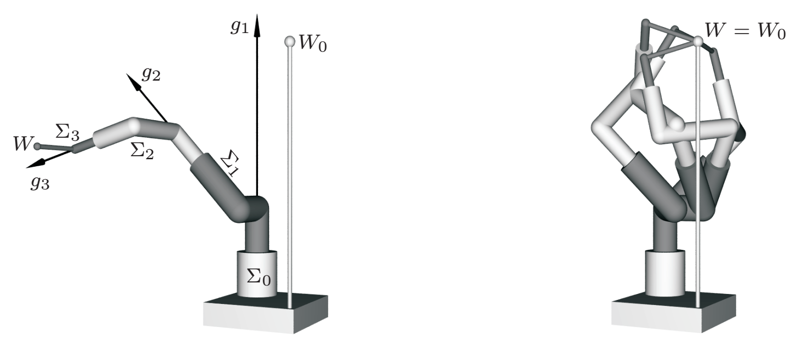

Fig. 1 a) 3R chain in its home position b) Four solutions for the IK

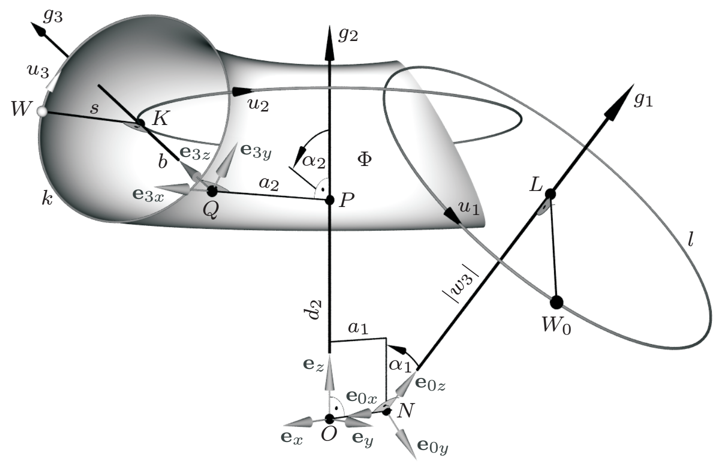

Fig. 2 The geometric skeleton of the 3R chain with local coordinate systems, the circles k and l and a section of the cyclid Φ.

This kinematic chain represents half of a serial 6R wrist partitioned robot where the last three axes g_4, g_5, g_6 contain *wrist center point* $W \in \Sigma_3$. The inverse kinematics (IK) of such a 6R robot can be split into

- the *wrist center positioning problem* (WP) to find joint angles u_1, u_2, u_3 to place the wrist center W on a given point $W_0 \in \Sigma_0$ and
- finding three remaining angles u_4, u_5, u_6 according to the given orientation; only a quadratic problem.

3 Geometric solution of (WP)

Assume that g_2, g_3 are skew lines, i.e., $a_2 \neq 0$, $\sin\alpha_2 \neq 0$. The special cases where g_2, g_3 either intersect or are parallel will be treated separately. To solve, consider Σ_1 is fixed, with respect to the observer, and introduce a third coordinate system $S := \{O; \mathbf{e}_x, \mathbf{e}_y, \mathbf{e}_z\}$ attached to Σ_1 as shown in Fig. 2. Motion Σ_0/Σ_1 is pure rotation about axis g_1 and angle $-u_1$, given by

$$\begin{bmatrix} 1 \\ \mathbf{x} \end{bmatrix} = \mathbf{C}_1^{-1} \cdot \mathbf{R}_z(-u_1) \cdot \begin{bmatrix} 1 \\ \mathbf{x}_0 \end{bmatrix} \tag{2}$$

where $\mathbf{x} = [x, y, z]^\top$ and $\mathbf{x}_0 = [x_0, y_0, z_0]^\top$ denote the position vector of a point w.r.t. S and S_0, respectively. Wrist center target point $W_0 \in \Sigma_0$ sweeps a circle l in Σ_1 centered on $L \in g_1$, where L is the pedal point of W_0 w.r.t g_1. Parametrization of l with u_1 is obtained by substituting the position vector $\mathbf{x}_0 = \mathbf{w}_0 = [w_1, w_2, w_3]^\top$ of W_0 into (Eq. 2):

