10. Multidimensional continued fractions in the sense of Klein (15 June 2011)

10.1. **Definition of multidimensional continued fractions.** The problem of generalizing ordinary continued fractions to the higher-dimensional case was posed by C. Hermite in 1839. A large number of attempts to solve this problem lead to the birth of several different remarkable theories of multidimensional continued fractions. We consider the geometrical generalization of ordinary continued fractions to the multidimensional case presented by F. Klein in 1895. The interest to multidimensional continued fractions in modern mathematics is mostly due to V. I. Arnold who presented several surveys of geometrical problems and theorems associated with one-dimensional and multidimensional continued fractions.

Consider a set of $n+1$ hyperplanes of $\mathbb{R}^{n+1}$ passing through the origin in general position. The complement to the union of these hyperplanes consists of 2^{n+1} open orthants. Let us choose an arbitrary orthant among them.

Definition 10.1. The boundary of the convex hull of all integer points except the origin in the closure of the orthant is called the *sail*. The set of all 2^{n+1} sails of the space $\mathbb{R}^{n+1}$ is called the *n-dimensional continued fraction* associated to the given $n+1$ hyperplanes in general position in $(n+1)$-dimensional space.

Two n-dimensional continued fractions are said to be *equivalent* if there exists a linear transformation that preserves the integer lattice of the $(n+1)$-dimensional space and takes the sails of the first continued fraction to the sails of the other.

Multidimensional continued fractions in the sense of Klein have many connections with other branches of mathematics. For example, H. Tsuchihashi found the relationship between periodic multidimensional continued fractions and multidimensional cusp singularities, which generalizes the relationship between ordinary continued fractions and two-dimensional cusp singularities. Later J.-O. Moussafir and O. N. German studied the connection between the sails of multidimensional continued fractions and Hilbert bases. M. L. Kontsevich and Yu. M. Suhov discussed the statistical properties of the boundary of a random multidimensional continued fraction.

10.2. **Several invariants in multidimensional integer geometry.** Let us define notions of integer volume and integer distance between planes.

An *integer volume* of an integer k-dimensional polyhedron P in some k-dimensional plane is a ration between the Euclidean volume of the polyhedron and the minimal possible volume of the integer k-dimensional tetrahedron in this plane. We denote it by $\mathrm{lV}(P)$.

Consider an integer plane π and an integer point p in the complement to π. The *integer distance* from p to π_2 is the index of a sublattice generated by all integer vectors starting at p and pointing to all integer points of π in the lattice of all integer vectors in the plane that spans p and π. Denote it by $\mathrm{ld}(p, \pi)$.

Let us show an interesting relation between integer volumes and integer distances from planes to points.

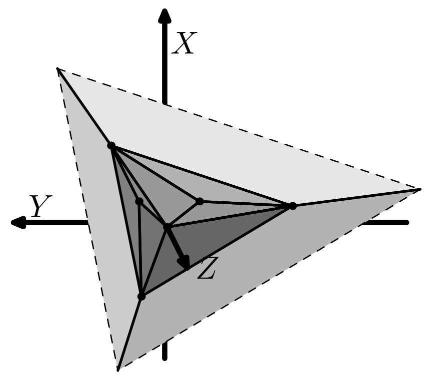


FIGURE 1. A sail of a finite continued fraction.

Proposition 10.2. *Let points $A_1, \ldots, A_{k+1}$ span a k-dimensional plane π. Then for any integer point in the complement to π we have*

$$\mathrm{ld}(A, \pi) = \frac{\mathrm{lV}(AA_1 \ldots A_{k+1})}{\mathrm{lV}(A_1 \ldots A_{k+1})}.$$

□

10.3. **Finite continued fractions.** Consider a cone (defined by hyperplanes) that contains integer points on all $n+1$ of its edges distinct to the origin. The sail corresponding to this cone has finitely many faces of all the dimension, actually it is contained completely in the pyramid $OA_1A_2 \ldots A_{n+1}$, where A_i is an integer point on the i-th edge. We call such sails *finite*. If a continued fraction has a finite sail then all its sails are also finite, we call such continued fraction *finite*.

