5. Integer trigonometry for integer angles (05 April 2011, part 2)

In this section we mostly work with rational angles and R-irrational angles, i.e. angles with integer vertices whose first edges have integer points distinct to the vertices. For rational angles we define integer sines, cosines, and tangents. For R-irrational angles we have only a definition of integer tangents.

5.1. Definition of trigonometric functions.

We start with definition of integer tangent for rational and R-irrational ordinary angles.

Definition 5.1. Consider a rational or R-irrational angle ABC. Let A, B, and C be not in a line. Suppose the LLS-sequence of ABC is $(a_0, a_1, a_2, \ldots)$ (finite or infinite). Then the *integer tangent* of the angle ABC equals

$$\operatorname{ltan}(ABC) = [a_0 : a_1; a_2; \ldots].$$

In case when the points A, B, and C are in a line (but the points A and C are distinct to B) we say that $\operatorname{ltan}(ABC) = 0$.

Consider a rational angle ABC and remind that

$$\operatorname{lsin}(ABC) = \frac{\operatorname{lS}(ABC)}{\operatorname{l\ell}(AB)\operatorname{l\ell}(BC)}.$$

Definition 5.2. For a rational angle α we define

$$\operatorname{lcos}\alpha = \frac{\operatorname{lsin}\alpha}{\operatorname{ltan}\alpha}.$$

It is clear that an integer sine, an integer tangent, and, therefore, an integer cosine are invariants of $\mathit{Aff}(2, \mathbb{Z})$.

Now we give an inverse function to integer tangent.

Definition 5.3. Consider a real $s \geq 1$ or $s = 0$. The *integer arctangent* of s is the angle with vertex at the origin and edges

$$\{y = 0 | x \geq 0\} \qquad \text{and} \qquad \{y = sx | x \geq 0\}.$$

We define zero angle as $\operatorname{larctan} 0$. The angle π is the angle ABC where $A = (1, 0)$, $B = (0, 0)$, and $C = (-1, 0)$.

5.2. Basic properties of integer trigonometry.

First we show that integer tangent and arctangent are really inverse to eachother.

Proposition 5.4. a). *For any real $s \geq 1$, we have:* $\operatorname{ltan}(\operatorname{larctan} s) = s$.
b). *For any ordinary rational or R-irrational angle α with $\sin\alpha \neq 0$ the following holds:*

$$\operatorname{larctan}(\operatorname{ltan}\alpha) \cong \alpha.$$

Proof. Item **a.** From Corollary 2.13 and Theorem 2.16 we have that the elements of the LLS-sequence for $\operatorname{larctan} s$ coincides with the elements of ordinary continued fraction for s. Hence the statement holds by definition of the integer tangent.

Item **b.** Both angles $\operatorname{larctan}(\operatorname{ltan}\alpha)$ and α have the same LLS-sequences. Therefore, they are congruent by Theorem 3.9. □

In the following proposition we collect several trigonometric properties.

Proposition 5.5. a). *Integer trigonometric functions are invariants of angles under the action of Aff*$(2,\mathbb{Z})$.
b). *For any rational angle α not contained in a line the values* $\operatorname{lsin}\alpha$ *and* $\operatorname{lcos}\alpha$ *are positive relatively prime integers.*
c). *For any rational angle α not contained in a line the following inequalities hold:*

$$\operatorname{lsin}\alpha \geq \operatorname{lcos}\alpha, \quad \textit{and} \quad \operatorname{ltan}\alpha \geq 1.$$

The equalities hold if and only if the lattice vectors of the angle rays generate the whole lattice.
d). *Two ordinary angles α and β are integer congruent if and only if* $\operatorname{ltan}\alpha = \operatorname{ltan}\beta$.

Proof. Item **a.** It is a direct corollary of the fact that integer sine and tangent are defined only by values of certain indexes.

Item **b.** By Proposition 5.4 it is enough to prove for angles $\operatorname{larctan}\alpha$ for rational $\alpha \geq 1$.

Consider $\frac{m}{n} \geq 1$, where m and n are relatively prime integers. Then the sail of the angle would contain the point (n,m). It is also clear that the lattice distance between the point (n,m) and the line $y=0$ is m, and hence $\operatorname{lsin}\left(\operatorname{larctan}\frac{m}{n}\right) = m$.

