2. On integer geometry (22 March 2011)

2.1. **Basic notions and definitions.** A notion of geometry in general can be interpreted in many different ways. In our course we think of geometry as of a *set of objects* and a *congruence relation* which is normally defined by some group of transformations. For instance, in Euclidean geometry in the plane we study points, lines, segments, polygons, circles, etc, the congruence relation is defined by the group of all length preserving transformations $O(2,\mathbb{R})$ (or the orthogonal group).

2.1.1. *Objects and congruence relation of integer geometry.* While saying about planar lattice geometry we think of a lattice $\mathbb{Z}^2$ in $\mathbb{R}^2$. Any two lattices are isomorphic, so it does not matter which lattice to work with. While changing one lattice to another one should be careful only with Euclidean invariants that are not preserved by the isomorphisms, all the lattice invariants are preserved. For simplicity we mostly consider the integer lattice (containing points with both integer coordinates). The objects of *integer* (*lattice*) *geometry* are integer points, *integer segments and polygons* having all vertices in the integer lattice, *integer lines* passing through couples of lattice points, *integer angles* with vertices in lattice points. The congruence relation is defined by the group of affine transformations preserving the lattice, i.e. $\mathrm{Aff}(2,\mathbb{Z})$. This group is a semidirect product of $GL(2,\mathbb{Z})$ and the group of translations on integer vectors. We use "$\cong$" to indicate that two objects are lattice congruent.

The congruence relation is slightly different to the Euclidean case. We illustrate with integer congruent triangles on Figure 1.

2.1.2. *Invariants of integer geometry.* As long as we have objects and a group of transformations that acts on the objects, we get the *invariants* — the quantities if objects that are preserved by transformations. Usually the study of these invariants is the main subject of the corresponding geometry. For instance in Euclidean geometry the invariants are lengths of segments, areas of polyhedra, measures of angles, etc.

So *what are the invariants in integer geometry?* There are two different type of invariants that present in integer geometry: affine invariants and lattice invariants. Affine invariants are intrinsic to affine geometries in general. Affine transformations preserves

– the set of lines;

– the property that a point in a line is in between two other points;

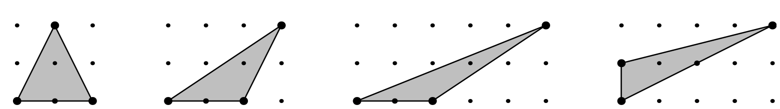


Figure 1. The following triangles are integer congruent.

– the property that two points are in one halfplane with respect to a line.
Lattice invariants are induced by the group structure of the vectors in the lattice. The majority of lattice invariants are indices of certain subgroups (i.e., sublattices).

2.1.3. *Index of sublattices.* Recall several notions of group theory. Let H be a subgroup of G. Consider an element $g \in G$. The sets

$$gH = \{gh | h \in H\} \qquad \text{and} \qquad Hg = \{hg | h \in H\}$$

are called *left* and *right cosets* of H in G.

The *index* of the subgroup H in the group G is the number of all left cosets of H in G, denote it by $|G : H|$. (For, example for a positive integer n we have $|\mathbb{Z} : n\mathbb{Z}| = n$.) It turns out that the number of left cosets coincides with the number of right cosets. We mostly work with commutative groups where for any g it holds $gH = Hg$.

In integer geometry it is natural to consider the additive group of integer vectors, which is called the *integer lattice.* Any subgroup of integer lattice is called an *integer sublattice.* Let A be an integer point and G by an integer sublattice. We say that the set of points

$$\{A + g | g \in G\}$$

is the *integer affine sublattice.* For any two integer affine sublattices $L_1 \subset L_2$ one can consider the index $|G_2 : G_1|$ of the corresponding integer sublattices G_1 and G_2 in the affine sublattices L_1 and L_2. Any integer transformation acts on the group of integer vectors as a group isomorphism. Therefore, all the indices for integer sublattices are preserved.

One of the simple ways to calculate the index of a sublattice is by counting integer points in a basis parallelogram for the sublattice.

