## 3. Geometry of ordinary continued fractions (29 March 2011)

In this lecture we study two geometrical constructions related to ordinary continued fractions.

3.1. **Sails of angles.** We start with a general definition of a sail for an arbitrary angle.

**Definition 3.1.** The boundary of the convex hull of all integer points except the vertex inside the angle is called the *sail* of this angle.

*Remark* 3.2. We mostly work with angles having an integer vertices (we call them *integer* angles). The sail of an integer angle is a broken line homeomorphic to $\mathbb{R}$. It could either contain one or two straight rays or do not contain them. Actually these rays appear in the case when edges of angles contain integer points other than the vertex of the corresponding angle. We say that the broken line of the sail excluding the open rays is the *principal part* of the sail.

**Example 3.3.** On Figure 1 we show an example of an angle $AOC$ with $A = (3, 2)$, $O = (0, 0)$, and $B = (3, -1)$. Since both rays contain integer points distinct to $O$, the principal part is a finite broken line. In this example it consists of two segments.

3.2. **On vertices of sails.** In this subsection we show a classical geometric description of continued fractions via integer invariants of sails. Consider an arbitrary number $\alpha \geq 1$. The line $y = \alpha x$ divides the first quadrant $\{(x, y)|x, y \geq 0\}$ into two angles. Denote

$$\omega_\alpha^- = A_0OC \qquad \text{and} \qquad \omega_\alpha^+ = B_0OC,$$

where $A_0 = (1, 0)$, $B_0 = (1, 0)$, and $C = (1, \alpha)$. Consider the sails for the angles $\omega_\alpha^-$ and $\omega_\alpha^+$. In case of rational $\alpha$ both sails consist of finitely many segments and two rays. Denote the vertices of the broken line in the sail of the angle containing $(1, 0)$ by $A_0, \ldots, A_n$ starting with $A_0 = (1, 0)$. In the same way we denote the vertices of the broken line in the sail of the angle containing $(0, 1)$ by $B_0, \ldots, B_m$ starting with $B_0 = (0, 1)$. As we show later either $m = n - 1$ or $m = n$. (See an example of $\alpha = 7/5$ on Figure 2.)

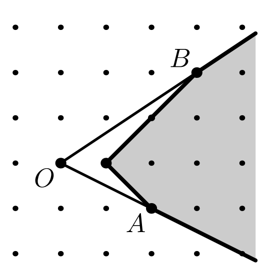
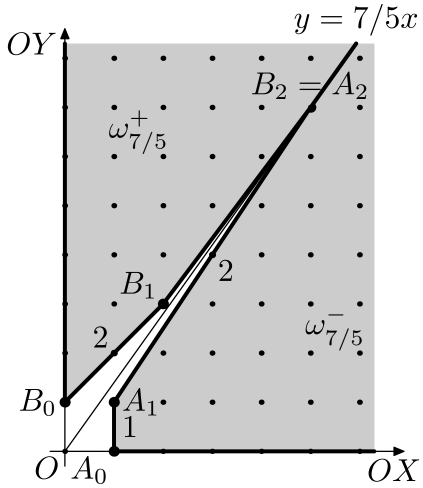



Figure 1. The sails for and angle $AOB$.




$a_0 = \mathrm{l}\ell(A_0A_1) = 1;$

$a_1 = \mathrm{l}\ell(B_0B_1) = 2;$

$a_2 = \mathrm{l}\ell(A_1A_2) = 2.$

$7/5 = [1; 2 : 2].$

FIGURE 2. The sails of the angles $\omega^-_\alpha$ and $\omega^+_\alpha$ for $\alpha = 7/5$.

In the case of irrational $\alpha$ each of the sails is a union of one ray and an infinite broken line. Denote the broken line starting from $(1, 0)$ by $A_0A_1\ldots$, and the broken line starting from $(0, 1)$ by $B_0B_1\ldots$ respectively.

In the next theorem we show the relation between $A_i$ and $B_i$ and the convergents.