Example 10.3. On Figure 1 we show an example of a sail for a cone defined by three vectors $(3, 2, 3)$, $(1, -4, 3)$, and $(-2, 1, 3)$.

We remind that the classification of geometric one-dimensional sails is provided by LLS-sequences for both finite and infinite cases.

The problem of description of two-dimensional finite sails is relatively new. Let us study the sails that have a unique face, this face is triangular.

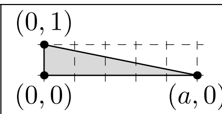
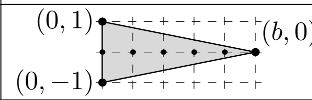
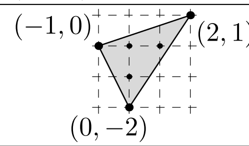
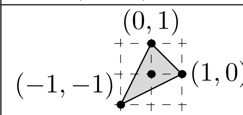
Theorem 10.4. *Let the sail have a unique triangular face.*

i). Suppose that the triangle is on unite integer distance, then any integer triangle is realized as such sail. Two sails are equivalent if and only if their triangular faces are integer congruent.

ii). Suppose that the triangle is on integer distance greater than 1. Then the corresponding sail is homeomorphic to exactly one of the sail of the list "T-W".

□

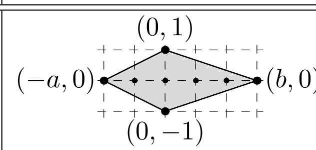
The first item of the theorem is straightforward. The second item is a direct corollary of the following more general fact on empty pyramids.

The list **"T-W"**	*Parameters*	*Coordinates of the face*	*Integer-affine type of the face*
$T^{\xi}_{a,r}$	$a{\geq}1,\ r{\geq}2,$ $0{<}\xi{\leq}r/2,$ $\gcd(\xi,r){=}1$	$(\xi,r{-}1,-r),$ $(a{+}\xi,r{-}1,-r),$ $(\xi,r,-r)$	
U_b	$b{\geq}2$	$(2,1,b{-}1),\ (2,2,-1),$ $(2,0,-1)$	
V		$(2,-2,1),$ $(2,-1,-1),$ $(2,1,2)$	
W		$(3,0,2),\ (3,1,1),$ $(3,2,3)$	

We say that a pyramid is *marked* if the vertex of the pyramid is chosen (this is actual for pyramids that are tetrahedra). A pyramid in $\mathbb{R}^3$ is called *integer* if its vertex is an integer point and its base is an integer polygon. An integer pyramid is called *completely empty* if any integer point inside the pyramid is either contained in the base or coincides with the vertex of the pyramid.

An integer pyramid is called one-story (multistory) if the integer distance from the vertex to the base is 1 (≥ 1).

Theorem 10.5. *Any multistory completely empty convex three-dimensional marked pyramid is integer-affine equivalent exactly to one of the marked pyramids from the list "T-W"* (*related to triangular sails*) *or to one of the following quadrangular pyramids:*

	Parameters	*Coords. of the vertex*	*Coordinates of the base*	*Integer-affine type of the base*
$M_{a,b}$	$b{\geq}a{\geq}1$	$(0,0,0)$	$(2,-1,0),$ $(2,-a{-}1,1),$ $(2,-1,2),\ (2,b{-}1,1)$	

□

In the next subsection we show how to classify empty pyramids with an empty face in the base. It is based on a famous White's theorem, that we formulate later. We skip the complete technical proof Theorem 10.5 (it is contained in paper **??**).

To conclude this subsection we formulate the following problem.

Problem 1. Describe all finite two-dimensional continued fractions.