Proposition 2.1. *The index of a sublattice generated by a pair of integer vectors v and w in $\mathbb{Z}^2$ equals the number of all integer points P satisfying*

$$AP = \alpha v + \beta w \quad \textit{with} \quad 0 \le \alpha, \beta < 1,$$

where A is an arbitrary integer point.

Proof. Let H be a subgroup of $\mathbb{Z}^2$ generated by v and w. Denote

$$\mathrm{Par}(v, w) = \{\alpha v + \beta w | 0 \le \alpha, \beta < 1\}.$$

First, we show that for any integer vector g there exists a point $P \in \mathrm{Par}(v, w)$ such that $AP \in gH$. The point P is constructed as follows. Let

$$g = \lambda_1 v + \lambda_2 w,$$

Consider

$$P = (\lambda_1 - \lfloor \lambda_1 \rfloor) v + (\lambda_2 - \lfloor \lambda_2 \rfloor) w + A.$$

Since

$$0 \le \lambda_1 - \lfloor \lambda_1 \rfloor, \lambda_2 - \lfloor \lambda_2 \rfloor < 1,$$

the point P is inside the parallelogram and $AP \in gH$.

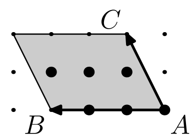
Second, we prove the uniqueness of P. Suppose that for two integer points $P_1, P_2 \in \mathrm{Par}(v,w)$ we have $AP_1 \in gH$ and $AP_2 \in gH$. Hence the vector P_1P_2 is an element in H. The only element of H of type

$$\alpha v + \beta w \quad \text{with} \quad 0 \le \alpha, \beta < 1,$$

is zero vector. Hence $P_1 = P_2$.

Therefore, the integer points of $\mathrm{Par}(v,w)$ are in one-to-one correspondence to the cosets of H in $\mathbb{Z}^2$. □

Example 2.2. Consider the points A, B, and C as follows:

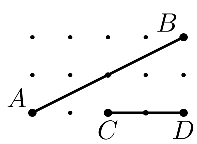


Then we have 6 points satisfying the above condition. Therefore, the index of the sublattice generated by AB and AC is 6.

2.1.4. *Integer length of integer segments.* Now we are ready to define several integer invariants. We start with an integer length.

Definition 2.3. An *integer length* of an integer segment AB is the number of integer points in the interior of AB plus one. We denote it by $\mathrm{l}\ell(AB)$.

The integer lengths of the segments AB and CD are both equal 2.



$$l\ell(AB) = 2, \quad l\ell(CD) = 2$$

The alternative invariant definition is as follows: an integer length of an integer segment AB is a sublattice generated by vector AB in the lattice of integer points in the line AB.

In Euclidean geometry the length is a complete invariant of congruence classes of segments. The same is true in lattice geometry. For instance the above segments AB and CD has the same integer lengths, and hence they are congruent. In general we have the following.

Proposition 2.4. *Two integer segments are congruent if and only if they have the same integer length.* □

2.1.5. *Integer distance to integer lines.* There is no simple definition of orthogonal vectors, since the scalar product of vectors is not $GL(2,\mathbb{Z})$-invariant. Still it is possible to give a natural definition of an integer distance from a point to a line.

Definition 2.5. Consider integer points A, B, and C that do not lie in one line. An *integer distance* from the point A to the integer segment (line) BC is the index of a sublattice generated by all integer vectors AV where V is in an integer point of the line BC in the integer lattice. We denote it by $\mathrm{ld}(A, BC)$.

