1. Introductional lecture (15 March 2011)

In this lecture we bring together some basic definitions and facts on continued fractions. After several technical preliminary statements and discussions we prove convergency theorem for infinite ordinary continued fractions. Further we show existence and uniqueness (odd and even continued fractions in rational case) of continued fractions for a given number. Finally we formulate two theorems on approximation rates by convergents and Lagrange theorem on periodic continued fractions and quadratic irrationalities. The most part of the material is taken from the book by A. Ya. Khinchin "Continued Fractions".

1.1. **Euclidean algorithm.** The story of continued fractions starts with an Euclidean algorithm named after the Greek mathematician Euclid, who described it in his Elements (Books VII and X). Actually the algorithm was known before Euclid, it was mentioned in the Topics of Aristotle.

The task of the algorithm is *to find the greatest common divisor for a pair of integers.* Let us describe the algorithm in a few words.

Consider two nonzero integers p and q, let us find their greatest common divisor (usually denoted by $\gcd(p,q)$). To do this we make several iterative steps. We describe them inductively.

Step 1. Let us find integer numbers a_0 and r_1 where $q > r_1 \geq 0$ such that

$$p = a_0 q + r_1.$$

Step k. Suppose we have completed $k-1$ steps and get the integers a_{k-2} and r_{k-1}. Let us find a_{k-1} and r_k where $r_{k-1} > r_k \geq 0$ such that

$$r_{k-2} = a_{k-1} r_{k-1} + r_k.$$

The algorithm stops at Step n when $r_n = 0$. Since the sequence (r_k) is decreasing sequence of positive integers, the algorithm always stops.

1.2. **Definition of a continued fraction.** In order to give a definition of a continued fraction we slightly modify the Euclidean algorithm. We start with a rational number α. On each step we define a pair of integer numbers (a_{k-1}, r_k).

Step 1. Let us subtract the integer part $\lfloor \alpha \rfloor$ and invert the remainder, i.e.,

$$\alpha = \lfloor \alpha \rfloor + \frac{1}{1/(\alpha - \lfloor \alpha \rfloor)}.$$

Denote $a_0 = \lfloor \alpha \rfloor$, and continue with the reminding part $r_1 = 1/(\alpha - a_0)$.

Step k. Suppose we have completed $k-1$ steps and get the number a_{k-2} and r_{k-1}. Let us find a_{k-1} and r_k

$$r_{k-1} = \lfloor r_{k-1} \rfloor + \frac{1}{1/(r_{k-1} - \lfloor r_{k-1} \rfloor)}.$$

According to this expression we denote $a_{k-1} = \lfloor \alpha \rfloor$ and $r_k = 1/(r_{k-1} - a_{k-1})$.

The algorithm stops at Step n when $r_n = 0$.

Let $r_i = p_i/q_i$ with positive integers p_i, q_i for $i \geq 1$. Then for any $k \geq 1$ we have

$$\frac{p_{k+1}}{q_{k+1}} = r_{k+1} = \frac{1}{r_k - \lfloor r_k \rfloor} = \frac{1}{\frac{p_k}{q_k} - \lfloor p_k/q_k \rfloor} = \frac{q_k}{p_k - q_k \lfloor p_k/q_k \rfloor}.$$

Since $r_k > 1$, its denominator is less than its numerator (i.e., $q_{k+1} < q_k$). Hence the sequence of denominators q_k decreases with growth of k. Therefore, the algorithm stops in a finite number of steps.

The described decomposition of a rational number α can be written as follows.

$$a_0 + \cfrac{1}{a_1 + \cfrac{1}{a_2 + \cfrac{1}{\ddots + \cfrac{1}{a_n}}}} \tag{1}$$

This expression is considered as a *continued fraction* of α.

Remark 1.1. In general there is no restriction to the elements of continued fractions, they can be any real numbers. To avoid the annoying consideration of different cases of zeroes and infinities we propose to add an element ∞ to the field of real numbers $\mathbb{R}$ and define the following operations:

$$\infty + a = a + \infty = \infty, \quad \frac{1}{0} = \infty, \quad \frac{1}{\infty} = 0.$$

Denote the resulting set by $\overline{\mathbb{R}}$.

