6. Integer angles of integer triangles
(12 April 2011)

In this lecture we study integer triangles. First we show trigonometric relations for integer triangles. Then we study criterions of equivalence of integer triangles and show several examples. Further we verify which triples of angles can be taken as angles of integer triangle. This section is concluded with examples of integer triangles of small area.

6.1. **Integer sine formula.** In this subsection we show the integer sine formula for angles and edges of integer triangles. Further we learn how to find the integer tangents of all angles and the integer lengths of all edges of any integer triangle knowing the integer lengths of two edges and the integer tangent of the angle between them.

Let A, B, C be three distinct and not collinear integer points. We denote the integer triangle with the vertices A, B, and C by $\triangle ABC$.

Proposition 6.1. (The sine formula for integer triangles.) *For any integer triangle $\triangle ABC$ the following holds.*

$$\frac{\mathrm{l}\ell(AB)}{\mathrm{lsin}\, BCA} = \frac{\mathrm{l}\ell(BC)}{\mathrm{lsin}\, CAB} = \frac{\mathrm{l}\ell(CA)}{\mathrm{lsin}\, ABC} = \frac{\mathrm{l}\ell(AB)\,\mathrm{l}\ell(BC)\,\mathrm{l}\ell(CA)}{\mathrm{lS}(\triangle ABC)}.$$

Proof. We have

$$\mathrm{lS}(ABC) = \mathrm{l}\ell(AB)\,\mathrm{l}\ell(AC)\,\mathrm{lsin}\, CAB = \mathrm{l}\ell(BA)\,\mathrm{l}\ell(BC)\,\mathrm{lsin}\, BCA = \mathrm{l}\ell(CB)\,\mathrm{l}\ell(CA)\,\mathrm{lsin}\, ABC,$$

After inversion of the expressions and multiplication by all three integer lengths we get the statement of the proposition. □

The following open problem is actual here.

Problem 1. Find an integer analog of the cosine formula for triangles.

Recall that the cosine formula for Euclidean triangle $|ABC|$ with $a = |BC|$, $b = |AC|$, $c = |AB|$ and $\alpha = BAC$ is as follows

$$a^2 = b^2 + c^2 = 2bc\cos\alpha.$$

It seems that there is no an easy opportunity to find integer cosine formula, since currently there is no theorems related to addition in integer trigonometry.

6.2. **Angles and segments of integer triangles.** Suppose that we know the integer lengths of the edges AB, AC and the integer tangent of the angle BAC in the triangle $\triangle ABC$. Let us show how to restore the integer length and the integer tangents for the remaining edge and the rational angles of the triangle.

For simplicity we fix some integer basis and use the system of coordinates OXY corresponding to this basis (denoted $(*, *)$).

Theorem 6.2. *Consider some triangle $\triangle ABC$. Let*

$$\mathrm{l}\ell(AB) = c, \qquad \mathrm{l}\ell(AC) = b, \quad \textit{and} \quad CAB \cong \alpha.$$

Then the angles BCA and ABC are defined in the following way.

$$BCA \cong \begin{cases} \pi - \mathrm{larctan}\left(\frac{c\,\mathrm{lsin}\,\alpha}{c\,\mathrm{lcos}\,\alpha - b}\right) & \textit{if} \quad c\,\mathrm{lcos}\,\alpha > b \\ \mathrm{larctan}(1) & \textit{if} \quad c\,\mathrm{lcos}\,\alpha = b \\ \mathrm{larctan}^t\left(\frac{c\,\mathrm{lsin}\,\alpha}{b - c\,\mathrm{lcos}\,\alpha}\right) & \textit{if} \quad c\,\mathrm{lcos}\,\alpha < b \end{cases},$$

$$ABC \cong \begin{cases} \pi - \mathrm{larctan}^t\left(\frac{b\,\mathrm{lsin}(\alpha^t)}{b\,\mathrm{lcos}(\alpha^t) - c}\right) & \textit{if} \quad b\,\mathrm{lcos}(\alpha^t) > c \\ \mathrm{larctan}(1) & \textit{if} \quad b\,\mathrm{lcos}(\alpha^t) = c \\ \mathrm{larctan}\left(\frac{b\,\mathrm{lsin}(\alpha^t)}{c - b\,\mathrm{lcos}(\alpha^t)}\right) & \textit{if} \quad b\,\mathrm{lcos}(\alpha^t) < c \end{cases}.$$