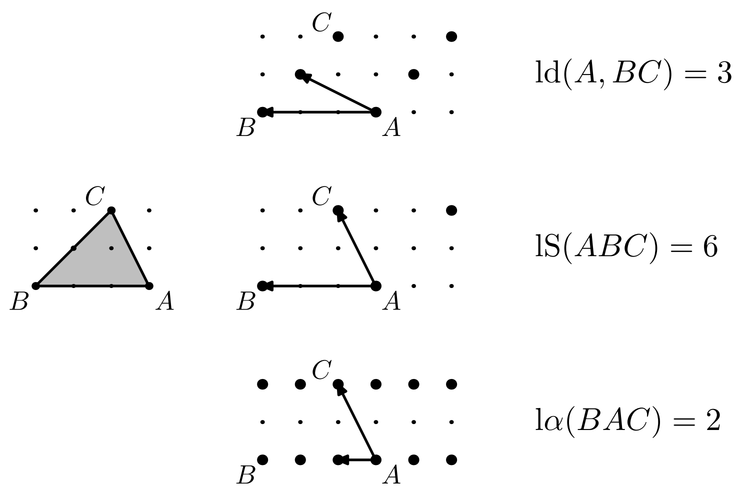
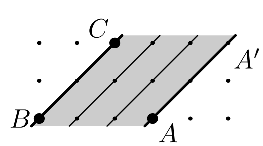


FIGURE 2. A triangle ABC and the sublattices for calculation of $\mathrm{ld}(A, BC)$, $\mathrm{lS}(ABC)$, and $\mathrm{l}\alpha(BAC)$.

In case of the points A, B, and C are in a line we agree to say that the integer distance from A to BC is zero.

One of the geometrical interpretations of integer distance from a point A to BC is as follows. Draw all integer lines parallel to BC. One of them contains the point A (let us call it AA'). The integer distance $\mathrm{ld}(A, BC)$ is the amount of lines in the region bounded by the lines AB and AA' plus one. For the example of Figure 2, we have



There are two integer lines parallel to AB and lying in the region with boundary lines AB and AA'. Hence, $\mathrm{ld}(A, BC) = 2 + 1 = 3$.

2.1.6. *Integer area of integer triangles.* Let us start with the notion of integer area for integer triangles. We will define the integer areas for polygons later in the next subsection.

Definition 2.6. An *integer area* of an integer triangle ABC is the index of the sublattice generated by the vectors AB and AC in the integer lattice. Denote $\mathrm{lS}(ABC)$.
For the points O, A, and B being in one line we say that $\mathrm{lS}(ABC) = 0$.

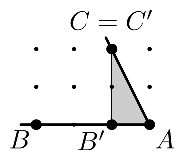
For instance the triangle on Figure 2 has the integer area equal to 6.

Like in Euclidean geometry integer area does not uniquely determine the congruence class of triangles. Nevertheless, all integer triangles of unit integer area are congruent since the vectors of the edges of such triangle generate the integer lattice.

2.1.7. *Integer area of rational angles.* An angle is called *rational* if its vertex is integer and both its edges contain integer points other that the vertex.

Definition 2.7. The *index* of a rational angle BAC is the index of the sublattice generated by all integers vectors of the lines AB and AC in the integer lattice. Denote $\mathrm{l}\alpha(AOB)$. In addition if the points A, O, and B are in a line we say that $\mathrm{l}\alpha(AOB)=0$.

Geometrically the index of an angle BAC is the integer area of the smallest triangle $AB'C'$ whose edges AB' and AC' generate the sublattices of lines containing the corresponding edges. For instance the angle BAC on Figure 2 has the index equal to 2:



Let us conclude this subsection with a general remark.

Remark 2.8. We write $\mathrm{l}\ell$, ld, lS and $\mathrm{l}\alpha$ (starting with letter "l") to indicate that all these notions are well-defined for any lattice, and not necessary for integer lattice.

2.2. **Empty triangles, their integer and Euclidean areas.** Let us completely study the case of the following "smallest" triangles.

Definition 2.9. An integer triangle is called *empty* if it does not contain integer points other that its vertices.

We show that empty triangles are exactly the triangles of integer area 1 and Euclidean area 1/2. In particular, this means that all empty triangles are integer congruent (as we show later, this is not true in multidimensional case for empty tetrahedra).

Denote by $S(ABC)$ the Euclidean area of the triangle ABC. Recall that

$$S(ABC)=\frac{1}{2}\Big|\det(AB,AC)\Big|.$$

Proposition 2.10. *Consider an integer triangle ABC. Then the following statements are equivalent:*
a) ABC is empty;
b) $\mathrm{lS}(ABC)=1$;
c) $S(ABC)=1/2$.