Next step in the direction to the solution of this problem is to study the sails that are unions of two triangles, three triangles, etc.

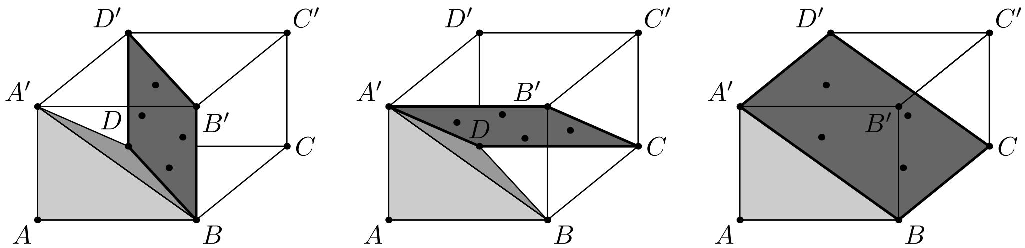


FIGURE 2. If $ADBA'$ is empty, then all integer points are in one of the following three planes. We show an example with 4 points inside.

10.4. **White's theorem.**

10.4.1. *Formulation of White's theorem and classification of empty integer tetrahedra.* In this subsection we describe integer congruence classes of empty three-dimensional integer tetrahedra. We remind that an integer tetrahedron is called *empty* if there is no integer points in it except for the vertices.

In two-dimensional case there is a unique class of empty triangles. *Any empty integer triangle is congruent to the triangle with vertices* $(0,0)$, $(1,0)$, *and* $(0,1)$.

Let us study the situation in three-dimensional case. We start with a White's theorem on geometric properties of empty tetrahedra.

Let $ABCD$ be a tetrahedron with enumerated vertices. Denote by $P(ABCD)$ the following parallelepiped:

$$\{A + \alpha AB + \beta AC + \gamma AD | \ 0 \leq \alpha \leq 1, 0 \leq \beta \leq 1, 0 \leq \gamma \leq 1\}.$$

Theorem 10.6. (White's theorem.) *Let $ADBA'$ be an empty tetrahedron. Then all integer points in $P(ADBA')$ except the vertices are contained in one of the planes passing trough centrally symmetric edges distinct to the edges of the tetrahedron.*

Remark 10.7. For any parallelepiped there exist exactly three planes as in the theorem. Se an example wit 4 inner points on Figure 2.

Corollary 10.8. *An integer empty tetrahedron with a marked vertex is integer congruent to exactly one of the following marked pyramids:*
— *a pyramid with vertices* $(0,0,0)$, $(1,0,0)$, $(1,0,1)$, *and* $(1,1,0)$*;*
— $T^{\xi}_{1,r}$ *of the list "T-W", where* $r{\geq}2$, $0{<}\xi{\leq}r/2$, *and* $\gcd(\xi,r){=}1$.
The point $(0,0,0)$ *is the marked vertex for all the tetrahedra in the list.*

It would be more convenient to show how to deduce Corollary 10.8 from White's theorem after the proof of White's theorem, so we do it a little later.

We conclude the main part of this subsection with the following open problem.

Problem 2. Find a description of empty tetrahedra in $\mathbb{R}^4$.

There is almost nothing known about the answer to the problem. Experiments show that direct generalizations of White's theorem do not work or help in four dimensional case.

10.4.2. *Preliminary definitions and notation.*

Definition 10.9. Let A, B, C, and D be integer points that span $\mathbb{R}^3$ (the collection of points is ordered). Let us construct a system of coordinates related to these points. Let b, c, and d be the integer distances from the points B, C, and D to the planes of the faces ACD, ABD, and ACD respectively. Consider the point A as an origin. Put the coordinates of B, C, and D to be equal $(b,0,0)$, $(0,c,0)$, and $(0,0,d)$ respectively. The coordinates of the other points in $\mathbb{R}^3$ are defined by linearity. We say that this system of coordinates is an *integer-distance coordinate system with respect to $ABCD$* or (*IDC-system* for short).

