4. Complete invariant of integer angles. (05 April 2011)

We study the following types of angles. If a lattice angle ABC has lattice points distinct to B on both edges AB and BC we call it *rational*. If the angle has a lattice point distinct to B in AB but not in BC (in BC but not in AB) we call the angle *ordinary R-irrational* (or respectively *ordinary L-irrational*) angle. In case if both edges of an angle do not contain lattice points other than B the angle is called *LR-irrational*. We will show later that ordinary R-irrational and L-irrational angles would have infinite LLS-sequences to the right and to the left respectively.

4.1. **Integer sines of rational angles.** Before to study geometric continued fraction we introduce the notation of the integer sine function for ordinary rational angles. We do this using an analog of sine-formula of Euclidean case.

Definition 4.1. Let ABC be an ordinary rational angle where A and C are lattice points distinct to B. The *integer sine* of the angle is the following number

$$\frac{\mathrm{lS}(ABC)}{\mathrm{l}\ell(AB)\,\mathrm{l}\ell(BC)}.$$

Denote it by $\mathrm{lsin}\,ABC$.

Remark 4.2. Notice that the integer sine is well-defined, it does not depend on the choice of points B and C. We leave to check this as an exercise for the reader.

There is a small difference to the Euclidean case where for the sin function we have

$$\sin ABC = \frac{2S(ABC)}{|AB||BC|}.$$

In Euclidean geometry we take twice the area since Euclidean area of the basis vector equals $1/2$, while its integer area is 1, see Proposition 2.6. Still the difference between the sin and lsin is much stronger, we illustrate this in the following proposition.

Proposition 4.3. *The integer sine of a rational angle coincides with the index of the angle.*

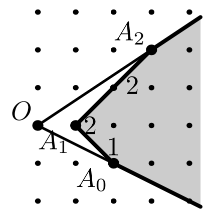
In particular this implies that the integer sine takes all nonnegative integer values.

Proof. Consider a rational ordinary angle ABC with A, B, C not in one line. Let A' and C' be the closest integer points to B in the open rays BA and BC respectively. Then from definition of integer length we get

$$\mathrm{l}\ell(BA') = \mathrm{l}\ell(BC') = 1.$$

Hence,

$$\mathrm{lsin}(AB'C') = \mathrm{lS}(AB'C').$$



$$a_2 = \mathrm{l}\ell A_1A_2 = 2$$
$$a_1 = \mathrm{lsin} A_0A_1A_2 = 2$$
$$a_0 = \mathrm{l}\ell A_0A_1 = 1$$

FIGURE 1. The LLS-sequence of the following angle is $(1, 2, 2)$.

The integer area of $AB'C'$ is the index of the sublattice generated by BA' and BC' in $\mathbb{Z}^2$. Since vectors BA' and BC' generate all integer points of the lines AB and AC, the integer area of $AB'C'$ is equivalent to the index of the angle. Therefore, we get

$$\mathrm{lsin}(ABC) = \mathrm{l}\alpha(ABC).$$

If A, B, and C are in one line then

$$\mathrm{lsin}\, ABC = \mathrm{l}\alpha(ABC) = 0.$$

This concludes the proof. □

4.2. **LLS-sequences of sails.** Let us now define a very important characteristic of a sail for an angle.

Definition 4.4. Consider an arbitrary angle with integer vertex. Let the sail for this angle be a broken line with the sequence of vertices (A_i). Denote:

$$a_{2k} = \mathrm{l}\ell\, A_kA_{k+1},$$
$$a_{2k-1} = \mathrm{lsin}\, A_{k-1}A_kA_{k+1}$$

for all admissible indices. The *lattice length-sine sequence* (*LLS-sequence*, for short) for the sail is the sequence (a_n) (See an example on Figure 1).

The LLS-sequence can be either finite or infinite in one or both sides.