Since $\operatorname{ltan}\left(\operatorname{larctan}\frac{m}{n}\right) = \frac{m}{n}$ and $\operatorname{lsin}\left(\operatorname{larctan}\frac{m}{n}\right) = m$, we have $\operatorname{lcos}\left(\operatorname{larctan}\frac{m}{n}\right) = n$.

Item **c.** This is true for all integer arctangent angles. Therefore, it is true for all angles.

Item **d.** The LLS-sequence is uniquely defined by the integer tangent. Therefore, the statement follows from Theorem 3.9. □

5.3. **Transpose ordinary integer angles.** Let us give a definition of the ordinary integer angle transpose to a given one.

Definition 5.6. An ordinary integer angle $\angle BOA$ is said to be *transpose* to the ordinary integer angle $\angle AOB$. We denote it by $(\angle AOB)^t$.

It immediately follows from the definition, that for any ordinary integer angle α we have

$$(\alpha^t)^t \cong \alpha.$$

Further we will use the following notion. Suppose that some arbitrary integers a, b and c, where $c \geq 1$, satisfy the following: $ab \equiv 1 (\operatorname{mod} c)$. Then we denote

$$a \equiv b^{-1} (\operatorname{mod} c).$$

For the trigonometric functions of transpose ordinary integer angles the following relations hold.

Theorem 5.7. Trigonometric relations for transpose angles. *Let an ordinary integer angle α be neither straight, nor zero. Then*
1) If $\alpha \cong \operatorname{larctan}(1)$ *then* $\alpha^t \cong \operatorname{larctan}(1)$.
2) If $\alpha \ncong \operatorname{larctan}(1)$ *then*

$$\operatorname{lsin}(\alpha^t) = \operatorname{lsin}\alpha, \quad \operatorname{lcos}(\alpha^t) \equiv (\operatorname{lcos}\alpha)^{-1} (\operatorname{mod} \operatorname{lsin}\alpha).$$

The statement of Theorem 5.7 was first introduced in the terms of ordinary continued fractions and, so-called, *zig-zags* by P. Popescu-Pampu. Here we give an alternative proof using integer trigonometry.

Proof. Consider an ordinary integer angle α. Let $\operatorname{ltan}\alpha = p/q$ where $\gcd(p,q) = 1$. By Proposition 5.4.b we have $\alpha \cong \operatorname{larctan}(p/q)$. The case $p/q = 1$ is trivial. Consider the case $p/q > 1$.

Let $A = (1,0)$, $B = (p,q)$, and $O = (0,0)$. Suppose that an integer point $C = (q', p')$ of the sail for the ordinary angle $\operatorname{larctan}(p/q)$ is the closest integer point to the endpoint B. Both coordinates of C are positive integers as far as $p/q > 1$. Since the triangle $\triangle BOC$ is empty, and the orientation of the couple of vectors (OB, OC) does not coincide with the orientation of the couple of vectors (OA, OB), we have

$$\det \begin{pmatrix} p & p' \\ q & q' \end{pmatrix} = -1.$$

Consider a linear transformation ξ of the two-dimensional plane

$$\xi = \begin{pmatrix} p - p' & q' - q \\ p & -q \end{pmatrix}.$$

Since $\det(\xi) = -1$, the transformation ξ is integer-linear and changes the orientation. Direct calculations show that the transformation ξ takes the ordinary angle $\operatorname{larctan}^t(p/q)$ to the ordinary angle $\operatorname{larctan}(p/(p-p'))$. (See example on Figure 1.)

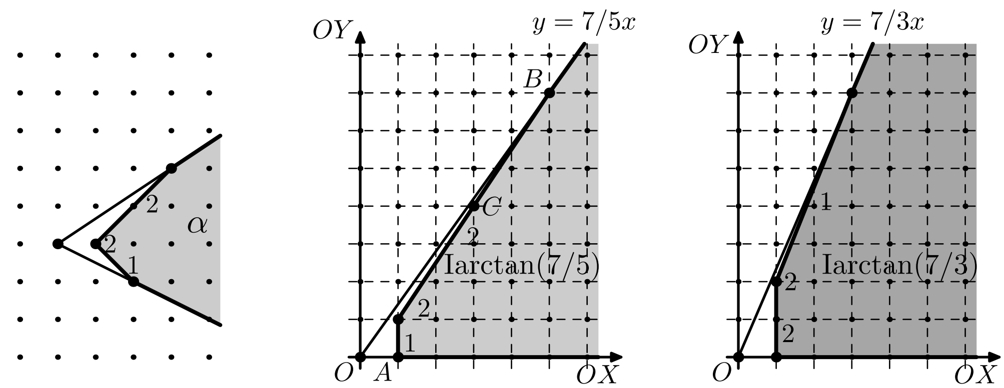

FIGURE 1. If α is integer-congruent to $\operatorname{larctan}(7/5)$ then α^t is integer-congruent to $\operatorname{larctan}(7/3)$.