Proof. **a) ⇒ b).** Let an integer triangle ABC be empty. Then from symmetry reasons a parallelogram with edges AB and AC does not contain integer points except for its vertices as well. Therefore, by Proposition 2.1 there is only one coset for the subgroup generated by AB and AC. Hence, AB and AC generate the integer lattice, and $\mathrm{lS}(ABC)=1$.

b) ⇒ c). Let $\mathrm{lS}(ABC)=1$. Hence the vectors AB and AC generate the integer lattice. Thus any integer point is an integer combination of them, so

$$(1,0)=\lambda_1 AB+\lambda_2 AC \quad \text{and} \quad (0,1)=\mu_1 AB+\mu_2 AC.$$

with integers $\lambda_1, \lambda_2, \mu_1, \mu_2$. Let also

$$AB = b_1(1,0) + b_2(0,1) \quad \text{and} \quad AC = c_1(1,0) + c_2(0,1)$$

for some integers a_1, a_2, b_1, b_2. Therefore,

$$\begin{pmatrix} b_1 & c_1 \\ b_2 & c_2 \end{pmatrix} \begin{pmatrix} \lambda_1 & \mu_1 \\ \lambda_2 & \mu_2 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

Since these matrices are integer, both their determinants equal to either to 1 or to -1. Hence the area of the triangle ABC coincides with the area of the triangle with vertices $(0,0)$, $(1,0)$, and $(0,1)$ and equals $1/2$.

c) $\Rightarrow$ a). Consider an integer triangle of Euclidean area $1/2$. Suppose it has an integer point in the interior or on its sides. Then there exists an integer triangle with the Euclidean area smaller than 2, which is impossible (since the determinant of two integer vectors is integer). □

2.3. **Integer area of polygons.** In this Subsection we show that any integer polygon is decomposable into empty triangles. Then we define integer area of polygons as the amount of empty triangles in the decomposition. Additionally, we show that this definition of integer area is well-defined and coincides with the definition of the integer area of triangles.

Consider a closed broken line $A_0A_1\ldots A_{n-1}A_n$ ($A_n = A_0$) with finitely many vertices and without self-intersections in $\mathbb{R}^2$. By polygonal Jordan curve theorem (REF) this broken line separates the plane into two regions: one of them is homeomorphic to a disk and another to the annulus. An *n-gon* (or just a *polygon*) $A_1\ldots A_n$ is the closure of the region homeomorphic to a disk.

Proposition 2.11. (i) *Any integer polygon admits a decomposition into empty triangles.*

(ii) *If a polyhedron has several different decompositions into empty triangles, then the amounts of empty triangles in them are the same.*

(iii) *The number of empty triangles in a decomposition of a triangle ABC equals $lS(ABC)$.*

Remark 2.12. Proposition 2.11 introduces a natural extension of the integer area to the case of polygons. An integer area of the *polygon* is the number of empty triangles in its decomposition into empty triangles.

Proof. (i). We prove the statement for the integer n-gons by induction in n.

Base if induction. We start with triangles. To prove the statement for triangles we use another induction on the number of integer points in the closure triangle.

If there is only three integer points in the closure of a triangle then they should be its vertices (since the triangle is integer). Hence this triangle is empty, and the decomposition consists of one empty triangle.

Suppose any triangle with $k' < k$ integer points ($k \geq 3$) is decomposable into empty triangles. Consider an arbitrary integer triangle ABC with k integer point. Since $k \geq 3$ there exists an integer point P in the triangle distinct to vertices. Decompose the triangle into ABP, BCP, and CAP (excluding triangles of zero area). This decomposition consists of at least two triangles, and each of these triangles has at most $k-1$ integer points. By the induction assumption any of these triangles admits a decomposition into empty triangles. Hence ABC is decomposable as well.

Therefore, any integer triangle admits a decomposition into integer triangles.