Definition 1.2. Let $a_0, \ldots, a_n$ be arbitrary real numbers. Expression (1) is called a *finite continued fraction*, and denoted by $[a_0; a_1 : \ldots : a_n]$. It corresponds to some α in $\overline{\mathbb{R}}$. The numbers $a_0, \ldots, a_n$ are the *elements* of this continued fraction.

Definition 1.3. A continued fraction is *odd* (*even*) if it has odd (even) number of elements.

Notice that the term *continued fractions* was used for the first time by John Wallis in 1695.

Example 1.4. Let us study one example:

$$\frac{9}{7} = 1 + \frac{1}{\left(\frac{7}{2}\right)} = 1 + \frac{1}{3 + \frac{1}{2}}$$

So we get the continued fraction $[1; 3 : 2]$.

Remark 1.5. The Euclidean algorithm actually generates the elements of a continued fraction. We always have (see the previous subsection)

$$\frac{p}{q} = [a_0; a_1 : \ldots : a_n].$$

Definition 1.6. An infinite continued fraction with an infinite sequence of elements $a_0, a_1, \ldots$ is the following limit (in case of existence)

$$\lim_{k\to\infty}[a_0; a_1 : \ldots : a_k].$$

We denote it by $[a_0; a_1 : \ldots]$.

The number $[a_0; a_1 : \ldots : a_{k-1}]$ is called the *k-convergent* (or just *convergent*) to the finite or infinite continued fraction $[a_0; a_1 : \ldots]$.

Definition 1.7. A continued fraction (finite or infinite) is called an *ordinary continued fraction* if its zero element is integer and the rest are positive integers.

1.3. **Convergence of infinite ordinary continued fractions.** In this subsection we show that a sequence of k-convergents of any infinite ordinary continued fraction converges to some real number.

There exists a unique pair of polynomials P_k and Q_k in variables $x_0, \ldots, x_k$ with non-negative integer coefficients such that

$$\frac{P_k(x_0,\ldots,x_k)}{Q_k(x_0,\ldots,x_k)} = [x_0; x_1 : \ldots : x_k], \quad \text{and} \quad P_k(0,\ldots,0)+Q_k(0,\ldots,0)=1.$$

Actually, the first condition defines the polynomials up to a multiplicative, so the second condition is a necessary normalization condition.

Example 1.8. For instance we have

$$\begin{array}{ll} P_0(x_0)=x_0, & Q_0(x_0)=1; \\ P_1(x_0,x_1)=x_0x_1+1, & Q_1(x_0,x_1)=x_1; \\ P_2(x_0,x_1,x_2)=x_0x_1x_2+x_2+x_0, & Q_2(x_0,x_1,x_2)=x_1x_2+1; \\ \ldots & \end{array}$$

Consider a finite (or infinite) continued fraction $[a_0; a_1 : \ldots : a_n]$ with $n \geq k$ (or $[a_0; a_1 : \ldots]$ respectively). We denote

$$p_k = P_k(a_0,\ldots,a_k) \quad \text{and} \quad q_k = Q_k(a_0,\ldots,a_k);$$

As we show later in Proposition 1.10 the integers p_k and q_k are relatively prime for any k. For the next several propositions we need an additional notation.

$$\hat{p}_k = P_{k-1}(a_1,\ldots,a_k) \quad \text{and} \quad \hat{q}_k = Q_{k-1}(a_1,\ldots,a_k).$$

We start with the following lemma.

Lemma 1.9. *The following holds*

$$\left\{ \begin{array}{l} p_k = a_0\hat{p}_k + \hat{q}_k \\ q_k = \hat{p}_k \end{array} \right. .$$

Proof. Since

$$\frac{\hat{p}_k}{\hat{q}_k} = [a_1; a_2 : \ldots : a_k].$$

we get

$$\frac{p_k}{q_k} = a_0 + \frac{1}{\hat{p}_k/\hat{q}_k} = \frac{a_0\hat{p}_k + \hat{q}_k}{\hat{p}_k}.$$

Therefore,

$$\begin{cases} p_k = \lambda(a_0\hat{p}_k + \hat{q}_k) \\ q_k = \lambda\hat{p}_k \end{cases}.$$

Now we rewrite the second condition for polynomials P_k and Q_k:

$$1 = P_k(0,\ldots,0)+Q_k(0,\ldots,0) = \lambda(0P_{k-1}(0,\ldots,0)+Q_{k-1}(0,\ldots,0))+\lambda P_{k-1}(0,\ldots,0) = \lambda.$$

Therefore, $\lambda = 1$. □

Proposition 1.10. *Let $[a_0; a_1 : \ldots : a_n]$ be a continued fraction with integer elements, then the corresponding integers p_k and q_k are relatively prime.*

Proof. We prove the statement by the induction in k.