We call the points with integer coordinates in an IDC-system the *IDC-nodes*. We say that an IDC-node is *integer* if it is integer in the original integer lattice.

Notice that any integer point in the old system is an IDC-node. The converse is not necessary true, i.e. it is not necessary that any ICD-node is integer. For integer nodes the coordinates coincide to the integer distances to the coordinate planes in IDC-system.

10.4.3. *A lemma on sections of an integer parallelepiped.* Let a and b be two integer vectors, denote by $L_{a,b}$ the lattice generated by a and b. We start with a small discussion of the factor-lattice

$$\mathcal{L} = \mathbb{Z}^3 / L_{a,b}.$$

Notice that any equivalence class of $\mathcal{L}$ is completely contained in one of the planes parallel to the plane spanned by a and b. Any such plane contains only finitely many equivalence classes, which is equivalent to the index of $L_{a,b}$ in the integer sublattice contained in the space spanned by a and b.

Lemma 10.10. *Consider an integer parallelepiped $ABCDA'B'C'D'$ and a plane π parallel to $ABCD$. Let π intersects the parallelepiped (by some parallelogram). Then the following two statements hold.*

i). The intersection of π and the parallelepiped contains the representatives of all the equivalence classes of $\mathcal{L}$ in π.

ii). For any equivalence class of $\mathcal{L}$ in π its intersection with the parallelepiped is
a) a point inside the parallelogram of intersection;
b) two points on opposite edges of the parallelogram of intersection;
c) four points coincide with the vertices of the parallelogram of intersection. □

We leave the proof of this lemma as a simple exercise for the reader.

10.4.4. *A corollary on integer distances between the vertices and the opposite faces of a tetrahedron with empty faces.*

Corollary 10.11. *All integer distances from the vertices of an integer (three-dimensional) tetrahedron with empty faces to the opposite faces are all equivalent.*

Proof. Consider an integer tetrahedron $OABC$ with empty faces. Suppose that

$$\mathrm{ld}(A, OBC) = n.$$

Let us show that $\mathrm{ld}(B, OAC) = \mathrm{ld}(C, OAB) = n$.

Consider the parallelepiped $P(OABC)$. Since the triangles OBC, OAB, and OAC are empty, the corresponding faces of $P(OABC)$ are empty as well. Hence all faces of the parallelepiped are empty. Consider all the integer planes parallel to OBC that divide the parallelepiped into two nonempty parts. Since $\mathrm{ld}(A, OBC) = n$ the number of these planes equals $n - 1$.

By Lemma 10.10 any of such integer planes contains exactly one equivalence class of the lattice $\mathbb{Z}^3/L_{OB,OC}$. Since the faces of the parallelepiped are empty, the intersection of this class with a parallelepiped is one point in the interior. Hence the parallelepiped $P(OABC)$ contains exactly $n - 1$ integer points.

Suppose that $\mathrm{ld}(B, OAC) = m$, and $\mathrm{ld}(B, OAC) = k$. Then by similar reasons the parallelepiped contains $m - 1$ and $k - 1$ integer points inside respectively. Therefore, we have $k = m = n$. □

10.4.5. *Lemma on one integer node.* For the proof of Theorem 10.6 we need the following lemma.

Let $\mathrm{ld}(B, ACD) = r$ for some $r > 1$. Consider the IDC-system with respect to $ADBA'$ (with integer-distance coordinates (x, y, z)). Denote by B', C, C', and D' the vertices with coordinates $(0, r, r)$, $(r, r, 0)$, (r, r, r), and $(r, 0, r)$ respectively.

Lemma 10.12. *There is a unique integer node in the interior of the intersection of the plane* $x+y+z = r+1$ *and the parallelepiped* (*here we restrict ourselves to the case* $r > 1$).