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} -a_1 \\ w_3 \sin\alpha_1 \\ w_3 \cos\alpha_1 \end{bmatrix} + \cos u_1 \begin{bmatrix} w_1 \\ w_2 \cos\alpha_1 \\ -w_2 \sin\alpha_1 \end{bmatrix} + \sin u_1 \begin{bmatrix} w_2 \\ -w_1 \cos\alpha_1 \\ w_1 \sin\alpha_1 \end{bmatrix} \tag{3}$$

On the other hand the motion Σ_3/Σ_1 is the result of the two rotations of a 2R dyad with axes g_2 and g_3. The two parameters are the joint angles u_2, u_3. The analytic description of this motion is

$$\begin{bmatrix} 1 \\ \mathbf{x} \end{bmatrix} = \mathbf{R}_z(u_2) \cdot \mathbf{C}_2 \cdot \mathbf{R}_z(u_3) \cdot \begin{bmatrix} 1 \\ \mathbf{x_3} \end{bmatrix}. \tag{4}$$

Wrist center $W \in \Sigma_3$ w.r.t. the rotation Σ_3/Σ_2 moves on a circle $k \subset \Sigma_2$ with radius $s = \text{dist}(W, g_3)$ centered in the pedal point K of W w.r.t. g_3. Together with rotation about g_2 through angle u_2, the circle k generates Φ a surface of revolution called a *cyclid*. The parametrization of Φ w.r.t. S is obtained by substitution of the position vector $\mathbf{x}_3 = \mathbf{w}_3 = [s, 0, b]^\top$ of W in (Eq. 4):

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} \cos u_2 (a_2 + s\cos u_3) - \sin u_2 (-b \sin\alpha_2 + s\cos\alpha_2 \sin u_3) \\ \sin u_2 (a_2 + s\cos u_3) + \cos u_2 (-b \sin\alpha_2 + s\cos\alpha_2 \sin u_3) \\ d_2 + b\cos\alpha_2 + s\sin\alpha_2 \sin u_3 \end{bmatrix} \tag{5}$$

Herein b denotes the (oriented) distance of Q and K.

Hence, one sees that:

Placing the wrist center W on the target point W_0 means finding the intersection points of circle l and cyclid Φ.

To find these intersection points one uses an equation for Φ. By squaring and adding the first two lines of (Eq. 5)

$$x^2 + y^2 = (a_2 + s\cos u_3)^2 + (-b\sin\alpha_2 + s\cos\alpha_2 \sin u_3)^2 \tag{6}$$

is obtained. The third line yields

$$\sin u_3 = \frac{z - d_2 - b\cos\alpha_2}{s\sin\alpha_2}. \tag{7}$$

Substitution of (Eq. 7) in (Eq. 6) produces the equation of Φ:

$$\begin{aligned} &\left[(x^2 + y^2 + z^2 - 2d_2 z - a_2^2 + d_2^2 - b^2 - s^2)\sin\alpha_2\right]^2 + \\ &\qquad \left[(z - d_2 - b\cos\alpha_2)^2 - s^2 \sin^2\alpha_2\right] 4a_2^2 = 0 \end{aligned} \tag{8}$$