Since $\gcd(p, p') = 1$ and $p > p - p'$, the following holds

$$\begin{cases} \operatorname{lsin}(\alpha^t) &= p \\ \operatorname{lcos}(\alpha^t) &= p - p' \end{cases}.$$

Since $pq' - qp' = -1$, we have $qp' \equiv -1 (\operatorname{mod} p)$. Therefore,

$$\operatorname{lcos}\alpha \operatorname{lcos}(\alpha^t) = q(p - p') \equiv 1 (\operatorname{mod} p).$$

From that we have

$$\begin{cases} \operatorname{lsin}(\alpha^t) &= \operatorname{lsin}\alpha \\ \operatorname{lcos}(\alpha^t) &\equiv (\operatorname{lcos}\alpha)^{-1} (\operatorname{mod} \operatorname{lsin}\alpha) \end{cases}.$$

This concludes the proof of Theorem 5.7. □

5.4. **Adjacent ordinary integer angles.** Let us define an ordinary integer angle adjacent to the given one.

Definition 5.8. An ordinary integer angle $\angle BOA'$ is said to be *adjacent* to an ordinary integer angle $\angle AOB$ if the points A, O, and A' are contained in the same straight line. We denote the ordinary angle $\angle BOA'$ by $\pi - \angle AOB$.

For the trigonometric functions of adjacent ordinary integer angles the following relations hold.

Theorem 5.9. Trigonometric relations for adjacent ordinary angles. *Let α be some ordinary integer angle. Then one of the following conditions holds.*
1) If α is zero ordinary angle then $\pi - \alpha \cong \pi$.
2) If α is straight ordinary angle then $\pi - \alpha \cong 0$.
3) If $\alpha \cong \operatorname{larctan}(1)$ then $\pi - \alpha \cong \operatorname{larctan}(1)$.
4) If α is neither zero, nor straight, nor integer-congruent to $\operatorname{larctan}(1)$ then

$$\pi - \alpha \cong \operatorname{larctan}^t \left(\frac{\operatorname{ltan}\alpha}{\operatorname{ltan}(\alpha) - 1} \right),$$
$$\operatorname{lsin}(\pi - \alpha) = \operatorname{lsin}\alpha, \quad \operatorname{lcos}(\pi - \alpha) \equiv (-\operatorname{lcos}\alpha)^{-1} (\operatorname{mod} \operatorname{lsin}\alpha).$$

The statement of Theorem 5.9 was first introduced in the terms of ordinary continued fractions and, so-called, *zig-zags* by P. Popescu-Pampu. Here we give an alternative proof using integer trigonometry.

Remark 5.10. Suppose that an ordinary integer angle α is neither zero, nor straight. Then the conditions

$$\begin{cases} \operatorname{lcos}(\pi - \alpha) \equiv (-\operatorname{lcos}\alpha)^{-1} (\operatorname{mod} \operatorname{lsin}\alpha) \\ 0 < \operatorname{lcos}(\pi - \alpha) \leq \operatorname{lsin}\alpha \end{cases}$$

uniquely determine the value $\operatorname{lcos}(\pi - \alpha)$.

Proof. Consider an ordinary integer angle α. Directly from definitions it follows, that if $\alpha \cong 0$ then $\pi - \alpha \cong \pi$, and if $\alpha \cong \pi$ then $\pi - \alpha \cong 0$.

Suppose that $\operatorname{ltan}\alpha = p/q > 0$, where $\gcd(p, q) = 1$. Then by Proposition 5.4.b we have $\alpha \cong \operatorname{larctan}(p/q)$. Therefore,

$$\pi - \alpha \cong \pi - \operatorname{larctan}(p/q).$$

It follows immediately, that if $p/q = 1$ then $\pi{-}\alpha \cong \operatorname{larctan}(1)$.