Proof. Since $\mathrm{ld}(A, A'BD) = r$ and the parallelogram $A'B'CD$ is contained in the plane $x + y + z = r$, for any integer n the plane $x + y + z = n$ is integer. So the plane $x + y + z = r + 1$ is also integer.

Consider a parallelogram generated by vectors $A'B$ and $A'D$ at point $A' + (0, 0, 1)$ (see on Figure 3). Its points in IDC-system are:

$$(r, 0, 1), \quad (0, r, 1), \quad (0, 0, r + 1), \quad (r, r, r - 1)$$

This parallelogram is contained in the plane $x + y + z = r + 1$. The faces of this parallelogram are on the planes of faces of the parallelepiped and they do not contain vertices. Hence there exists a unique point in the interior of the shifted parallelogram, denote in by $K(x_0, y_0, z_0)$. Therefore, we have at most one point satisfying the conditions of the lemma.

Let us show that $K(x_0, y_0, z_0)$ is inside the parallelepiped. The shifted parallelogram is defined by the following four inequalities and one equation:

$$0 \leq x \leq r, \qquad 0 \leq y \leq r, \quad \text{and} \quad x + y + z = r + 1.$$

Since the point $(0, 0, r + 1)$ is not integer ($r + 1$ is not divisible by r), we have $z_0 \leq r$.

Suppose that $z_0 \leq 0$. In this case the vector KC is integer, hence the point $K' = A+KC$ is an integer node (see on Figure 3). Notice that the point K' is in the tetrahedron

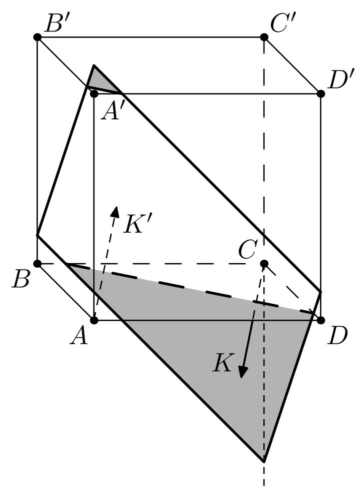


FIGURE 3. The points K and K' are the integer nodes.

$ADBA'$ and it is distinct to the vertices of the tetrahedron. So $ADBA'$ is not empty, that contradicts the conditions of the lemma. Therefore, we obtain $z_0 > 0$.

Hence we get: $0 \le x_0 \le r$, $0 \le y_0 \le r$, and $0 \le z_0 \le r$. Since the edges of the shifted parallelogram do not pass through the vertices of the parallelepiped, the point K is in the interior of the parallelepiped. □

10.4.6. *Proof of White's theorem.* We prove the statement of the theorem by induction in number of points inside the parallelepiped (or equivalently in the value of $\mathrm{ld}(A', ABD)$.

Base of induction. Suppose that $\mathrm{ld}(A', ABD) = 2$. Then the unique integer node inside the parallelepiped is the center of symmetry of the parallelepiped. It is in the intersection of all planes passing through opposite edges.

Step of induction. Suppose that the statement of the theorem holds for all empty tetrahedra with integer distance from the vertices to the opposite edges less than r. Let us prove the theorem for an arbitrary empty tetrahedron $A'ABD$ with $\mathrm{ld}(A', ABD) = r$.

By Lemma 10.12 there exists a unique integer node on the plane $x + y + z = r + 1$ inside the parallelepiped $P(A'ABD)$, denote it by $A''(x_0, y_0, z_0)$ (in the IDC-system with respect to the tetrahedron $ABDA'$). Without loss of generality we suppose that $z_0 \ge x_0$, and $z_0 \ge y_0$. The tetrahedron $ADBA''$ is empty integer tetrahedron, since all the nodes inside $ADBA''$ except A'' are inside $ADCA'$ as well. We get $\mathrm{ld}(A'', ADB) < r$ and hence by induction we have that all integer nodes of $P(ADBA'')$ are either in the plane $A''CD$, or in the plane $A''BC$, or in the plane $B''BD$ (where $B'' = A'' + AB$). Consider the intersections of these planes with the plane $z = 1$. One of these intersections should contained an integer node inside the parallelepiped $P(ADBA'')$.