Hence Φ is algebraic of degree 4 and doubly contains the absolute conic $k_\infty : (x^2 + y^2 + z^2)^2 = 0$. To find the intersection points of l and Φ the parametrization (Eq. 3) of l is substituted into (Eq. 8). Manipulation results in an equation in terms of only $1, \cos u_1, \sin u_1, \cos^2 u_1, \sin^2 u_1$ and $\cos u_1 \cdot \sin u_1$. It can be written as

$$[1, \cos u_1, \sin u_1] \cdot \mathbf{C} \cdot \begin{bmatrix} 1 \\ \cos u_1 \\ \sin u_1 \end{bmatrix} = 0 \tag{9}$$

with a symmetric 3×3-matrix $\mathbf{C} = [c]_{ij}$. The constant entries of $\mathbf{C}$ are

$$\begin{aligned}
c_{11} &= e^2 \sin^2 \alpha_2 + 4a_2^2 f \\
c_{12} &= 2\left[e \sin^2 \alpha_2 (d_2 w_2 \sin \alpha_1 - a_1 w_1) - 2a_2^2 w_2 \sin \alpha_1 (w_3 \cos \alpha_1 - d_2 - b \cos \alpha_2)\right] \\
c_{13} &= 2\left[-e \sin^2 \alpha_2 (d_2 w_1 \sin \alpha_1 + a_1 w_2) + 2a_2^2 w_1 \sin \alpha_1 (w_3 \cos \alpha_1 - d_2 - b \cos \alpha_2)\right] \\
c_{22} &= 4\left[\sin^2 \alpha_2 \cdot (d_2 w_2 \sin \alpha_1 - a_1 w_1)^2 + a_2^2 w_2^2 \sin^2 \alpha_1\right] \\
c_{23} &= 4\left[-\sin^2 \alpha_2 \cdot (d_2 w_1 \sin \alpha_1 + a_1 w_2)(d_2 w_2 \sin \alpha_1 - a_1 w_1) - a_2^2 w_1 w_2 \sin^2 \alpha_1\right] \\
c_{33} &= 4\left[\sin^2 \alpha_2 \cdot (d_2 w_1 \sin \alpha_1 + a_1 w_2)^2 + a_2^2 w_1^2 \sin^2 \alpha_1\right]
\end{aligned}$$

where

$$\begin{aligned}
e &= a_1^2 - a_2^2 + d_2^2 - b^2 - s^2 + w_1^2 + w_2^2 + w_3^2 - 2d_2 w_3 \cos \alpha_1 \\
f &= (w_3 \cos \alpha_1 - d_2 - b \cos \alpha_2)^2 - s^2 \sin^2 \alpha_2
\end{aligned}$$

After half-angle substitution $\cos u_1 = \frac{1-\tau^2}{1+\tau^2}$, $\sin u_1 = \frac{2\tau}{1+\tau^2}$ and multiplication by the denominator $(1+\tau^2)^2$ (Eq. 9) becomes

$$\rho(\tau) = [1+\tau^2, 1-\tau^2, 2\tau] \cdot \mathbf{C} \cdot \left[1+\tau^2,\ 1-\tau^2,\ 2\tau\right]^\top = 0 \tag{10}$$

i.e., a *quartic polynomial in* τ. The geometric reason for the algebraic degree four of the resulting univariate $\rho(\tau)$ is that four of the eight intersection points of l and Φ always fall into the absolute points $I, \bar{I}$ of l. Each of these two points has to be counted twice since the absolute conic is a double curve on Φ. Each real solution τ^* of (Eq. 10) yields a solution u_1^* for the first joint angle. By substitution of $u_1 = u_1^*$ in (Eq. 3) we obtain the coordinates x^*, y^*, z^* of the corresponding intersection point of Φ and l. By means of (Eq. 7) we get

$$\sin u_3^* = \frac{z^* - d_2 - b \cos \alpha_2}{s \sin \alpha_2},\ \cos u_3^* \ = \ \pm\sqrt{1 - \sin u_3^{*2}} \tag{11}$$

