11. Periodicity of Klein polyhedra (28 June 2011)

11.1. **Algebraic irrationalities and periodicity of sails.** Let $A \in GL(n+1, \mathbb{R})$ be an operator all of whose eigenvalues are real and distinct. Let us take the n-dimensional spaces that spans all possible subsets of n linearly independent eigenvectors of the operator A. The spans of every n eigenvectors defines $n{+}1$ hyperspaces passing through the origin in general position. These hyperspaces define a multidimensional continued fraction. Such continued fraction is said to be *associated to the operator* A.

Definition 11.1. Consider an operator $A \in GL(n+1, \mathbb{Z})$ with irreducible characteristic polynomial over $\mathbb{Q}$ that does not have non-real roots. The n-dimensional continued fraction associated to $A \in GL(n+1, \mathbb{Z})$ is called *the n-dimensional continued fraction of $(n+1)$-algebraic irrationality.* The case of $n = 1(2)$ corresponds to *one*(*two*)*-dimensional continued fractions of quadratic* (*cubic*) *irrationalities.*

The following statement holds.

Proposition 11.2. *Let A and B be operators of $GL(n+1, \mathbb{R})$ with distinct real irrational eigenvalues. The continued fractions associated to A and B are congruent if and only if there exists an integer operator X with unit determinant such that $\tilde{A}$ obtained from A by means of the conjugation by the operator X commutes with B.*

Proof. Let A and B satisfy the conditions of the proposition and suppose that their continued fractions are integer congruent. So there exists a linear integer lattice preserving transformation of the space that maps the continued fraction of the operator A to the continued fraction of the operator B. Under such transformation the operator A conjugates by some integer operator X with unit determinant. All eigenvalues of the obtained operator A are distinct and real (since conjugations preserve the characteristic polynomial of the operator). As far as the orthants of the first continued fraction maps to the orthants of the second one, the sets of the proper directions for the operators $\tilde{A}$ and B coincide. Hence these operators are diagonalizable together in some basis and therefore they commute.

Let us prove the converse. Suppose that there exists an integer lattice preserving operator X, such that the operator

$$\tilde{A} = XAX^{-1}$$

commutes with B. Note that the eigenvalues of the operators A and $\tilde{A}$ coincide. Therefore, all eigenvalues of the operator $\tilde{A}$ (just as for the operator B) are real and distinct. Let us consider a basis in which the operator $\tilde{A}$ is diagonal. Simple verification shows that the operator B is also diagonal in this basis. Hence the operators $\tilde{A}$ and B define the same orthant decomposition of $\mathbb{R}^{n+1}$, and thus the same continued fraction as well. □

Now let us formulate the notion of a periodic continued fraction associated to an algebraic irrationality. Suppose the characteristic polynomial of the operator A with all real roots is irreducible over the field of rational numbers. Consider the group of integer operators with unit determinant and positive eigenvalues that commutes with A. These

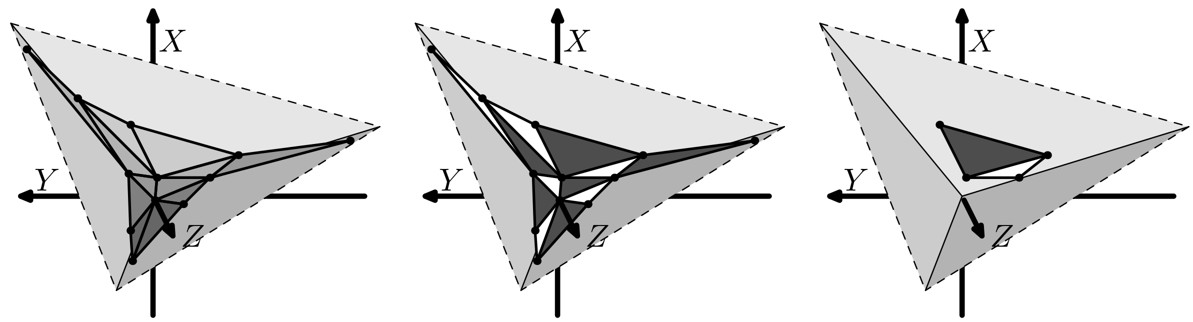


FIGURE 1. The sail of an algebraic continued fraction (left); its periodic structure (middle); a fundamental domain (right).