Let now $\alpha \ncong \operatorname{larctan}(1)$, and hence $p/q > 1$. Consider a linear transformation ξ_1 of the two-dimensional plane

$$\xi_1 = \begin{pmatrix} -1 & 1 \\ 0 & 1 \end{pmatrix}.$$

Since $\det(\xi_1) = -1$, the transformation ξ_1 is integer-linear and changes the orientation. Direct calculations show that the transformation ξ_1 takes the cone corresponding to the ordinary angle $\pi{-}\operatorname{larctan}(p/q)$ to the cone corresponding to the ordinary angle $\operatorname{larctan}^t(p/(p{-}q))$. Since ξ_1 changes the orientation we have to transpose $\operatorname{larctan}^t(p/(p{-}q))$. (See an example on Figure 2). Therefore,

$$\pi{-}\alpha \cong \operatorname{larctan}^t\left(\frac{\operatorname{ltan}\alpha}{\operatorname{ltan}(\alpha)-1}\right).$$

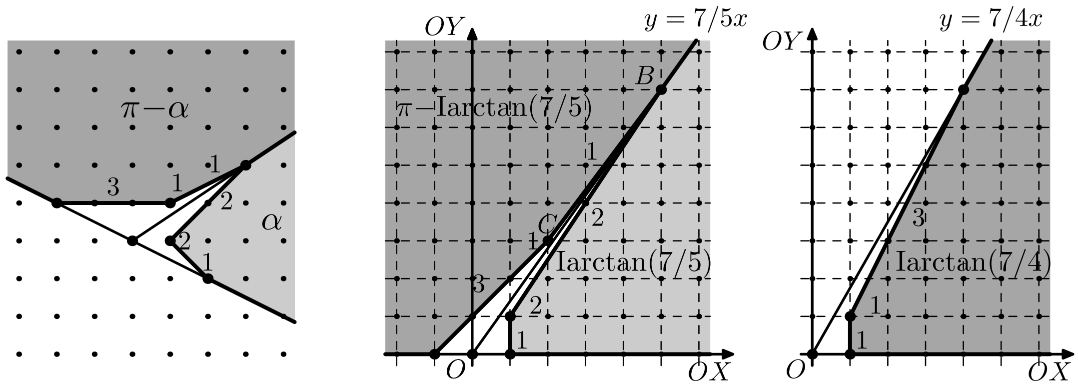

Figure 2. If α is integer-congruent to larctan(7/5) then $\pi{-}\alpha$ is integer-congruent to larctan(7/4).

Now we show that

$$\begin{cases} \operatorname{lsin}(\pi{-}\alpha) & = & \operatorname{lsin}\alpha \\ \operatorname{lcos}(\pi{-}\alpha) & \equiv & (-\operatorname{lcos}\alpha)^{-1}(\operatorname{mod}\operatorname{lsin}\alpha) \end{cases}.$$

Let $A = (1,0)$, $B = (q,p)$, and $O = (0,0)$. Consider the sail for the for the ordinary angle $\pi{-}\operatorname{larctan}(p/q)$ that is integer-congruent to the $\pi{-}\alpha$. Suppose that an integer point $C = (q',p')$ of the sail for $\pi{-}\operatorname{larctan}(p/q)$ is the closest integer point to the endpoint of the sail B. (Or, equivalently, the segment BC is in the sail and is of the unit integer length). The coordinate p' is positive as far as $p/q > 1$. Since the triangle $\triangle BOC$ is empty and the vectors OB and OC defines the same orientation as OA and OB,

$$\det\begin{pmatrix} p & p' \\ q & q' \end{pmatrix} = 1.$$

Consider a linear transformation ξ_2 of the two-dimensional plane

$$\xi_2 = \begin{pmatrix} p'-p & q-q' \\ -p & q \end{pmatrix}.$$

Since $\det(\xi_2) = 1$, the transformation ξ_2 is integer-linear and orientation-preserving. Direct calculations show that the transformation ξ_2 takes the ordinary integer angle $\pi -\operatorname{larctan}(p/q)$ to the ordinary integer angle $\operatorname{larctan}(p/(p-p'))$. Since $\gcd(p,p') = 1$, we have $\operatorname{lsin}(\pi-\alpha) = p$. Since $\gcd(p,p') = 1$ and $p > p-p'$, the following holds:

$$\operatorname{lcos}(\pi-\alpha) = \operatorname{lcos}\left(\operatorname{larctan}\Big(\frac{p}{p-p'}\Big)\right) = p-p'.$$