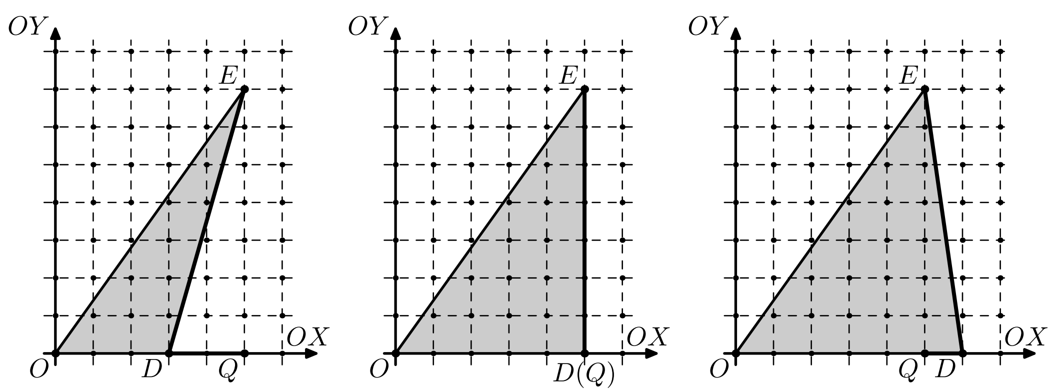
For the integer length of the edge CB we have

$$\mathrm{l}\ell(CB) = \frac{\mathrm{lsin}\,\alpha}{\mathrm{lsin}\,ABC} b = \frac{\mathrm{lsin}\,\alpha}{\mathrm{lsin}\,BCA} c.$$

Proof. We start with proving formula for the angle BCA. Let $\alpha \cong \mathrm{larctan}(p/q)$, where $\gcd(p, q) = 1$. Then $\triangle CAB \cong \triangle DOE$ where $D = (b, 0)$, $O = (0, 0)$, and $E = (qc, pc)$. Let us express the angle DEO. Denote by Q the point $(qc, 0)$. If $qc - b = 0$, then $BCA = \angle DEO \cong \mathrm{larctan}\,1$. If $qc - b \neq 0$, then we have

$$QDE \cong \mathrm{larctan}\left(\frac{cp}{cq - b}\right) \cong \mathrm{larctan}\left(\frac{c\,\mathrm{lsin}\,\alpha}{c\,\mathrm{lcos}\,\alpha - b}\right).$$

The expression for the angle BCA follows directly from the above expression for QDE, since $BCA \cong QDE$. (See Figure 1: here $\mathrm{l}\ell(OD) = b$, $\mathrm{l}\ell(OQ) = c\,\mathrm{lcos}\,\alpha$, and therefore $\mathrm{l}\ell(DQ) = |c\,\mathrm{lcos}\,\alpha - b|$.)



$c \cdot \mathrm{lcos}\alpha > b$ $\qquad$ $c \cdot \mathrm{lcos}\alpha = b$ $\qquad$ $c \cdot \mathrm{lcos}\alpha < b$

FIGURE 1. Three possible configuration of points O, D, and Q.

To obtain the expression for the angle ABC we consider the triangle $\triangle BAC$. Calculate the angle CBA and then transpose all angles in the expression. Finally, the integer length of CB is defined from integer sine formula. □

6.3. **On integer congruence criterions for triangles.** We start with the study the situation with integer analogs for the first, the second, and the third Euclidean criterions of triangle congruence.