operators form an Abelian group, which we call *positive Dirichlet group*, denoted by $\Xi(A)$. From Dirichlet unity elements theorem it follows that

$$\Xi(A) = \mathbb{Z}^n.$$

Each operator of this group preserves the integer lattice and the union of all $n+1$ hyperplanes, and hence it preserves the n-dimensional continued fraction. Since all eigenvalues are positive, the sails are mapped to themselves as well. The group $\Xi(A)$ acts free on any sail. The factor of a sail under such group action is isomorphic to n-dimensional torus.

Remark 11.3. All periodic continued fractions correspond to algebraic case. This statement is known in different variations as *generalized Lagrange theorem*. It was studied by E. Korkina, G. Lachaud, O. N. German and E. L. Lakshtanov.

On Figure 1 we show an example of periodic continued fraction associated with an operator of two-dimensional golden ratio (that we study a little later):

$$\begin{pmatrix} 1 & 1 & 1 \\ 1 & 2 & 2 \\ 1 & 2 & 3 \end{pmatrix}.$$

On the left picture we see a fragment of the sail. In the middle we see triangles of two different orbits (white ones and dark ones). Finally on the right we show one of the possible fundamental domains for with respect to the action of the positive Dirichlet group.

11.2. **Torus decompositions of periodic sails in $\mathbb{R}^3$.** Any one-dimensional sail is uniquely defined by its odd or infinite continued fraction, and the relation is one-to-one. The situation in multidimensional case is much harder, since the structure of the sail is very complicated. The complete invariant is not known even for sails in $\mathbb{R}^3$. Further in this section we discuss only periodic continued fractions, in this case all sails are always homeomorphic to $\mathbb{R}^2$.

Definition 11.4. We say that two objects in $\mathbb{R}^3$ have the same *integer affine type* if there exist an integer lattice preserving affine transformation of the space that maps one of them to the other.

Let us discuss the case of $\mathbb{R}^3$. It is not very hard to get an invariant that distinguish all periodic sails. For instance, *if we know*

— *integer affine types of all faces in the sail;*

— *integer affine types of the pyramids at all vertices of the sail;*

— *an additional data about one of the vertices: integer affine type of the union of the pyramid and the origin*

then we uniquely reconstruct the sail, in case of existence.

Let us show in a few words how to do this. Using the third condition we construct the first pyramid, then using the first condition we find all polygonal faces of the pyramid, they are now uniquely defined. Now we proceed with the pyramids at vertex of the constructed boundary where two constructed faces come together, etc. In this manner we uniquely (up to lattice preserving linear transformation) build the whole sail.

In the periodic case we need to store only the finite amount of data since integer affine types of the objects are also periodic.

Suppose that we are given by a collection of mentioned invariants. *How to find if this collection is realizable*? The proposed invariant has many monodromy conditions, so the majority of cases will be non-realizable. Recently V. I. Arnold proposed a weaker invariant to distinguish the periodic sails.

Definition 11.5. A *torus decomposition* corresponding to the periodic two-dimensional sail is the factor-torus face decomposition of the sail equipped with integer affine types of faces and integer distances to them.

V. I. Arnold conjectured that torus decompositions of noncongruent sails are distinct. For all checked noncongruent sails this conjecture is true. (See Conjecture 1.)

In multidimensional case one can define similar torus decompositions, still the actual question is which decompositions are realizable and which are not realizable.

By *a fundamental region* of the sail we call a union of some faces that contains exactly one face from each equivalence class.

11.3. **Three single examples of torus decompositions.** In this and the next subsection we describe study several examples. We construct one of the sails for each example.

Example 11.6. We start with the simplest example. Consider two-dimensional fraction that generalizes the ordinary continued fraction corresponding to the golden ratio $\frac{1+\sqrt{5}}{2}$. All four one-dimensional sails of the one-dimensional continued fraction corresponding to the golden ratio are congruent. This fraction is given by the following operator:

$$\begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix}$$

The associated continued fraction has sails whose LLS-sequences are $[\ldots,1,1,1\ldots]$ – the simplest possible periodic LLS-sequence. The corresponding circle decompositions consist of one vertex and one edge.