Since $a_2 \neq 0$ (the axes g_2 and g_3 were assumed to be skew in the general case) the sign of $\cos u_3^*$ is uniquely determined by (Eq. 6),

$$x^{*2} + y^{*2} = (a_2 + s \cos u_3^*)^2 + (-b \sin \alpha_2 + s \cos \alpha_2 \sin u_3^*)^2$$

and so u_3^* is uniquely determined. After substitution of $u_3 = u_3^*, x = x^*, y = y^*$ into the first two lines of (Eq. 5) one gets $\sin u_2^*$, $\cos u_2^*$ and thereby u_2^*, uniquely:

$$\cos u_2^* = \frac{x^*(a_2 + s\cos u_3^*) + y^*(-b\sin\alpha_2 + s\cos\alpha_2 \sin u_3^*)}{x^{*2} + y^{*2}}$$

$$\sin u_2^* = \frac{y^*(a_2 + s\cos u_3^*) - x^*(-b\sin\alpha_2 + s\cos\alpha_2 \sin u_3^*)}{x^{*2} + y^{*2}}$$

When joint axes g_2, g_3 are, in general, skew (WP) has ≤ 4 real solutions. To obtain the solution angle triples (u_1^*, u_2^*, u_3^*)

- Determine the roots of a quartic univariate polynomial to get the first angle u_1^*.
- Then compute the corresponding other two angles u_1^*, u_2^* by linear routines.

Architecture and the four poses shown in Fig. 1, b) were produced with the following data. Below these appear matrices that contain the coordinates (x^*, y^*, z^*) of the four points of intersection between circle and cyclid and the four corresponding joint angle triples (u_1^*, u_2^*, u_3^*). All angles are given in degrees.

$a_1 = 2.0, a_2 = 3.5, \alpha_1 = 45, \alpha_2 = 60, d_2 = 5.0, b = 3.4, s = 2.5, \mathbf{w}_0 = [3,3,7]^\top$

$$\begin{bmatrix} x^* & y^* & z^* \\ 2.2094 & 5.3245 & 4.5750 \\ 0.8782 & 2.7456 & 7.1539 \\ -1.8043 & 1.9529 & 7.9466 \\ -5.7428 & 3.5370 & 6.3625 \end{bmatrix}, \quad \begin{bmatrix} u_1^* & u_2^* & u_3^* \\ 37.825 & 113.817 & 281.043 \\ 92.282 & 140.784 & 167.900 \\ 132.356 & 189.533 & 144.847 \\ 196.906 & 176.111 & 351.032 \end{bmatrix}$$

4 The case of intersecting joint axes g_2 and g_3

If g_2 intersects g_3 in the point $P = Q$ then the surface generated by circle k by means of rotation around g_2 is a sphere Φ centered in $P = Q$ and radius $_+\sqrt{b^2 + s^2}$. The equation of Φ follows immediately by setting $a_2 = 0$ in (Eq. 8):

$$x^2 + y^2 + z^2 - 2d_2 z + d_2^2 - b^2 - s^2 = 0 \tag{12}$$

One may say, more precisely, that k generates *only a strip on the sphere* Φ which is bounded by the two latitude circles with z-coordinates $z = d_2 + b\cos\alpha_2 \mp s\sin\alpha_2$.

As in the general case one has to determine the common points of the other circle l with Φ: Substitution of (Eq. 3) yields an equation in the terms $1, \cos u_1, \sin u_1$ which can have at most two solutions for u_1 in the interval $[0, 2\pi[$ in accordance with the fact that a circle intersects a sphere in at most two points. To obtain a polynomial equation one can again apply half-angle substitution as before but the resulting polynomial $\rho = \rho(\tau)$ is only of degree 2. Via (Eq. 3) one can again determine the coordinates x^*, y^*, z^* of the intersection point belonging to a solution u_1^* for the first joint angle. A valid (WP) solution occurs only if this point lies within the mentioned strip region of Φ, *i.e.*,

$$d_2 + b\cos\alpha_2 - s\sin\alpha_2 \le z^* \le d_2 + b\cos\alpha_2 + s\sin\alpha_2.$$

In contrast to the general case the angle u_3^* corresponding to a solution u_1^* for the first angle is no longer unique:
The sine of this angle is again given uniquely by means of (Eq. 11) but by having set $a_2 = 0$ (Eq. 6) determines cosine *magnitude* only. Hence, for any solution u_1^* there are two possible angles u_3^* and $\tilde{u}_3^*$ with $u_3^* + \tilde{u}_3^* = \pi$.
The angle u_2^* belonging to an already determined pair u_1^*, u_3^* can again be computed linearly as in the main case.