4.3. **On complete invariant of angles with integer vertex.**

Proposition 4.5. *Integer length, integer area, index and integer sine of a rational angle are invariant under the action of the group of integer affine transformations.*

Proof. Any integer affine transformation preserves indexes of integer lattice subgroups in $\mathbb{Z}^2$. Therefore, the integer area and the index (and, therefore, the integer sine) of a rational angle are also preserved. The integer lengths is preserved since all the inner integer points map to inner integer points. □

Corollary 4.6. *The LLS-sequence is an invariant of lattice angles with respect to* $\mathit{Aff}(2, \mathbb{Z})$.

Proof. First note that convex hulls are preserved by the elements of $\mathit{Aff}(2, \mathbb{Z})$. Then the statement follows directly from the fact that the integer length and the index are invariants of $\mathit{Aff}(2, \mathbb{Z})$. □

Before to formulate the next theorem we make an important remark on integer congruence. It is not necessary that the angles ABC and CBA are integer congruent. For example the LLS-sequence of the angle A_0OA_2 on Figure 1 is $(1,2,2)$. While the LLS-sequence of A_2OA_0 is $(2,2,1)$. Hence A_0OA_2 and A_2OA_0 are not integer congruent.

Theorem 4.7. *Two angles with vertices at integer points are lattice congruent if and only if they have the same LLS-sequences.*

Proof. From Corollary 4.6 we know that the LLS-sequence is an integer invariant of angles. It remains to prove that if two LLS-sequences coincide then the corresponding angles are integer congruent.

Consider two angles α and β with sails (A_i) and (B_i). Let the corresponding LLS-sequences coincide and be equivalent to (a_i). Let the vertices of the angles α and β be O_α and O_β respectively. Consider an affine transformation taking O_β to O_α, B_0 to A_0 and B_1 to A_1 this affine transformation is integer since $\mathrm{l}\ell(A_0A_1) = \mathrm{l}\ell(B_0B_1) = a_0$ and $\mathrm{ld}(O_\alpha, A_0A_1) = \mathrm{ld}(O_\beta, B_0B_1)$. Suppose the angle β is taken to some angle γ with the sail (C_i) (we already know that the vertex of γ is O_α, $C_0 = A_0$, and $C_1 = A_1$).

Let us prove that the broken lines $A_0A_1A_2\ldots$ and $C_0C_1C_2\ldots$ coincide by induction. Suppose $A_0A_1\ldots A_{k-1}$ coincides with $C_0C_1\ldots C_{k-1}$. Let us prove that $A_k = C_k$.

First, we know that

$$\begin{aligned}&\mathrm{lsin}(A_{k-2}A_{k-1}A_k) = \mathrm{lsin}(C_{k-2}C_{k-1}C_k) = a_{2k-3} \quad \text{and}\\ &\mathrm{l}\ell(A_{k-1}A_k) = \mathrm{l}\ell(C_{k-1}C_k) = a_{2k-2}.\end{aligned}$$

Hence we have

$$\mathrm{ld}(A_k, A_{k-2}A_{k-1}) = \mathrm{ld}(C_k, C_{k-2}C_{k-1}) = a_{2k-2}a_{2k-3}.$$

This follows from the simple fact that

$$\mathrm{ld}(A, BC) = \frac{\mathrm{lS}(ABC)}{\mathrm{l}\ell(BC)}.$$

Notice that $A_{k-2}A_{k-1}A_k$ and $C_{k-2}C_{k-1}C_k$ are two parts of convex sails, and hence the points A_k and C_k are on the other side from the point O_α with respect the line $A_{k-2}A_{k-1} = C_{k-2}C_{k-1}$. Hence A_k and C_k are both on a line l_1 parallel to line $A_{k-2}A_{k-1}$ containing points on integer distance $a_{2k-2}a_{2k-3}$ from the line $A_{k-2}A_{k-1}$.