The multidimensional continued fraction associated to the operator

$$\begin{pmatrix} 1 & 1 & 1 & \cdots & 1 \\ 1 & 2 & 2 & \cdots & 2 \\ 1 & 2 & 3 & \cdots & 3 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & 2 & 3 & \cdots & n \end{pmatrix}$$

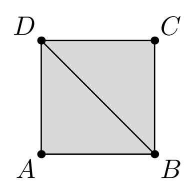
is called *the generalized golden ratio.*

Remark 11.7. Unfortunately these operators are not always give the simplest periodic continued fractions in all dimensions, since the characteristic polynomials are not always irreducible. For instance there are nonconstant factors if the dimension $n = 4$, 7, 10, 12, 13, 16 ,17, 19 for $n \leq 20$. Further we study the case $n = 3$ which corresponds to an irreducible operator.

Here we consider the continued fraction associated to the generalized golden ratio operator:

$$\begin{pmatrix} 1 & 1 & 1 \\ 1 & 2 & 2 \\ 1 & 2 & 3 \end{pmatrix}.$$

The torus decomposition corresponding to this operator is homeomorphic to the following one:



Here the segment AB is identified with the segment DC and the segment AD to the segment BC.

On the picture above we show only homeomorphic types of the faces (without lattice structure). The integer affine types of the corresponding faces are given on the next picture.

Both triangles have integer affine types of the simplest triangle:

The positive Dirichlet group $\Xi(M)$ is generated by the following two operators:

$$\begin{pmatrix} 1 & 1 & 1 \\ 1 & 2 & 2 \\ 1 & 2 & 3 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} 1 & 0 & 1 \\ 0 & 2 & 1 \\ 1 & 1 & 2 \end{pmatrix}$$

The integer distance from the triangle ABD to the origin equals 2, and from the triangle BCD equals 1.

This torus decomposition was found by E. I. Korkina, G. Lachaud and A. D. Bryuno and V. I. Parusnikov approximately at the same time.

Let us give an idea how to prove that the constructed torus triangulation is correct. To do this we need to check the following:

i). All the points A, B, C, and D are in the same cone. (Actually this can be seen from the fact that all the points are in the same orbit of the sail with respect to the positive Dirichlet group).

ii). All the pyramids with vertex at the origin and with faces ABD and BCD do not contain integer points except the origin and integer points of the base.

iii). The sail is convex at edges AB, BD, and DA.

Let us give the comments to the last step. We say that a polyhedral surface is *convex* at edge AB with respect the origin if the origin is the plane of the first face separates the origin from the points of the second face and the plane of the second face separates the origin from the points of the first face.

Note that to check convexity at AB we need an additional triangle ABC' that is in the same orbit with respect to the Dirichlet group as the triangle DCB.

We leave all the calculations to the reader. Later we discuss the algorithms to verify if a certain set of faces is a fundamental domain of a certain algebraic operator.

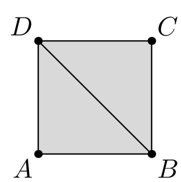
Example 11.8. The second example was given by A. D. Bryuno and V. I. Parusnikov. They construct the continued fraction that associated to the following operator:

$$M = \begin{pmatrix} 1 & 1 & 1 \\ 1 & -1 & 0 \\ 1 & 0 & 0 \end{pmatrix}.$$

The positive Dirichlet group $\Xi(M_a)$ is generated by the following two operators:

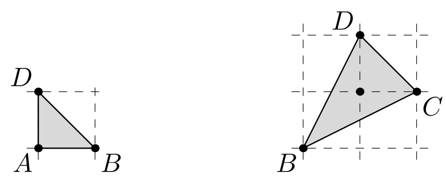
$$X = M^2, \qquad Y = 2I - M^2.$$

The torus decomposition corresponding to this operator is also homeomorphic to the following one:



Here the segment AB is identified with the segment DC and the segment AD to the segment BC.

The triangle ABD has an integer affine type of the simplest triangle, The triangle BCD has an integer affine type of the triangle with the vertices $(-1,-1)$, $(0,1)$, and $(1,0)$.