Step of inductions. Suppose that any integer k'-gon for $k' < k$ is decomposable into empty triangles. Let us find a decomposition for an arbitrary integer k-gons. Let $A_1 \ldots A_k$ be an integer k-gon. If there exists s such that A_s, A_{s+1}, A_{s+2} are in a line then we just remove one of the points and reduce the number of vertices. Suppose that is not the case. Consider the ray with vertex at A_1 and containing A_1A_2 and start to turn it in the direction of the vertex A_3. We stop when we reach A_3 (then $A_s = A_3$) or at the moment when the ray contains some vertex A_s at the same halfplane as A_1 with respect to the line A_2A_3 for the first time. Now we decompose the polygon $A_1 \ldots A_k$ into polygons $A_1 \ldots A_s$ and $A_1A_s \ldots A_k$. Both of these polygons has less than k vertices. By the induction assumption any of these polygons admits a decomposition into empty triangles. Hence $A_1 \ldots A_k$ is decomposable as well.

This concludes the proof of (i).

(ii). Let a polygon P has two decompositions into n_1 and n_2 empty tetrahedra. Then by Proposition 2.10 we have

$$\frac{n_1}{2} = S(P) = \frac{n_2}{2},$$

since Euclidean area is additive. Hence, $n_1 = n_2$.

(iii). Consider an integer triangle ABC. Denote

$$\mathrm{Par}(AB, AC) = \{A + \alpha v + \beta w | 0 \leq \alpha, \beta < 1\}.$$

Denote by $\#(ABC)$ the number of empty triangles in the decomposition (by the items (i) and (ii) it is well defined). Let us prove that if $\mathrm{lS}(ABC) = n$ then $\#(ABC) = n$ by induction in n.

Base of induction. Suppose that $\mathrm{lS}(ABC) = 1$, then by Proposition 2.10 we have $\#(ABC) = 1$.

Step of induction. Suppose that the statement holds for any $k' < k$ for all $k > 1$. Let us prove it for k. Consider a triangle ABC such that $\mathrm{lS}(ABC) = k$. By Proposition 2.1 there exists an integer point P in the parallelogram $\mathrm{Par}(AB, AC)$ distinct from the vertices. Without loss of generality we suppose that P is not on the edge AB. Denote $Q = P + AB$ and $D = C + AB$.

The triangle BQD is obtained from the triangle APC by a shift on an integer vector AB (see on Figure 3). Hence the total number of orbits of integer points with respect to the shift on vectors multiple to AB in the parallelograms $\mathrm{Par}(AB, AP)$ and $\mathrm{Par}(PQ, PC)$ equals the number of such orbits in the parallelogram $\mathrm{Par}(AB, AC)$. Therefore, from Proposition 2.1, we get

$$\mathrm{lS}(ABC) = \mathrm{lS}(APB) + \mathrm{lS}(CPD).$$

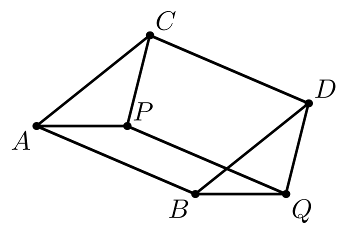


FIGURE 3. The integer area is additive.

By Proposition 2.10 we have

$$\#(ABC) = 2S(ABC) = S(ABCD) = S(ABQP) + S(CPQD)$$
$$= 2S(ABP) + 2S(CPD) = \#(APB) + \#(CPD).$$

Since P is not on AB, we have $\#(APB) < k$ and $\#(CPD) < k$. Therefore, by the assumption of induction we get

$$l\mathrm{S}(ABC) = l\mathrm{S}(APB) + l\mathrm{S}(CPD) = \#(APB) + \#(CPD) = \#(ABC).$$

The proof is completed by induction. □

Corollary 2.13. *The integer area of polygons in the plane is twice the Euclidean area.*

Proof. The statement holds for empty triangles by Proposition 2.10. Now the statement of the proposition follows directly from Proposition 2.11 and the definition of integer area. □

Remark 2.14. Corollary 2.13 implies that the index of an integer sublattice generated by vectors AB and AC in the integer lattice equals

$$|\det(AB, AC)|.$$

2.4. **Pick's formula.** We conclude this section with a nice formula that describes a relation between integer points in the polygon and its Euclidean area.