It is clear that $p_0 = a_0$ and $q_0 = 1$ are relatively prime.

Suppose that the statement holds for $k-1$. Then $\hat{p}_k$ and $\hat{q}_k$ are relatively prime by the induction assumption. Now the statement holds directly from the equalities of Lemma 1.9. □

Proposition 1.11. *For any integer k we get*

$$\begin{cases} p_k = a_k p_{k-1} + p_{k-2} \\ q_k = a_k q_{k-1} + q_{k-2} \end{cases}.$$

Proof. We prove the statement by induction on k.

For $k = 2$ the statement holds since

$$\frac{p_0}{q_0} = \frac{a_0}{1} \quad \text{and} \quad \frac{p_1}{q_1} = \frac{a_0 a_1 + 1}{a_1}$$

and, therefore,

$$\frac{p_2}{q_2} = \frac{a_2 + a_0 a_1 a_2 + a_0}{a_1 a_2 + 1} = \frac{a_2 p_1 + p_0}{a_2 q_1 + q_0}.$$

Suppose the statement holds for $k-1$ let us prove for k.

$$\frac{p_k}{q_k} = a_0 + \frac{1}{\hat{p}_k/\hat{q}_k} = a_0 + \frac{1}{\frac{a_k\hat{p}_{k-1}+\hat{p}_{k-2}}{a_k\hat{q}_{k-1}+\hat{q}_{k-2}}} = \frac{a_0(a_k\hat{p}_{k-1} + \hat{p}_{k-2}) + a_k\hat{q}_{k-1} + \hat{q}_{k-2}}{a_k\hat{p}_{k-1} + \hat{p}_{k-2}}$$
$$= \frac{a_k(a_0\hat{p}_{k-1} + \hat{q}_{k-1}) + (a_0\hat{p}_{k-2} + \hat{q}_{k-2})}{a_k\hat{p}_{k-1} + \hat{p}_{k-2}} = \frac{a_k p_{k-1} + p_{k-2}}{a_k q_{k-1} + q_{k-2}}.$$

(The last equality holds by Lemma 1.9.) Therefore, the relations of the system hold. □

Denote by F_n the Fibonacci numbers (defined by $F_1 = F_2 = 1$, and $F_n = F_{n-1} + F_{n-2}$).

Corollary 1.12. *For ordinary continued fractions the following estimates hold*

$$|p_k| \geq F_k \quad \textit{and} \quad q_k \geq F_{k+1}$$

Proof. We prove the statement by induction in k. Direct calculations show that

$$\begin{aligned} &|p_0| \geq 0, \quad |p_1| \geq 1, \quad \text{and} \quad |p_2| \geq 1, \\ &|q_0| \geq 1, \quad |q_1| \geq 1, \quad \text{and} \quad |q_2| \geq 2. \end{aligned}$$

Let the statement holds for $k-2$ and $k-1$, we prove it for k. Notice that q_k are all positive and p_k are either all negative, or all non-negative. We apply Proposition 1.11

$$\begin{aligned} |p_k| = a_k|p_{k-1}| + |p_{k-2}| \geq 1 \cdot F_{k-1} + F_{k-2} = F_k, \quad \text{and} \\ q_k = a_k q_{k-1} + q_{k-2} \geq 1 \cdot F_k + F_{k-1} = F_{k+1} \end{aligned}$$

This concludes the proof. □

Proposition 1.13. *For any $k \geq 1$ the following holds*

$$\frac{p_{k-1}}{q_{k-1}} - \frac{p_k}{q_k} = \frac{(-1)^k}{q_{k-1}q_k}.$$

Proof. Let us multiply both sides by $q_{k-1}q_k$, we get

$$p_{k-1}q_k - p_k q_{k-1} = (-1)^k.$$

We prove this by induction in k.