**Theorem 3.4.** *Consider $\alpha \geq 1$. Let $A_0A_1A_2\ldots$ and $B_0B_1B_2\ldots$ be the principal parts of the sails (finite or infinite) for the angles $\omega^-_\alpha$ and $\omega^+_\alpha$. Then*

$$A_i = (p_{2i-2}, q_{2i-2}) \quad \text{and} \quad B_i = (p_{2i-1}, q_{2i-1}), \quad i = 1, 2, \ldots$$

*where $p_k/q_k$ are convergents. In addition for the rational case the last vertices of the principal parts for both sails coincides with $(p_n, q_n)$, where $p_n/q_n$ is the last convergent, i.e., $\alpha = p_n/q_n$.*

We start the proof with the following lemma.

**Lemma 3.5.** (*i*) *The segment with endpoints $(p_{2k-2}, q_{2k-2})$ and $(p_{2k}, q_{2k})$ is in the principal part of $\omega^-_\alpha$ (i.e., in $A_0A_1A_2\ldots$).*
(*ii*) *The segment with endpoints $(p_{2k-1}, q_{2k-1})$ and $(p_{2k+1}, q_{2k+1})$ is in the principal part of $\omega^+_\alpha$ (i.e., in $B_0B_1B_2\ldots$).*

*Proof.* (*i*). By Proposition **??** we have

$$\alpha > \frac{p_{2k-2}}{q_{2k-2}} \quad \text{and} \quad \alpha > \frac{p_{2k}}{q_{2k}},$$

hence both points $(p_{2k-2}, q_{2k-2})$ and $(p_{2k}, q_{2k})$ are in $\omega^-_\alpha$.

Consider the line $l$ passing through the points $(p_{2k-2}, q_{2k-2})$ and $(p_{2k}, q_{2k})$. From Proposition 1.11 the set of integer points in $l$ is as follows

$$\{(p_{2k-2}, q_{2k-2}) + \lambda(p_{2k-1}, q_{2k-1}) | \lambda \in \mathbb{Z}\}.$$

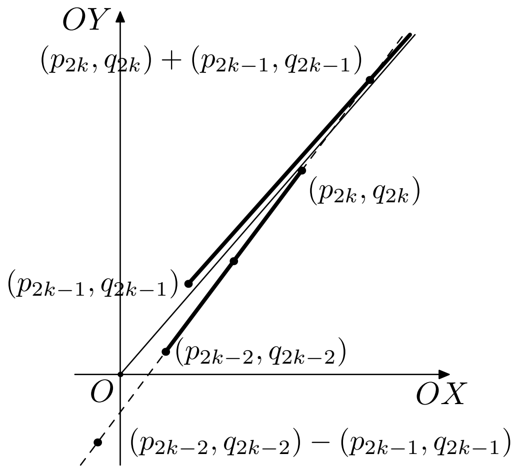



FIGURE 3. The segment with endpoints $(p_{2k-2}, q_{2k-2})$ and $(p_{2k}, q_{2k})$ is in the sail.

Let us prove that the line $l$ is on unit integer distance from the origin. The integer distance is equivalent to the index of integer sublattice generated by the vectors $(p_{2k-2}, q_{2k-2})$ and $(p_{2k-1}, q_{2k-1})$. From Proposition 1.13 it follows that

$$|p_{2k-2}q_{2k-1} - p_{2k-1}q_{2k-2}| = 1,$$

i.e. the Euclidean area of the corresponding triangle is 1/2. Hence by Proposition 2.9 the integer Area is 1 (i.e., the index of the corresponding sublattice), hence these vectors generate the integer lattice and $\mathrm{ld}\big((0,0), l\big) = 1$.

Therefore, there is no integer point in the interior of the region between $l$ and the parallel to $l$ line passing through the origin.

From Proposition **??** the point $(p_{2k-2}, q_{2k-2}) - (p_{2k-1}, q_{2k-1})$ has a non-positive second coordinate coordinates and, therefore, it is not in the angle $\omega_\alpha^-$ which is contained in the first quadrant.