5 The case of parallel joint axes g_2 and g_3

This case is characterized by $\sin\alpha_2 = 0$. The cyclid Φ degenerates to an annulus in the plane

$$z = d_2 \pm b \tag{13}$$

centered in $M(0,0,d_2 \pm b)$, with "+" if $\alpha_2 = 0$ and "−" if $\alpha_2 = \pi$. The interior and exterior radii of the ring are $a_2 \mp s$. Substitution of (Eq. 3) into (Eq. 13) yields the condition

$$d_2 \pm b - w_3\cos\alpha_1 + \sin\alpha_1(w_2\cos u_1 - w_1\sin u_1) = 0$$

which can again be transformed into a quadratic polynomial by means of half-angle substitution giving at most two real solutions for u_1 (intersection of a circle with a plane).
A solution u_1^* is only valid if the corresponding point x^*, y^*, z^* which is computed via (Eq. 3) lies within the annulus, i.e., the inequality

$$a_2 - s \le \sqrt{x^{*2} + y^{*2}} \le a_2 + s$$

must be fulfilled.

To determine the other joint angles u_2^*, u_3^* that correspond to a valid solution of the first angle u_1^* one substitutes x^*, y^* into the first two lines of (Eq. 5) recalling that $\alpha_2 = 0, \pi$:

$$x^* = s\cos(u_2 \pm u_3) + a_2\cos u_2, \quad y^* = s\sin(u_2 \pm u_3) + a_2\sin u_2$$

After squaring and adding we obtain the cosine of angle u_3:

$$\cos u_3 = \frac{x^{*2} + y^{*2} - a_2^2 - s^2}{2a_2 s}$$

To each angle u_1^* there are two corresponding angles u_3^*, $\tilde{u}_3^*$ within the interval $[0, 2\pi[$ with $u_3^* + \tilde{u}_3^* = 2\pi$. The angle u_2^* is again computed linearly, as in the general case.

Remark. If instead of g_2, g_3 the axes g_1, g_2 are intersecting or parallel, it is obvious that one can again reduce (WP) to quadratic problems, by means of role reversal of the circles k and l: Instead of letting k rotate around g_2 and intersecting the surface Φ with l one can alternately let l revolve about g_2 and determine the intersection points of the resulting surface Ψ with k.

If one of the axis pairs (g_1, g_2) or (g_2, g_3) is intersecting or parallel then (WP) has again at most four solutions, but the solution angle triples (u_1^*, u_2^*, u_3^*) can be determined by solving quadratic equations.

6 Conclusion

An alternative solution to the inverse kinematic point positioning problem of a general 3R chain was presented. A geometric approach showed that the problem is equivalent to finding the intersection points of a special surface Φ (cyclid) with a circle l. This fact clearly shows that where adjacent revolute joint axes are skew the problem leads to a UVP of degree four. On the other hand, if two adjacent revolute joint axes are intersecting or parallel the cyclid is replaced by a sphere or a plane, which means that the resulting UVP is *quadratic*. Although these results are not new the geometric formulation is novel and is believed to convey new insight. Up to four real solutions are contained in the intersection of three easy to visualize surfaces. A cyclid intersects a plane and sphere. The latter two make up the spatial circle.

Acknowledgements This research is supported by a Natural Sciences and Engineering Research Canada (NSERC) "Discovery" grant.

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