The integer distance from the triangle ABD to the origin equals 2, and from the triangle BCD equals 1.

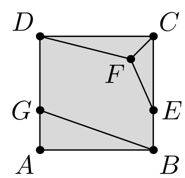
Example 11.9. The third example was given by V. I. Parusnikov. This continued fraction is associated to the following operator:

$$M = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 1 & -3 \end{pmatrix}.$$

The positive Dirichlet group $\Xi(M)$ is generated by the following two operators:

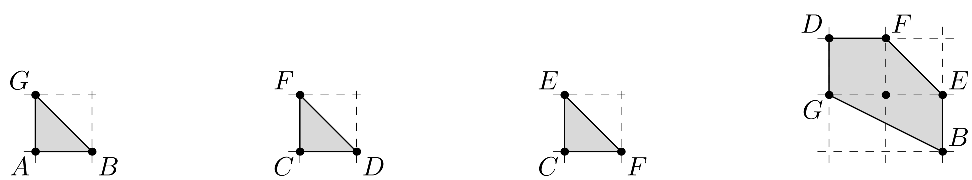
$$X = M^2, \qquad Y = 3I - 2M^{-1}.$$

The torus decomposition corresponding to this operator is also homeomorphic to the following one:



Here the segment AB is identified with the segment DC and the polygonal line AGD — to the polygonal line BEC (the point G is identified with the point E).

All triangles have an integer affine type of the simplest triangle, The pentagon $BEFDG$ has an integer affine type of the pentagon with the vertices $(-1,0)$, $(-1,1)$, $(0,1)$ $(1,0)$ and $(1,-1)$:



The integer distances from the triangle ABC and the pentagon $BEFDG$ to the origin equal 1, from the triangle CDF to the origin equals 2, and from the triangle CFE equals 3.

The continued fractions constructed in Examples 11.6, 11.8 and 11.9 are also known as the continued fractions corresponding to the first, the second and the third Davenport form.

Let us now consider the following norm for the space of matrices: the sum of absolute values of all the coefficients for the matrix.

Theorem 11.10. *There is no hyperbolic matrix among the set of matrices with integer coefficients and irrational eigenvalues, and hence there is no continued fraction associated to the operator with such norm.*

If the norm of the hyperbolic matrix with integer coefficients and irrational eigenvectors equal to five (their number is 48), then the corresponding continued fraction is congruent *to the generalization of the ordinary fraction for the golden ratio. This fraction is shown in Example 11.6.*

The amount of such matrices with norm equals six is 912, and only three different two-dimensional continued fractions associated to them: 480 continued fractions are congruent to the fraction of Example 11.6, 240 continued fractions are congruent to the fraction of Example 11.9 and 192 — to the fraction of Example 11.8.

The classification of two-dimensional continued fractions with the norm equals seven or more is unknown.

11.4. **Infinite sequences of torus decomposition examples.** Now we continue the description of examples with the infinite sequences of the torus decompositions. The first two infinite sequences of torus decompositions was calculated by E. I. Korkina. One of this sequences is shown below.

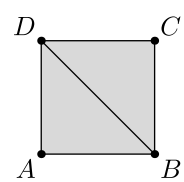
Example 11.11. This sequence of continued fractions is associated to the following operators for $m \geq 0$:

$$M_a = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & -a-5 \\ 0 & 1 & a+6 \end{pmatrix}.$$

The positive Dirichlet group $\Xi(M_a)$ is generated by the following two operators:

$$X_a = M_a, \qquad Y_a = (M_{a,b} - I)^2.$$

The torus decomposition corresponding to this operator is homeomorphic to the following one:



Here the segment AB is identified with the segment DC and the segment AD to the segment BC.

The integer distance from the triangle ABD to the origin equals to $m+2$, and from the triangle BCD it equals 1.

All two triangles have integer affine types of the simplest triangle:

Many other examples of infinite sequence of continued fractions were done by the author of this issue. Now we show some of them. The following sequence generalize the one from the previous example.