Consider now the point

$$(p_{2k-2}, q_{2k-2}) + (a_{2k}+1)(p_{2k-1}, q_{2k-1}) = (p_{2k}, q_{2k}) + (p_{2k-1}, q_{2k-1}).$$

Notice that the point $(p_{2k}, q_{2k}) + (p_{2k-1}, q_{2k-1})$ belongs to the segment with endpoints

$$(p_{2k-1}, q_{2k-1}) \quad \text{and} \quad (p_{2k+1}, q_{2k+1}) = (p_{2k-1}, q_{2k-1}) + a_{2k+1}(p_{2k}, q_{2k})$$

which is contained in $\omega_\alpha^+$. Hence the point is not in $\omega_\alpha^-$.

Therefore, the segment with endpoints $(p_{2k-2}, q_{2k-2})$ and $(p_{2k}, q_{2k})$ is contained in the convex hull of all integer points in the angle $\omega_\alpha^-$ (see on Figure 3).

The proof of $(ii)$ repeats the proof for $(i)$, so we omit it. □

*Proof of Theorem 3.4.* Actually Lemma 3.5 almost proves the theorem. We should check only the endpoints of broken lines.

First, note that $A_1 = (\lfloor \alpha \rfloor, 1)$, i.e. $A_1 = (p_0, q_0)$.

Secondly, (if $\alpha$ is not an integer), we have

$$B_1 = (0,1) + a_2(\lfloor\alpha\rfloor, 1) = (p_1, q_1).$$

Finally, in the case of rational $\alpha$ we check if the segment with endpoints $(p_{n-1}, q_{n-1})$ and $(p_n, q_n)$ for the last two convergents is in one of two sails. The point $(p_{n-1}, q_{n-1})$ is in one of two sails by Lemma 3.5. The point $(p_n, q_n)$ is in the intersection of sails, i.e., in a line $y = \alpha x$. Let us show that the triangle with vertices $(0,0)$, $(p_{n-1}, q_{n-1})$, and $(p_n, q_n)$ is empty. From Proposition 1.13 it follows that an Euclidean area of the triangle is $1/2$, hence by Proposition 2.9 the triangle is empty. Therefore, the segment with endpoints $(p_{n-1}, q_{n-1})$ and $(p_n, q_n)$ is in the sail.

We have found all the segments of the principal parts of the sails for the angles $\omega_\alpha^-$ and $\omega_\alpha^+$, this concludes the proof. □

*Remark* 3.6. In the case of rational $\alpha$ with the principle parts $A_1 \dots A_k$ and $B_1 \dots B_m$ for $\omega_\alpha^-$ and $\omega_\alpha^+$ respectively we have the following. If the last element of the odd continued fraction is 1, then $k = m + 1$, otherwise $k = m$.

Let us formulate a similar theorem for the case $0 < \alpha < 1$.

**Theorem 3.7.** *Consider $0 < \alpha < 1$. Let $A_0A_1A_2\dots$ and $B_0B_1B_2\dots$ be the principal parts (finite or infinite) of the sails for the angles $\omega_\alpha^-$ and $\omega_\alpha^+$. Then*

$$A_i = (p_{2i}, q_{2i}) \quad \textit{and} \quad B_i = (p_{2i-1}, q_{2i-1}), \quad i = 1, 2, \dots$$

*where $p_k/q_k$ are convergents. The only exception for rational case is as follows: the last vertices of the principal parts for both sails coincide with $(p_n, q_n)$, where $p_n/q_n$ is the last convergent, i.e. $\alpha = p_n/q_n$.*

*Proof.* The proof repeats the proof of Theorem 3.4 except for the following difference. Since $a_0 = 0$, the points $A_0$ "should coincide" with $A_1$. This is the explanation for the shift in indices. □

3.3. **Geometric interpretation of the elements of continued fractions.** Let us first formulate a corollary of Theorem 3.4.