For $k = 1$ we have

$$p_0 q_1 - p_1 q_0 = a_1 a_0 - (a_1 a_0 + 1) = -1.$$

Suppose the statement holds for $k-1$ let us prove it for k. By Proposition 1.11 we get

$$\begin{aligned} p_{k-1}q_k - p_k q_{k-1} &= p_{k-1}(a_k q_{k-1} + q_{k-2}) - (a_k p_{k-1} + p_{k-2})q_{k-1} \\ &= -(p_{k-2}q_{k-1} - p_{k-1}q_{k-2}) = (-1)^k. \end{aligned}$$

Therefore, the statement holds. □

Now we are ready to proof the following fundamental theorem.

Theorem 1.14. *For any integer a_0 and positive integers a_k the ordinary infinite continued fraction $[a_0; a_1 : \ldots]$ exists (i.e., the corresponding sequence $\left(\frac{p_n}{q_n}\right)$ converges).*

Proof. Notice that from Proposition 1.13 and Corollary 1.12 it follows that

$$\left|\frac{p_{k-1}}{q_{k-1}} - \frac{p_k}{q_k}\right| \leq \frac{1}{F_k F_{k+1}}.$$

Since the sum

$$\sum_{k=1}^{\infty} \frac{1}{F_k F_{k+1}}$$

converges (we leave this statement as an exercise for the reader), the sequence $\left(\frac{p_k}{q_k}\right)$ is a Cauchy sequence. Therefore, $\left(\frac{p_k}{q_k}\right)$ converges. □

1.4. **Existence and uniqueness of ordinary continued fraction for a given number.** In the next theorem we show in particular that the limit for an infinite ordinary continued fraction always exists.

Proposition 1.15. *Any rational number has a unique odd and even ordinary continued fractions.*
Any irrational number has a unique infinite ordinary continued fraction.

For instance $\frac{9}{7} = [1; 3 : 2] = [1; 3 : 1 : 1]$ and $\pi = [3; 7 : 15 : 1 : 292 : 1 : 1 : 1 : 2 : \ldots]$.

Proof. **Existence.** In Subsection 1.2 we have shown how to construct an ordinary continued fraction $[a_0; a_1 : \ldots a_n]$ for a rational number α. Notice that if α is rational but not integer then $a_n > 1$ and, therefore,

$$\alpha = [a_0; a_1 : \ldots : a_n] = [a_0; a_1 : \ldots : a_n{-}1 : 1].$$

One of these continued fractions is odd and the other is even. For integer α we always get $\alpha = [\alpha] = [\alpha{-}1; 1]$.

In the case of irrational number α the algorithm works infinite time and generate the ordinary continued fraction $\alpha' = [a_0; a_1 : a_2 : \ldots]$ and the sequence of remainders $r_k > 1$, such that

$$\alpha = [a_0; a_1 : \ldots : a_{k-1} : r_k].$$

Let us show that $\alpha = \alpha'$. From Proposition 1.11 for $[a_0; a_1 : \ldots : a_{k-1} : r_k]$ and $[a_0; a_1 : \ldots : a_{k-1} : \ldots]$ we have

$$\alpha = \frac{p_{n-1} r_n + p_{n-2}}{q_{n-1} r_n + q_{n-2}} \qquad \text{and} \qquad \frac{p_n}{q_n} = \frac{p_{n-1} a_n + p_{n-2}}{q_{n-1} a_n + q_{n-2}}.$$

Using these expressions and the fact that $a_n = \lfloor r_n \rfloor$ we have

$$\left|\alpha - \frac{p_n}{q_n}\right| = \left|\frac{(p_{n-1}q_{n-2} - p_{n-2}q_{n-1})(r_n - a_n)}{(q_{n-1}r_n + q_{n-2})(q_{n-1}a_n + q_{n-2})}\right| < \left|\frac{1}{(q_{n-1}r_n + q_{n-2})(q_{n-1}a_n + q_{n-2})}\right|$$
$$< \frac{1}{q_{n-1}q_n} \le \frac{1}{F_n F_{n+1}}.$$

The last inequality follows from Corollary 1.12.