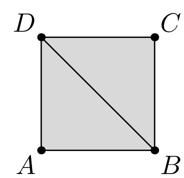
Example 11.12. This sequence depends on two integer parameters $a, b \geq 0$. The continued fractions of the sequence are associated to the following operators:

$$M_{a,b} = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 1+a-b & -(a+2)(b+1) \end{pmatrix}.$$

The positive Dirichlet group $\Xi(M_{a,b})$ is generated by the following two operators:

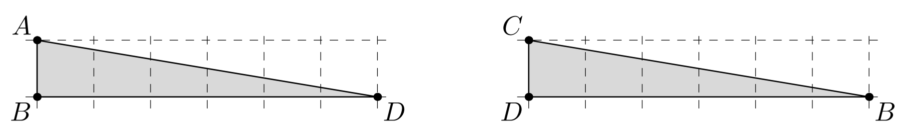
$$X_{a,b} = M_{a,b}^{-2}, \qquad Y_{a,b} = M_{a,b}^{-1}\big(M_{a,b}^{-1} - (b+1)I\big).$$

The torus decomposition corresponding to this operator is homeomorphic to the following one:



Here the segment AB is identified with the segment DC and the segment AD to the segment BC.

All two triangles have the same integer affine type of the triangle with the vertices $(0,0)$, $(0,1)$ and $(b+1,0)$ (b=5 at the picture):



The integer distance from the triangle ABD to the origin equals $a+2$, and from the triangle BCD equals 1.

Remark. If we pose $b = 0$, then we have the sequence of operators that have congruent continued fractions with the sequence of operators of Example 11.11.

Proposition 11.13. *The continued fractions associated to the following operators are congruent* (*for integer* $t \geq 0$)*:*

$$M_{a,0} = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 1+a & -a-2 \end{pmatrix} \quad M'_a = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & -a-5 \\ 0 & 1 & a+6 \end{pmatrix}.$$

The proof of this statement is straightforward. It bases on the fact that the operators $(I - M_{a,0})^{-1}$ and M'_a are conjugate by the operator X in the group $SL(3,\mathbb{Z})$ (here I — is the identity operator):

$$X = \begin{pmatrix} -1 & -1 & -2 \\ 0 & 0 & -1 \\ 1 & 0 & -1 \end{pmatrix}.$$

(i.e. $M'_a = X^{-1}(I - M_{a,0})^{-1}X$).

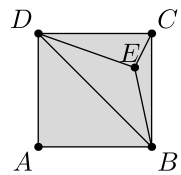
Example 11.14. The sequence of this example depends on an integer parameters $a \geq 1$. The continued fractions of the sequence are associated to the following operators:

$$M_{a,b} = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & a & -2a-3 \end{pmatrix}.$$

The positive Dirichlet group $\Xi(M_a)$ is generated by the following two operators:

$$X_a = M_a^{-2}, \qquad Y_a = \big(2I - M_a^{-2}\big)^{-1}.$$

The torus decomposition corresponding to this operator is homeomorphic to the following one:



Here the segment AB is identified with the segment DC and the segment AD to the segment BC.

All four triangles have integer affine types of the simplest triangle:

The integer distance from the triangle ABD to the origin is equivalent to $a+2$, from BDE — to $a+1$ and from BCE and CED — to 1.

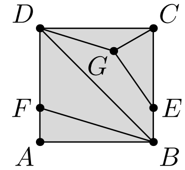
Example 11.15. The family of this example depend on two integer parameters $a > 0$ and $b \geq 0$. The continued fractions of the sequence are associated to following operators:

$$M_{a,b} = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & (a+2)(b+2)-3 & 3-(a+2)(b+3) \end{pmatrix}.$$

The positive Dirichlet group $\Xi(M_{a,b})$ is generated by the following two operators:

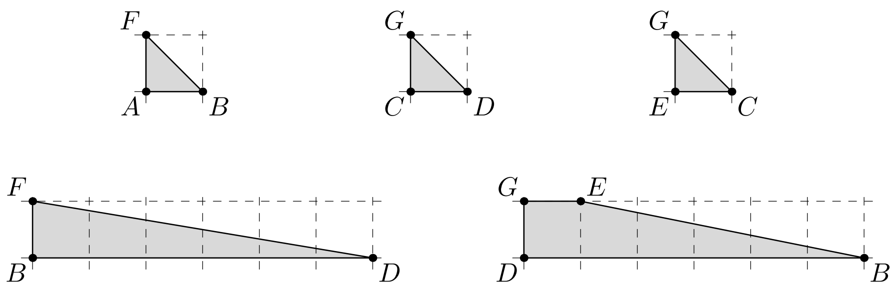
$$X_{a,b} = \big((b+3)I - (b+2)M_{a,b}^{-1}\big)M_{a,b}^{-2}, \qquad Y_{a,b} = M_{a,b}^{-2}.$$

The torus decomposition corresponding to this operator is homeomorphic to the following one:



Here the segment AB is identified with the segment DC and the polygonal line AFD — to the polygonal line BEC (the point F is identified with the point E).

The triangles ABF, CGE and CDG have an integer affine type of the simplest triangle; the triangle BDF has an integer affine type of the triangle with the vertices $(0,0)$, $(b+2,0)$ and $(0,1)$; the quadrangle $DBEC$ has an integer affine type of the quadrangle with the vertices $(0,0)$, $(b+2,0)$, $(1,1)$ and $(0,1)$:



The integer distance from the triangles ABF and BFD to the origin equals 1, from the triangle CDG — to $2+2a+2b+ab$, from the triangle CEG — to $3+2a+2b+ab$, from the quadrangle $DBEC$ — to 1.

11.5. **Some problems and conjectures.** In the last section we formulate several actual geometrical problems and conjectures on Klein multidimensional continued fractions.

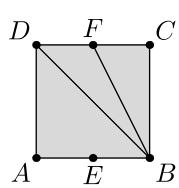
We start with a problem on a complete invariant of sails. In the one-dimensional case these characteristics are known, it is an LLS-sequence. There is no answer to this question in the two-dimensional case (in both periodic and general cases). For the periodic case we have the following conjecture and the problem. If the conjecture is true and the problem is solved then we have a complete invariant for two-dimensional periodic sails.

Conjecture 1. (V. I. Arnold) Torus decompositions of noncongruent sails are distinct.

Problem 2. (V. I. Arnold) Describe all torus decompositions that are possible for some periodic two-dimensional continued fraction.

Only a few is known in this direction. There is a lot of trivial examples for torus decompositions that do not corresponds to some sails.

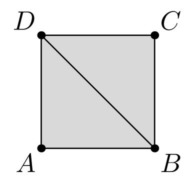
Example 11.16. The following torus decomposition does not correspond to any sail for the periodic continued fraction of cubic irrationality (there is one integer point in the interval AD, we denote it by E):



Here the polygonal line AEB maps to the polygonal line DFC under one of the operators of the group $SL(3,\mathbb{Z})$ that preserves the sail. As far AEB is a segment, DFC is also a segment. Therefore, the points B, C, F and D lie in the same plane. And hence BF is not a edge of some face.

Now we present the first nontrivial example of torus decomposition that was made by E. I. Korkina.

Example 11.17. Consider the simplest torus triangulation. It consists of two triangles with the simplest integer affine type of the triangle $(0,0)$, $(0,1)$ and $(1,0)$. The integer distances to both faces is 1. This decomposition does not correspond to any sail for the periodic continued fraction of cubic irrationality:



This example as far as a large number of sail calculations allow us to formulate the following conjecture.

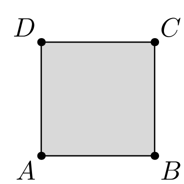
Conjecture 3. For any torus decomposition for the continued fraction of the cubic irrationality there exists some face with integer distance to the origin more than one.

On the other hand each of the calculated sails have some face with the integer distance to the origin equals one.

Conjecture 4. For any torus decomposition for the continued fraction of the cubic irrationality there exists some face with integer distance to the origin equals one.

The following example is on torus decomposition, that consist of one parallelogram.

Example 11.18. Consider the torus decomposition consisting of the only one face with integer affine type of the simplest parallelogram with the vertices $(0,0)$, $(0,1)$, $(1,1)$ and $(1,0)$. The integer distances to all of the faces equals some natural number n. This decomposition does not correspond to any sail for the periodic continued fraction of cubic irrationality:



It seems that the torus decomposition with one rectangular face is not possible for the decompositions associated to the two-dimensional continued fractions of cubic irrationalities. Moreover we conjecture the following.