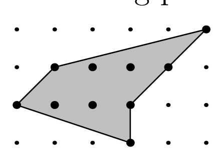
Theorem 2.15. (Pick's formula.) *A Euclidean area S of an integer polygon satisfies the following relation:*

$$S = I + E/2 - 1.$$

where I is the number of integer points in the interior of the polygon and E is the number of integer points in the edges. □

For the integer area one should multiply the right part of the formula by two.

Example 2.16. For instance, for the following pentagon



the number of inner vertices equals to 4 the number of vertices on the edges equals 6 and the area equals 6. So

$$6 = 4 + 6/2 - 1.$$

Proof. We prove Pick's formula by induction on area.

Base of induciont. If the Euclidean area of an integer polygon is 1/2, then the polygon is an empty triangle. For the empty triangle we have

$$S = 0 + 3/2 - 1 = 1/2.$$

Step of induction. Suppose the statement holds for any integer polygon of area $k'/2$ where $k < k'$. Let us prove the statement for polygons of area $k/2$. Let P be an integer polygon of area $k/2$. Then it can be decomposed into two integer polygons P_1 and P_2 (for instance as in the proof of Proposition 2.11 (i)) intersecting in a simple integer segment A_iA_j. Let P_i has I_i interior points and E_i boundary points. Let A_iA_j contains $\hat{E}$ integer points. Since the areas of P_1 and P_2 are less than $k/2$, by the assumption of induction we have

$$S(P) = S(P_1) + S(P_2) = I_1 + E_1/2 - 1 + I_2 + E_2/2 - 1 = (I_1 + I_2 + \hat{E}) + (E_1 + E_2 - 2\hat{E} - 2)/2 - 1.$$

Since the number of inner integer points of P is $I_1 + I_2 + \hat{E}$ and the number of boundary integer points is $E_1 + E_2 - 2\hat{E} - 2$, the Pick's formula holds for P. This concludes the step of induction. □

We have formulated Pick's theorem traditionally in Euclidean geometry. It is clear that this theorem is not true for all lattices. Still the lattice analog of the theorem holds for all lattices.

Exercises.

[1] Any subgroup of a free abelian group is a free abelian group.

[2] For any triangle ABC it holds

$$\mathrm{lS}(ABC) = \mathrm{l}\ell(AB)\,\mathrm{ld}(C, AB).$$

[3] Let ℓ_1, ℓ_2, and ℓ_3 be three parallel integer lines and $A_i \in l_i$ be integer points. Suppose also the lines ℓ_1 and ℓ_3 are in different halfplanes with respect to the line ℓ_2. Prove that

$$\mathrm{ld}(A_1, \ell_3) = \mathrm{ld}(A_1, \ell_2) + \mathrm{ld}(A_2, \ell_3).$$

Describe the edge-angle duality for the case $0 < \alpha < 1$.

[4] **Geometric interpretation of integer distance.** Let $\mathrm{ld}(O, AB) = k$. Prove that there are exactly $k - 1$ integer lines parallel to AB such that the point O and the segment AB are in different halfplanes with respect these lines.

[5] Find an examples of two integer noncongruent triangles that have the same integer area.

[6] Consider an empty tetrahedron in $\mathbb{R}^3$ with vertices in $\mathbb{Z}^3$. Suppose that it does not contain other points of lattice $\mathbb{Z}^3$ except the vertices. Is it true that the edges of the tetrahedron generate the whole lattice $\mathbb{Z}^3$?

[7] For any triangle ABC it holds

$$\mathrm{lS}(ABC) = \mathrm{l}\ell(AB)\,\mathrm{l}\ell(AC)\,\mathrm{l}\alpha(ABC).$$

[8] Prove that the index of a subgroup generated by v and w in integer lattice is the absolute value of $\det(v, w)$.

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