Therefore, the sequence $\left(\frac{p_n}{q_n}\right)$ converges to α and hence $\alpha = \alpha'$.

Uniqueness. Consider a rational α. Let us prove the uniqueness of a finite ordinary continued fraction $\alpha = [a_0; a_1 : \ldots : a_n]$ where $a_n \neq 1$. We prove this by reductio ad absurdum.

Suppose

$$\alpha = [a_0; a_1 : \ldots : a_k : a_{k+1} : \ldots : a_n] = [a_0; a_1 : \ldots : a_k : a'_{k+1} : \ldots : a'_m],$$

where $a_{k+1} \neq a'_{k+1}$. Then we have

$$\alpha = \frac{p_k r_{k+1} + p_{k-1}}{q_k r_{k+1} + q_{k-1}} = \frac{p'_k r'_{k+1} + p'_{k-1}}{q'_k r'_{k+1} + q'_{k-1}} = \frac{p_k r'_{k+1} + p_{k-1}}{q_k r'_{k+1} + q_{k-1}}.$$

Therefore, $r_{k+1} = r'_{k+1}$, and thus $a_{k+1} = \lfloor r_{k+1} \rfloor = \lfloor r'_{k+1} \rfloor = a'_{k+1}$. We come to the contradiction.

The statement on uniqueness of continued fractions for irrational numbers repeats the case of rational numbers. □

We conclude this subsection with a statement on a behavior of the sequence of convergents.

Proposition 1.16. (*i*) *The sequence* (p_{2k}/q_{2k}) *is increasing, and the sequence* (p_{2k+1}/q_{2k+1}) *is decreasing.*
(*ii*) *For any real* α *and a nonnegative integer* k *we have*

$$\frac{p_{2k}}{q_{2k}} \leq \alpha \quad \text{and} \quad \frac{p_{2k+1}}{q_{2k+1}} \geq \alpha,$$

the equality holds only for the last convergent in case of rational α*.*

Proof. (*i*). By Proposition 1.11 we have

$$\frac{p_{m-2}}{q_{m-2}} - \frac{p_m}{q_m} = \left(\frac{p_{m-2}}{q_{m-2}} - \frac{p_{m-1}}{q_{m-1}}\right) - \left(\frac{p_{m-1}}{q_{m-1}} - \frac{p_m}{q_m}\right) = \frac{(-1)^{m-1}}{q_{m-1}}\left(\frac{1}{q_{m-2}} - \frac{1}{q_m}\right).$$

Since the sequence of the denominators (q_k) is increasing (by Proposition 1.13), we have that p_{m-2}/q_{m-2} is greater that p_m/q_m for all even m and it is smaller for all odd m. This concludes the proof of (*i*).

(*ii*). The sequence of even (odd) convergents is increasing (decreasing) and tends to α in irrational case or end up with some $p_n/q_n = \alpha$ in rational case. This implies the second statement of the proposition. □

1.5. **Continued fractions and best approximations.** We say that a rational number a/b (where $b > 0$) is a *best approximation* of a real number α if for any other fraction c/d with $0 < d \leq b$ we get

$$\left|\alpha - \frac{c}{d}\right| \geq \left|\alpha - \frac{a}{b}\right|. \tag{2}$$

Theorem 1.17. *Consider a real number* α*. Let* $[a_0; a_1 : \ldots]$ (*or* $[a_0; a_1 : \ldots : a_n]$) *be an ordinary infinite* (*finite*) *continued fraction for* α*. Then the set of best approximations consists of convergents* $p_k/q_k = [a_0; a_1 : \ldots : a_k]$ *where* $k = 1, 2, \ldots$ (*In case of finite continued fraction we additionally have for* $[a_0; a_1 : \ldots : a_{n-1} : a_n{-}1]$ *as a best approximation*). □

Now we say a few words about the rate of approximations.