Conjecture 5. For any torus decomposition for the continued fraction of the cubic irrationality there exists the face that has an integer affine type of some triangle.

Let us discuss the relation between two-dimensional continued fractions of cubic irrationalities and cubic extensions of the field of rational numbers. For any such two-dimensional continued fraction there exist an operator in $SL(3,\mathbb{Z})$ with irreducible characteristic polynomial over the field $\mathbb{Q}$ such that the continued fraction is associated to the operator. This polynomial determines the cubic extension of the field of rational numbers. All integer operators that identifies the continued fraction commutes with A and hence have the same cubic extension. However the converse is not true.

Example 11.19. The following two operators having the same characteristic polynomial $x^3 + 11x^2 - 4x - 1$ (and hence the same cubic extension) define noncongruent continued fractions:

$$\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 1 & -2 \end{pmatrix}^3, \qquad \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 4 & -11 \end{pmatrix}.$$

So, the natural question here is as follows.

Problem 6. (V. I. Arnold) Classify continued fractions that corresponds to the same cubic extensions of the field of rational numbers.

It is known almost nothing here. For example, it is even unknown the finiteness of the number of possible continued fractions associated to the same extension. (For properties of cubic extensions of rational numbers see in the work of B. N. Delone and D. K. Faddeev.)

It seems that in the case of one-dimensional continued fraction similar question is also open.

In conclusion we show the table that was taken from my work [**?**] with squares filled with torus decomposition of the sails for continued fractions associated to the family of Frobenius operators

$$A_{m,n} := \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & -m & -n \end{pmatrix},$$

where m and n are integers, see Fig 2.

Note the following: if the characteristic polynomial $\chi_{A_{m,n}(x)}$ is irreducible over the field $\mathbb{Q}$ than the matrix for the left multiplication by the element x operator in the natural basis $\{1, x, x^2\}$ in the field $\mathbb{Q}[x]/\big(\chi_{A_{m,n}(x)}\big)$ coincides with the matrix $A_{m,n}$.

Let an operator $A \in SL(3,\mathbb{Z})$ has distinct real irrational eigenvectors. Let e_1 be some integer nonzero vector, $e_2 = A(e_1)$, $e_3 = A^2(e_1)$. Then the matrix of the operator in the basis (e_1, e_2, ce_3) for some rational c will be Frobenius. However the transition matrix here could be non-integer and the corresponding continued fraction is not congruent to initial one.

Example 11.20. The continued fraction constructed by the operator

$$A = \begin{pmatrix} 1 & 2 & 0 \\ 0 & 1 & 2 \\ -7 & 0 & 29 \end{pmatrix},$$

is not congruent to the continued fraction constructed by any Frobenius operator with the determinant equals to one.