Since $pq'-qp' = 1$, we have $qp' \equiv 1 (\operatorname{mod} p)$. Therefore,

$$\operatorname{lcos}\alpha\operatorname{lcos}(\pi-\alpha) = q(p-p') \equiv -1 (\operatorname{mod} p).$$

From that we have

$$\left\{\begin{array}{lcl} \operatorname{lsin}(\pi-\alpha) & = & \operatorname{lsin}\alpha \\ \operatorname{lcos}(\pi-\alpha) & \equiv & (-\operatorname{lcos}\alpha)^{-1}(\operatorname{mod}\operatorname{lsin}\alpha) \end{array}\right. .$$

This conclude the proof of Theorem 5.9. □

The following statement is an easy corollary of Theorem 5.9.

Corollary 5.11. *For any ordinary integer angle α the following holds:*

$$\pi - (\pi - \alpha) \cong \alpha.$$

□

5.5. **Right ordinary integer angles.** We define right ordinary integer angles by analogy with Euclidean angles using their symmetric properties.

Definition 5.12. The ordinary integer angle is said to be *right* if it is self-dual and integer-congruent to the adjacent ordinary angle.

It turns out that in integer geometry there exist exactly two integer non-equivalent right ordinary integer angles.

Proposition 5.13. *Any ordinary integer right angle is integer-congruent to exactly one of the following two angles:* $\operatorname{larctan}(1)$*, or* $\operatorname{larctan}(2)$*.*

Proof. Let α be an ordinary integer right angle.

Since $(\pi-0) \ncong 0$, and $(\pi-\pi) \ncong \pi$, we have $\operatorname{ltan}\alpha > 0$.

By the definition of ordinary integer right angles and Theorem 5.7, we obtain

$$\operatorname{lcos}(\alpha) \equiv \operatorname{lcos}(\alpha^t) \equiv (\operatorname{lcos}\alpha)^{-1}(\operatorname{mod}\operatorname{lsin}\alpha).$$

By the definition of ordinary integer right angles and Theorem 5.9, we obtain

$$\operatorname{lcos}(\alpha) \equiv \operatorname{lcos}(\pi-\alpha) \equiv (-\operatorname{lcos}\alpha)^{-1}(\operatorname{mod}\operatorname{lsin}\alpha).$$

Hence,

$$(\operatorname{lcos}\alpha)^{-1}(\operatorname{mod}\operatorname{lsin}\alpha) \equiv (-\operatorname{lcos}\alpha)^{-1}(\operatorname{mod}\operatorname{lsin}\alpha).$$

Therefore, $\operatorname{lsin}\alpha = 1$, or $\operatorname{lsin}\alpha = 2$.

The ordinary integer angles with $\operatorname{lsin}\alpha = 1$ are integer-congruent to $\operatorname{larctan}(1)$. The ordinary integer angles with $\operatorname{lsin}\alpha = 2$ are integer-congruent to $\operatorname{larctan}(2)$. The proof is completed. □

5.6. **Opposite interior ordinary angles.** First we give the definition of parallel lines. Two integer lines are said to be *parallel* iff there exist an integer shift of the plane by the integer vector taking the first line to the second.

Definition 5.14. Consider two integer parallel distinct lines AB and CD, where A, B, C, and D are integer points. Let the points A and D be in different open half-planes with respect to the line BC. Then the ordinary integer angle $\angle ABC$ is called *opposite interior* to the ordinary integer angle $\angle DCB$.

Further we use the following proposition on opposite interior ordinary integer angles.

Proposition 5.15. *Two opposite interior to each other ordinary integer angles are integer-congruent.*

Proof. Consider two integer parallel distinct lines AB and CD. Let the points A and D be in distinct open half-planes with respect to the line BC. Let us prove that $\angle ABC \cong \angle DCB$.

Consider the central symmetry S of the two-dimensional plane at the midpoint of the segment BC. Let P be an arbitrary integer point. Note that $S(P) = C + PB$. Since the point C and the vector PB are both integer, $S(P)$ is also integer. Since the central symmetry is self-inverse, the inverse map S^{-1} also takes the integer lattice to itself.

Therefore, the central symmetry S is integer-affine transformation. Since S takes the ordinary angle $\angle ABC$ to the ordinary angle $\angle DCB$, we obtain $\angle ABC \cong \angle DCB$. □

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