Theorem 1.18. *Consider an inequality*

$$\left|\alpha - \frac{p}{q}\right| < \frac{c}{q^2}$$

Let $c \geq \frac{1}{\sqrt{5}}$*. Then for any* α *the inequality has an infinite number of integer solutions* (p, q)*.* □

Proposition 1.19. *Let α be the golden ratio, i.e.,*

$$\alpha = \frac{1+\sqrt{5}}{2} = [1;1:1:1:1:\ldots]$$

If $c < \frac{1}{\sqrt{5}}$ then Equation 2 has only finitely many solutions.

Denote the golden ration by θ and its conjugate $(1-\sqrt{5})/2$ by $\overline{\theta}$.

Proof. First of all, let us show that it is enough to check only all the convergents p_k/q_k. From Theorem 1.17 it follows that best approximations are convergents to a number. Let p/q be a rational such that $q_k \le q < q_{k+1}$. Therefore,

$$q^2\Big|\alpha - \frac{p}{q}\Big| > q'^2\Big|\alpha - \frac{p'}{q'}\Big| \ge q^2\Big|\alpha - \frac{p'}{q'}\Big|.$$

Hence if p/q is a solution of Equation 2 then p_k/q_k is a solution of Equation 2 as well. Therefore, if there are infinitely many solutions of Equation 2 then there are infinitely many convergents to golden ration among them.

Secondly, we prove the statement for the convergents. From Proposition 1.11 it follows that the k-convergent to the golden ratio equals to F_{k+1}/F_k. Recall a general formula for Fibonacci numbers via golden ration and its conjugate:

$$F_k = \frac{\theta^k - \overline{\theta}^k}{\sqrt{5}}.$$

We have

$$\Big|\theta - \frac{p_{k-1}}{q_{k-1}}\Big| = \Big|\theta - \frac{F_{k+1}}{F_k}\Big| = \Big|\theta - \frac{\theta^{k+1} - \overline{\theta}^{k+1}}{\theta^k - \overline{\theta}^k}\Big| = \Big|\frac{\overline{\theta}^k}{\theta^k - \overline{\theta}^k}\Big| = \Big|\frac{1-\overline{\theta}^{2k}}{\left(\theta^k - \overline{\theta}^k\right)^2}\Big| = \frac{1}{\sqrt{5}F_k^2}\big|1-\overline{\theta}^{2k}\big|.$$

Since $|\overline{\theta}| < 1$, we have $\big|1-\overline{\theta}^{2k}\big| = 1 + o(1)$ and hence

$$\Big|\theta - \frac{p_{k-1}}{q_{k-1}}\Big| = \frac{1}{\sqrt{5}q_{k-1}^2} + o\Big(\frac{1}{q_{k-1}^2}\Big).$$

This implies the statement for convergents and concludes the proof of Proposition 1.19. $\square$

1.6. **Periodic continued fractions and quadratic irrationalities.** A continued fraction $[a_0;a_1:\ldots]$ is called *periodic* if there exists positive integers k_0 and h such that for any $k > k_0$

$$a_{k+h} = a_k.$$

we denote it by $[a_0;a_1:\ldots:a_{k_0}:(a_{k_0+1}:\ldots:a_{k_0+h})]$.

Theorem 1.20. (Lagrange.) *Any periodic ordinary continued fraction is a quadratic irrationality (i.e., $\dfrac{a+b\sqrt{c}}{d}$ for some integer a, b, c, and d where $b \neq 0$, $c > 1$, $d > 0$, and c is square free). The inverse is also true: any quadratic irrationality has a periodic ordinary continued fraction.*

Lemma 1.21. *For any quadratic irrationality ξ there exists an $SL(2,\mathbb{Z})$ operator such that one of its eigenvectors is $(1,\xi)$.*

Proof. Let ξ be a root of the equation $c_2x^2+c_1x+c_0=0$ with integer coefficients A, B, C. Consider an arbitrary operator

$$A=\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}$$

Its eigenvectors are

$$\left(1, \frac{a_{22}-a_{11}\pm\sqrt{(a_{22}-a_{11})^2+4a_{12}a_{21}}}{2a_{12}}\right).$$

Notice that ξ and its conjugate roots are of the form

$$\frac{-c_1\pm\sqrt{c_1^2-4c_0c_2}}{2c_2}.$$

So the operator A has a root $(1,\xi)$ if the following system satisfied

$$\begin{cases} nc_0=-a_{21} \\ nc_1=a_{11}-a_{22} \\ nc_2=a_{12} \end{cases}$$

for some $n\neq 0$.