Second, we have

$$\mathrm{ld}(A_k, O_\alpha A_{k-1}) = \mathrm{lS}(O_\alpha A_{k-1}A_k) = a_{2k-1} = \mathrm{lS}(O_\alpha C_{k-1}C_k) = \mathrm{ld}(C_k, O_\alpha C_{k-1}),$$

where $O_\alpha A_{k-1} = O_\alpha C_{k-1}$. Again from convexity reasons we know that the points A_k and C_k are in a different halfspace to the point $A_{k-2} = C_{k-2}$ with respect to the line $O_\alpha A_{k-1}$. Therefore, C_k and A_k are in a line l_2 parallel to $O_\alpha A_{k-1}$.

Since $O_\alpha A_{k-1}$ and $A_{k-2}A_{k-1}$ are not parallel, the intersection of lines l_1 and l_2 is a point coinciding both with A_k and C_k.

Therefore, the broken lines $A_0A_1A_2\ldots$ and $C_0C_1C_2\ldots$ coincide.

The fact that $\ldots A_{-1}A_0A_1$ and $\ldots C_{-1}C_0C_1$ follows from the considered case after performing a $GL(2,\mathbb{Z})$-transformation taking A_0 to A_1 and A_1 to A_0, which is as follows

$$\begin{pmatrix} 1 & 0 \\ a_0 & -1 \end{pmatrix}.$$

We apply this transformation to both sails (A_i) and (C_i) and get that the images of C_k and A_k coincide for any negative k. Therefore, the whole sails (A_i) and (B_i) coincide.

Suppose that the sails BOA and COD coincide but the angles do not coincide. Then there is some nonzero angle POQ in one of the angles but not in the other. This angle contain integer points, which are not in the convex hulls of the sails. That is impossible. Hence the angles BOA and COD coincide. □

Proposition 4.8. *For any sequence of positive integers (odd finite or infinite in one or both sides) there exists an angle with vertex at the origin whose LLS-sequence is a given one.*

Proof. First we consider the case of a sequence $(a_0, a_1, a_2, \ldots)$ (odd finite or infinite to the right). Let $A = (1,0)$, $B = (0,0)$, and $C = (1,\alpha)$, where $\alpha = [a_0 : a_1; a_2; \ldots]$. The angle ABC has the desired LLS-sequence, this follows directly from Corollary 2.13 and Theorem 2.16 (the angle ω_α^-).

In the case when the sequence is infinite to the left, we construct an angle ABC for the inverse sequence. Then the angle CBA is the angle with the prescribe LLS-sequence.

The remaining case is the case of both-side infinite sequences. Here we construct the angle for the part with nonnegative indexes. Then apply to this angle the transformation we used in the proof of the previous theorem and construct the angle for the remaining part of the sail and get the angle with the prescribed LLS-sequence. □

So LLS-sequence distinguish two non-congruent angles, and any sequence (odd, or infinite) of positive integers is realizable as an LLS-sequence for some angle.

Theorem 4.9. (On a complete invariant of lattice angles.) *The LLS-sequence is a complete invariant of lattice angles under the group of lattice affine transformations Aff$(2,\mathbb{Z})$.* □

Exercises.

[1] The integer sine of an ordinary rational angle ABC does not depend on the choice of integer points B and C on the edges.

[2] Prove that convex sets are taken to convex sets and their boundaries to boundaries under the affine transformations.

[3] Let $A = (3,-2)$, $B = (0,0)$, and $C = (2,1)$. Prove than ABC and CBA are not integer congruent.

[4] Prove that if the sails of two angles coincide then the angles itself coincide.

[5] Let d be a positive integer and l be an integer line. Prove that all lattice points lying on the integer distance are contained in two lines parallel to l on the same Euclidean distance from l.

[6] Draw a geometric continued fraction defined by the lines $x - 2y = 1$ and $3x + 4y = 3$ and calculate the LLS-sequences for all the sails.

[7] Prove that

$$\mathrm{ld}(A, BC) = \frac{\mathrm{lS}(ABC)}{\mathrm{l}\ell(BC)}.$$

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