Case 1. The integer nodes are in the plane $A''CD$. The plane $A''CD$ is defined by the equation

$$x + \frac{r - x_0}{z_0} z = r.$$

The intersection of the plane $A''CD$ with the plane $z = 1$ is the line $(x_1, t, 1)$, where t is a linear parameter of the line and x_1 is a constant equal to $r - \frac{r - x_0}{z_0}$. This plane contains integer nodes only if x_1 is integer. Let us estimate it (assuming that $y_0 \le z_0$). From one hand.

$$x_1 = r - \frac{r - x_0}{z_0} = r - \frac{r - (r + 1 - z_0 - y_0)}{z_0} = r + \frac{1}{z_0} - 1 - \frac{y_0}{z_0} \ge \\ r - 2 + \frac{1}{z_0} > r - 2.$$

From the other hand it holds $x_1 < r$. Since x_1 is integer and $r - 2 < x_1 < r$, we have $x_1 = r - 1$. Therefore, there exist integer nodes with coordinates $(r - 1, t, 1)$, and hence by Lemma 10.10 one of them (with coordinates $(r - 1, t_0, 1)$, for some integer t_0) is contained in the parallelepiped $P(ADBA')$. So the rest integer nodes of the parallelepiped $P(ADBA')$ have the coordinates $\big(r - k, (k \cdot t_0 \mod r), k\big)$ and hence they are all in the parallelogram $A'B'CD$.

Case 2. The integer nodes are in the plane $A''BC$. The plane $A''BC$ is defined by the equation

$$y + \frac{r - y_0}{z_0} z = r.$$

The intersection of the plane $A''BC$ with the plane $z = 1$ is the line $(t, y_1, 1)$, where t is a linear parameter of the line and y_1 is a constant equal to $r - \frac{r - y_0}{z_0}$. This plane contains integer nodes only if x_1 is integer. Let us estimate it (assuming that $x_0 \le z_0$). From one hand.

$$y_1 = r - \frac{r - y_0}{z_0} = r - \frac{r - (r + 1 - z_0 - x_0)}{z_0} = r + \frac{1}{z_0} - 1 - \frac{x_0}{z_0} \ge \\ r - 2 + \frac{1}{z_0} > r - 2.$$

From the other hand it holds $y_1 < r$. Since y_1 and $r - 2 < y_1 < r$, we have $y_1 = r - 1$. Therefore, there exist integer nodes with coordinates $(t, r - 1, 1)$, and hence by Lemma 10.10 one of them (with coordinates $(t_0, r - 1, 1)$, for some integer t_0) is contained in the parallelepiped $P(ADBA')$. So the rest integer nodes of the parallelepiped $P(ADBA')$ have the coordinates $((k \cdot t_0 \mod r), r - k, k)$ and hence they are all in the parallelogram $A'B'CD$.

The integer nodes are not in the plane $B''BD$. Let us show that the case of the plane $B''BD$ is an empty case. This plane is defined by the equation

$$x + y + \frac{r - y_0 - x_0}{z_0} z = r.$$

The intersection of the plane $B''BD$ with the plane $z = 1$ is the line $\left(\frac{a+t}{2}, \frac{a-t}{2}, 1\right)$, where t is a linear parameter for a line and a is a constant equivalent to $r - \frac{r - x_0 - y_0}{z_0}$. Such line contains integer nodes only if a is integer. From one hand

$$a = r - \frac{r - y_0 - x_0}{z_0} = r - \frac{z_0 - 1}{z_0} > r - 1.$$

From the other hand, we have $a < r$. Therefore, there is no integer nodes in the intersection of the parallelepiped $P(ADBA')$ (and of the parallelepiped $P(ADBA'')$ as well) with the plane $z = 1$, that is impossible. Therefore, the first two cases enumerates all realizable position of integer nodes.