Let us find an operator of $SL(2,\mathbb{Z})$ satisfying this system for some integer n. Since n is integer, the coefficients a_{12} and a_{21} are integers as well. Since $\det A=1$ we have

$$\det A=a_{11}a_{22}-a_{12}a_{21}=a_{11}(na_{11}-c_1)+n^2c_0c_2=1.$$

Therefore,

$$a_{11}=\frac{nc_1\pm\sqrt{n^2(c_1^2-4c_0c_2)-4}}{2}.$$

The coefficient a_{11} is integer if and only if there exist an integer n satisfying

$$m^2=n^2(c_1^2-4c_0c_2)-4.$$

Denote $D=c_1^2-4c_0c_2$, $m'=m/2$, and $n'=n/2$ and rewrite the equation

$$m'^2-Dn'^2=1.$$

So, we end up with Pell's equation. Since ξ is irrational the discriminant $D=c_1^2-4c_0c_2$ is not a square of some integer (since ξ is real, $D\geq 0$). Hence, by Theorem ??? it has an integer solution (m_0',n_0') with $n_0'\neq 0$. Hence the operator

$$\begin{pmatrix} m_0'-n_0'c_1 & 2n_0'c_2 \\ -2n_0'c_0 & m_0'+n_0'c_1 \end{pmatrix}$$

has integer elements and unit determinant. Therefore, it is in $SL(2,\mathbb{Z})$. $\square$

Proposition 1.22. *Let α_1, α_2, α_3 be distinct numbers and α_1 be irrational. Consider two angles defined by pairs of lines $(y = \alpha_1 x, y = \alpha_2 x)$ and $(y = \alpha_1 x, y = \alpha_3 x)$ lying in the half-plane $x > 0$. The LLS-sequences of these two angles coincide from some element (up to a sequence shift).*

Proof. ... □

Theorem 1.23. *For any integer d which is not a square of an integer Pell's equation*

$$m^2 - dn^2 = 1$$

has a solution (m_0, n_0) where $n_0 \neq 0$.

Proof. By Theorem 1.18 there are infinitely many integer solutions of the inequality

$$\left|\sqrt{d} - \frac{p}{q}\right| < \frac{1}{\sqrt{5}q^2}.$$

For these points we have

$$|q^2 d - p^2| = |q\sqrt{d} - p||q\sqrt{d} + p| = \left|\sqrt{d} - \frac{p}{q}\right|\left|2\sqrt{d} + \left(\sqrt{d} - \frac{p}{q}\right)\right|q^2 < \frac{2\sqrt{d}+1}{\sqrt{5}}.$$

Therefore, there exists an integer k such that the equation

$$q^2 d - p^2 = c$$

has infinitely many integer solutions. Choose among these solutions (m_1, n_1) and (m_2, n_2) such that

$$m_1 \equiv m_2 (\operatorname{mod} c) \qquad \text{and} \qquad n_1 \equiv n_2 (\operatorname{mod} c).$$

(This is possible since the amount of distinct reminders pairs is finite.)

Now take

$$\hat{m} = \frac{m_1 m_2 - d n_1 n_2}{c} \qquad \text{and} \qquad \hat{n} = \frac{m_2 n_1 - d m_1 n_2}{c}.$$

Notice that

$$\hat{m} \equiv m_1^2 - dn_1^2 = c \equiv 0 (\operatorname{mod} c) \qquad \text{and} \qquad \hat{n} \equiv m_1 n_1 - m_1 n_1 = 0 (\operatorname{mod} c).$$

Hence the point $(\hat{m}, \hat{n})$ is integer. Now let us consider

$$\hat{m}^2 - d\hat{n}^2 = (\hat{m} - \sqrt{d}\hat{n})(\hat{m} + \sqrt{d}\hat{n}) = \frac{m_1 - \sqrt{d}n_1}{m_2 - \sqrt{d}n_2} \cdot \frac{m_1 + \sqrt{d}n_1}{m_2 + \sqrt{d}n_2} = \frac{m_1^2 - dn_1^2}{m_2^2 - dn_2^2} = \frac{c}{c} = 1.$$

This concludes the proof. □

Proof of Lagrange theorem. First, let us show that *any periodic continued fraction is a quadratic irrationality.* Since the continued fraction is infinite the corresponding number is irrational.