**Corollary 3.8.** *Consider $\alpha \geq 1$. Let $A_0A_1A_2\dots$ and $B_0B_1B_2\dots$ be the principal parts (finite or infinite) of the sails for $\omega_\alpha^-$ and $\omega_\alpha^+$ respectively. Then*

$$\mathrm{l}\ell(A_iA_{i+1}) = a_{2i} \quad \textit{and} \quad \mathrm{l}\ell(B_iB_{i+1}) = a_{2i+1}, \quad i = 0, 1, 2, \dots$$

*where $[a_0; a_1 : a_2 : \dots]$ is an ordinary continued fraction for $\alpha$ with the last element not equal to 1 (in rational case).*

*For the rational case we have one additional edge: let $(p_n, q_n)$ be the common point of two sails, i.e. $\alpha = p_n/q_n$, then it is a vertex of two segments of distinct sails. The integer length of one of them is $a_n$ and of the other is 1.*

In the case of $0 < \alpha < 1$ the corollary also holds, the only difference is that $\mathrm{l}\ell(A_iA_{i+1}) = a_{2i+2}$ instead $a_{2i}$.

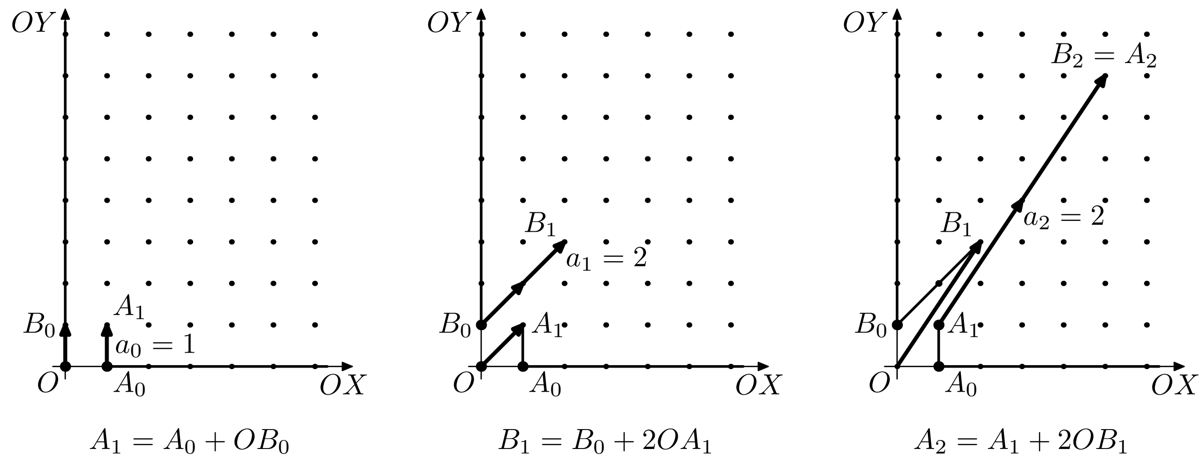



FIGURE 4. Construction of the sail for the continued fraction $[1; 2 : 2]$.

*Proof.* The corollary follows directly from explicit formulas for $A_k$ and $B_k$ of Theorem 3.4 after applying Proposition 1.13. □

Theorem 3.4 and Corollary 3.8 lead to an interesting algorithm to construct the sails for the angles $\omega_\alpha^-$ and $\omega_\alpha^+$.

**Algorithm to construct the sails by ordinary continued fractions.** Suppose we know an ordinary continued fraction $[a_0; a_1 : a_2 : \ldots]$ for a positive $\alpha$. Let us find geometrically the sails for the angles $\omega_\alpha^-$ and $\omega_\alpha^+$ without calculating the convex hulls.

Denote $O = (0,0)$. We start from $A_0 = (1,0)$ and $B_0 = (0,1)$ and construct

$$A_1 = A_0 + a_0 OB_0 \quad \text{and} \quad B_1 = B_0 + a_1 OA_1.$$

Suppose we have already constructed $A_0 \ldots A_k$ and $B_0 \ldots B_k$ then put

$$A_{k+1} = A_k + a_k OB_k \quad \text{and} \quad B_{k+1} = B_k + a_{k+1} OA_{k+1}$$

(see an illustration on Figure 4).

If $\alpha$ is rational, then the algorithm constructs both sails in a finite time. We leave the case $0 < \alpha < 1$ as an easy exercise for the reader.