Therefore, the statement holds for an arbitrary pyramid $ABCDA'$ with $\mathrm{ld}(A', ABD) = r$.

The proof of theorem is completed by induction. □

Remark 10.13. Notice that the third case in the proof is empty since we assume that the third coordinate of A'' is not less than the first and the second coordinates.

10.4.7. *Deduction of Corollary 10.8 from White's theorem.* Let us prove Corollary 10.8.

Proof. **Completeness of the list.** Consider an integer empty tetrahedron $A'ABD$ with a marked vertex A', let $\mathrm{ld}(A', ABD) = r$. If $r = 1$ then we get a pyramid integer congruent to $(0,0,0)$, $(1,0,0)$, $(1,0,1)$, and $(1,1,0)$.

Suppose that $r > 1$ then consider a point K on a unite integer distance from the plane ABD lying in the parallelepiped $P(ABDA')$. By White's theorem K is in one of the three diagonal planes, without loss of generality we assume that K is in $A'B'CD$.

So K has coordinates $(\xi, r-1, 1)$ in the IDC-system related with $ABDA'$. Let us rewrite the coordinates of the point in the basis AB, AC, and AK (notice that it is a basis of integer lattice). We have:

$$A = (0,0,0), \quad B = (1,0,0), \quad C = (0,1,0), \quad \text{and} \quad A' = (-\xi, 1-r, r).$$

Hence $ABDA'$ is integer congruent to $T^{\xi}_{1,r}$.

The parameter ξ is relatively prime with r, otherwise there exist integer points on the edges distinct to the vertices.

Further the pyramids $T^{\xi}_{1,r}$ and $T^{r-\xi}_{1,r}$ are integer congruent. Therefore, for uniqueness in the list we should restrict the consideration of the tetrahedra to the case $0<\xi\le r/2$.

Emptiness of the tetrahedra. All the tetrahedra are empty, since all integer points in $P(ABDA')$ are contained in one of the diagonal plains that does not intersect the tetrahedron in the interior of the parallelepiped.

Any two marked tetrahedra in the list are not integer congruent. Let us introduce the following integer invariant to distinguish the pyramids. Consider an arbitrary marked tetrahedron $ABDA'$ with vertex A' and the related tree-sided angle with vertex at A' and a section ABD. By Lemma 10.12 this plane contains a unique integer point (denoted by K) on the plane parallel to ABD on integer distance $r+1$ from A'. Integer distances to the faces of the tree-sided angle are 1, ξ, and $r-\xi$ for some ξ. The unordered collection $[1, \xi, r-\xi]$ is an integer invariant of the marked tetrahedron $A'ABD$. This invariant (together with $\mathrm{ld}(A', ABD) = r$) distinguishes all distinct pyramids in the list of the corollary. □

EXERCISES.

[1] Is an open segment convex?

[2] A set is convex if and only if it is an intersection of closed half-spaces.

[3] Find all n for which for any set P in $\mathbb{R}^n$ it holds

$$\operatorname{conv}(P) = \bigcup_{a,b\in P} [a,b].$$

[4] Describe integer empty pyramids in $\mathbb{R}^3$ up to integer congruence if the integer distances from the vertex to the base of such pyramids equal: **a)** 2; **b)** 3.

[5] Find the integer distance in $\mathbb{R}^4$ between the coordinate plane OXY and the plane

$$\left\{ \begin{array}{l} 2x + 3y + 4z + 5t = 6 \\ 6x + 5y + 4z + 3t = 2 \end{array} \right. .$$

[6] Classify all parallelepipeds associated with empty tetrahedra in $\mathbb{R}^4$, whose opposite faces are on integer distance no greater than 3 from each other.

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