Suppose the periodic continued fraction for ξ does not have a pre-period, i.e.,

$$\xi = [(a_0 : a_1 : \ldots : a_n)]$$

then

$$\xi = [a_0; a_1 : \ldots : a_n : \xi] = \frac{p_n\xi + p_{n-1}}{q_n\xi + q_{n-1}}.$$

(The last equality holds by Theorem 1.11.) Notice that the denominator $q_{n-1}\xi + q_{n-2}$ is nonzero, since ξ is irrational. Therefore ξ satisfy

$$q_{n-1}\xi^2 + (q_{n-1} - p_{n-1})\xi - p_{n-2} = 0.$$

Hence ξ is a quadratic irrationality.

Suppose now that

$$\xi = [a_0; a_1 : \ldots : a_n : (a_{n+1} : a_{n+2} : \ldots : a_{n+m})]$$

Denote

$$\hat{\xi} = [(a_{n+1} : a_{n+2} : \ldots : a_{n+m})].$$

Then by Theorem 1.11 we have

$$\xi = [a_0; a_1 : \ldots : a_n : \hat{\xi}] = \frac{p_n\hat{\xi} + p_{n-1}}{q_n\hat{\xi} + q_{n-1}}.$$

We have already showed that $\hat{\xi}$ is a quadratic irrationality, hence ξ is a quadratic irrationality as well.

Secondly, we prove that *any quadratic irrationality is a periodic continued fraction.* Let $\xi > 1$ be a quadratic irrationality. By Lemma 1.21 there exists an $SL(2,\mathbb{Z})$-operator A with an eigenvector $(1,\xi)$. By Theorem ??? the geometric continued fraction of A has periodic LLS-sequence. The LLS-sequence for the angle generated by two vectors $(1,0)$ and $(1,\xi)$ is one-Side infinite and by Lemma 1.22 from some element it coincides (up to a shift) with the LLS-sequence for operator A. Hence, the LLS-sequence for A is periodic and, therefore, by Theorem ??? the continued fraction for ξ is periodic.

Suppose now, $\xi < 1$. The number $\hat{\xi} = \xi - \lfloor \xi \rfloor + 1$ is quadratic and greater than 1, and hence it is periodic by the above. The continued fractions for ξ and $\hat{\xi}$ are distinct only in the first element. Hence the continued fraction for ξ is periodic as well. □

Exercises

[1] Show that the sum $\sum\limits_{k=1}^{\infty} \frac{1}{F_k F_{k+1}}$ converges.

[2] Prove that for any $k \geq 2$ we get

$$\frac{p_{k-2}}{q_{k-2}} - \frac{p_k}{q_k} = \frac{(-1)^{k-1}a_k}{q_k q_{k-2}}.$$

[3] Prove that the sequence $\left(\frac{p_{2k+1}}{q_{2k+1}}\right)$ is decreasing and the sequence $\left(\frac{p_{2k}}{q_{2k}}\right)$ is increasing.

[4] Prove that for any $k \geq 1$ we get

$$\frac{q_k}{q_{k-1}} = [a_k; a_{k-1} : \ldots, a_1].$$

[5] Prove that for any $k \geq 0$ we get

$$\frac{1}{q_k(q_{k+1})} \geq \left|\alpha - \frac{p_k}{q_k}\right| > \frac{1}{q_k(q_{k+1}+q_k)}.$$

[6] Prove the statement of Example 1.19.

[7] Prove that a) $\sqrt{2} = [1;(2)]$;
b) $\exp(1) = [2;1:2:1:1:4:1:1:6:1:1:8:1:1:10:\ldots]$.

[8] Consider an irrational number α.
a) Suppose that we know that $\alpha \approx 4,17$. Is it true that $\frac{417}{100}$ is its best approximation?
b) Find the set of all real numbers for which the rational number $[1;2:3:4]$ is one of best approximations.

[9] Construct an infinite continued fraction which has exactly two limit points: 1 and -1.

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