3.4. **Duality of sails.** In this subsection we show that there is a certain duality between edges and angles. The important consequence of this duality is that all the elements of the ordinary continued fraction can be read from one of the sails. We restrict ourselves only to the case of $\alpha \geq 1$. Reduction of the case $0 < \alpha < 1$ is straightforward, so we omit it.

**Theorem 3.9.** *Consider $\alpha \geq 1$. Let $A_0 A_1 A_2 \ldots$ and $B_0 B_1 B_2 \ldots$ be the principal parts (finite or infinite) of the corresponding sails for $\omega_\alpha^-$ and $\omega_\alpha^+$. Then*

$$\mathrm{l}\alpha(A_i A_{i+1} A_{i+2}) = a_{2i+1} \quad \textit{and} \quad \mathrm{l}\alpha(B_i B_{i+1} B_{i+2}) = a_{2i+2}$$

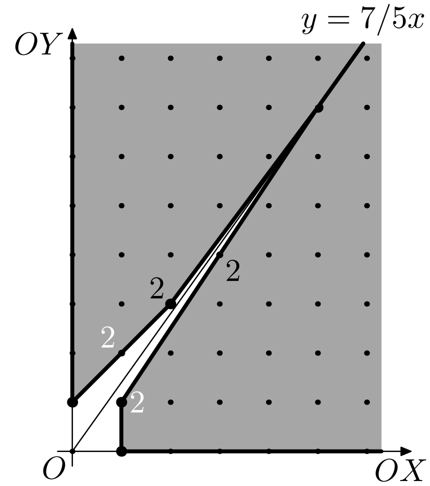



FIGURE 5. The edge-angle duality for $\alpha = 7/5$.

*for all admissible indices, where* $[a_0; a_1 : a_2 : \ldots]$ *is the ordinary continued fraction (whose last element does not equal to 1 in rational case).*

*Proof.* Let us calculate the index of a rational angle at some vertex of the principal part of one of the two sails. By Theorem 3.4 it is equals to the index of an angle between a pair of vectors $(p_{i-1}, q_{i-1})$ and $(p_{i+1}, q_{i+1})$.

$$\mathrm{l}\alpha\big((p_{i-1}, q_{i-1})(0,0)(p_{i+1}, q_{i+1})\big) = |p_{i-1}q_{i+1} - p_{i+1}q_{i-1}| =$$
$$|p_{i-1}(a_iq_i + q_{i-1}) - (a_ip_i + p_{i-1})q_{i-1}| = a_i|p_{i-1}q_i - p_iq_{i-1}| = a_i.$$

The second equality follows from Proposition 1.11, the last follows from Proposition 1.13. □

**Edge-angle duality.** So from Corollary 3.8 and Theorem 3.9 we get that

$$\mathrm{l}\alpha(A_iA_{i+1}A_{i+2}) = \mathrm{l}\ell(B_iB_{i+1}) \quad \text{and} \quad \mathrm{l}\alpha(B_iB_{i+1}B_{i+2}) = \mathrm{l}\ell(A_{i+1}A_{i+2}).$$

For $\alpha = 7/5$ (see on Figures 2 and 5) we get the following

$$\mathrm{l}\alpha(A_0A_1A_2) = \mathrm{l}\ell(B_0B_1) = a_1 = 2;$$
$$\mathrm{l}\alpha(B_0B_1B_2) = \mathrm{l}\ell(A_1A_2) = a_2 = 2.$$

EXERCISES.

**[1]** Describe the edge-angle duality for the case $0 < \alpha < 1$.

**[2]** Find relations between the index of angles, the integer lengths of edges and the integer area of an integer triangle.

**[3]** Construct the sails for angles $\omega_\alpha^\pm$ where an ordinary continued fraction for $\alpha$ is
a) $[1 : 2]$; b) $[2 : 2 : 3]$; c) $[1 : 1 : 1 : 2]$.

**[4]** Construct the first 4 vertices for both sails of the angles $\omega_\theta^\pm$, where $\theta=\frac{1+\sqrt{5}}{2}$.

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