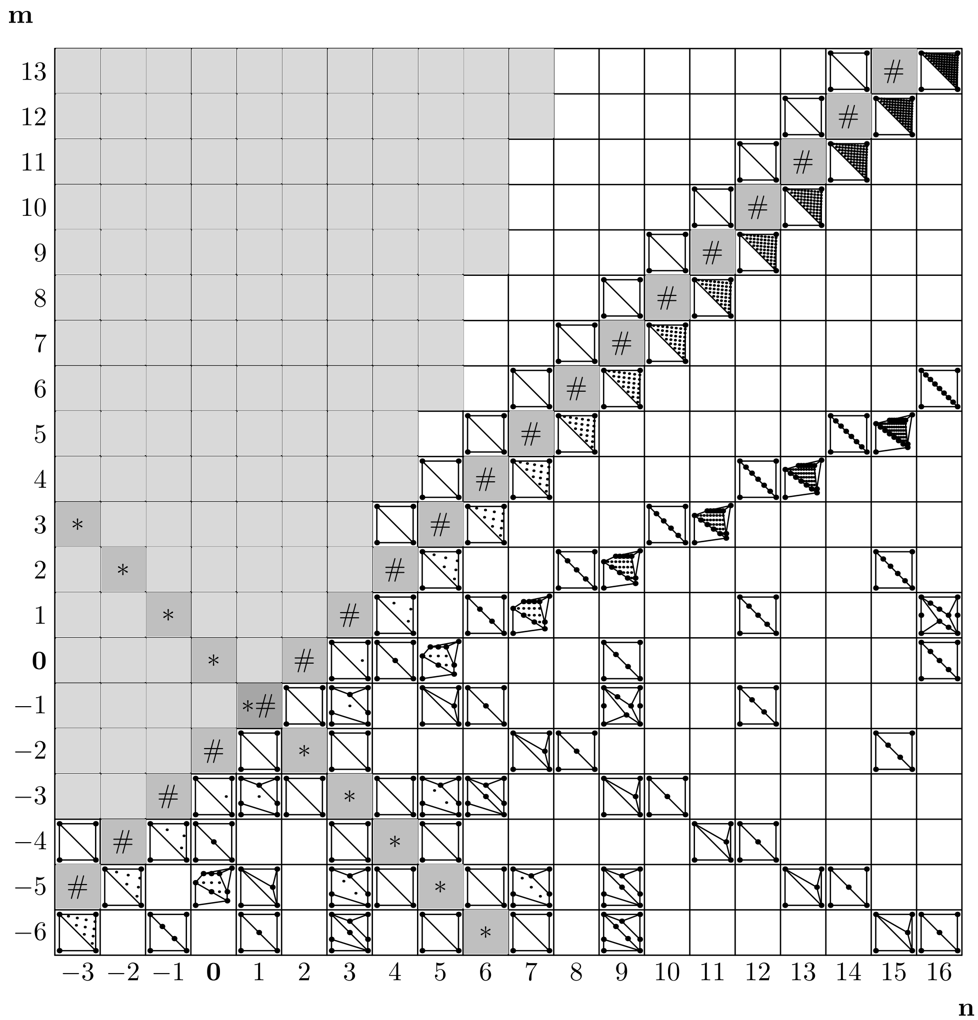


FIGURE 2. Torus decompositions for operators $A_{m,n}$.

Thereupon the following question is of interest. *How often the continued fractions that do not correspond to Frobenius operators can occur?*

In any case the family of Frobenius operators possesses some useful properties that allows us to construct whole subfamilies of noncongruent two-dimensional periodic continued fractions at once, that extremely actual itself.

It is easy to obtain the following statements.

Proposition 11.21. *The set Ω of operators $A_{m,n}$ having all eigenvalues real and distinct is defined by the inequality $n^2m^2-4m^3+4n^3-18mn-27\leq 0$. For the eigenvalues of the operators of the set to be irrational it is necessary to subtract extra two perpendicular lines in the integer plain: $A_{a,-a}$ and $A_{a,a+2}$, $a\in\mathbb{Z}$.*

Proposition 11.22. *The two-dimensional continued fractions for the cubic irrationalities constructed by the operators* $A_{m,n}$ *and* $A_{-n,-m}$ *are congruent.*

Further we will consider all statements modulo this symmetry.

Remark. Example 11.19 given below shows that among periodic continued fractions constructed by operators in the set Ω congruent continued fractions can happen.

Integer affine types of the faces of the torus decomposition for the sail of the two-dimensional continued fraction for the cubic irrationality, associated to the operator $A_{m,n}$ is shown in the square sited at the intersection of the string with number n and the column with number m. If one of the roots of characteristic polynomial for the operator equals 1 or -1 at that than the square (m, n) is marked with the sign $*$ or $\#$ correspondingly. The squares that correspond to the operators which characteristic polynomial has two complex conjugate roots are painted over with light gray color.

Exercises.

[1] Prove that torus triangulations in Examples 11.6-11.9 is correct.
[2] Prove that the method introduced in Example 11.6 works for all five examples in this section.
[3] Prove that the torus triangulation of Example 11.18 is not realizable.
[4] Construct a sail for a totally real $SL(3,\mathbb{Z})$-operator not considered in this chapter.
[5] Construct all 8 sails for certain totally real $SL(3,\mathbb{Z})$-operator.

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