Proposition 6.3. (The first criterion of integer triangle integer congruence.) *Consider integer triangles $\triangle ABC$ and $\triangle A'B'C'$. Suppose that*

$$AB \cong A'B', \quad AC \cong A'C', \quad \textit{and} \quad CAB \cong C'A'B',$$

then $\triangle A'B'C' \cong \triangle ABC$. □

Proof. Since $CAB \cong C'A'B'$ there exist an integer affine transformation taking the angle CAB to the angle $C'A'B'$. Since the integer lengths of the corresponding segments are the same, the transformation takes B and C to B' and C' respectively. □

It turns out that the second and the third criterions taken from Euclidean geometry do not hold. The following two examples illustrate these phenomena.

Example 6.4. The second criterion of triangle integer congruence does not hold in integer geometry. On Figure 2 we show two integer triangles $\triangle ABC$ and $\triangle A'B'C'$. We have

$$AB \cong A'B', \quad ABC \cong A'B'C' \cong \mathrm{l}\arctan(1), \quad \text{and} \quad CAB \cong C'A'B' \cong \mathrm{l}\arctan(1).$$

The triangle $\triangle A'B'C'$ is not integer congruent to the triangle $\triangle ABC$, since $\mathrm{lS}(\triangle ABC) = 4$ and $\mathrm{lS}(\triangle A'B'C') = 8$.

FIGURE 2. A counterexample to the second criterion of integer congruence for triangles.

Example 6.5. The third criterion of triangle integer congruence does not hold in integer geometry. On Figure 3 we show two integer triangles $\triangle ABC$ and $\triangle A'B'C'$. All edges of both triangles are integer congruent (of length one), but the triangles are not integer congruent, since $\mathrm{lS}(\triangle ABC) = 1$ and $\mathrm{lS}(\triangle A'B'C') = 3$.

FIGURE 3. A counterexample to the third criterion of integer congruence for triangles.

Instead of the second and the third criterions we have the following additional criterion.

Statement 6.6. (An additional criterion of integer triangle integer congruence.) *Consider two integer triangles $\triangle ABC$ and $\triangle A'B'C'$ of the same integer area. Suppose that*

$$ABC \cong A'B'C', \quad CAB \cong C'A'B', \quad BCA \cong B'C'A',$$

then $\triangle A'B'C' \cong \triangle ABC$.

Proof. All the integer lengths of the corresponding edges are the same by the sine formula. Hence the triangles are integer congruent by the first criterion. □

In the following example we show that the additional criterion of integer triangle integer congruence is not improvable.

Example 6.7. On Figure 4 we show an example of two integer non-equivalent triangles $\triangle ABC$ and $\triangle A'B'C'$ of the same integer area equals 4 and the congruent angles ABC, CAB, and $A'B'C'$, $C'A'B'$ all integer-equivalent to the angle larctan(1), but $\triangle ABC \ncong \triangle A'B'C'$.

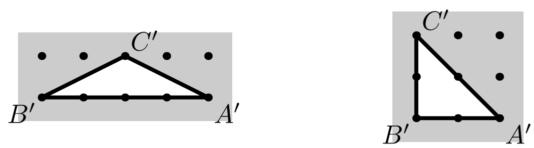


FIGURE 4. The additional criterion of integer congruence is not improvable.

6.4. **On sum of angles in triangles.** In Euclidean geometry a triple of angles is a triple of angles in some triangle if and only if their sum equals π. Let us introduce a generalization of this statement to the case of integers geometry.

First, we reformulate Euclidean criterion in the form of tan functions. *There exist a triangle with angles α, β, and γ exists if and only if*

$$\left\{ \begin{array}{l} \tan(\alpha+\beta+\gamma) = 0 \\ \tan(\alpha+\beta) \notin [0; \tan\alpha] \end{array} \right.$$

(*here without loss of generality we suppose that α is acute*). The next Theorem is a translation of this condition into integer case.

Second we give several preliminary definitions

Let n be an arbitrary positive integer, and $A = (x, y)$ be an arbitrary integer point. Denote by nA the point (nx, ny).

Definition 6.8. Consider an integer polygon or broken line with vertices $A_0, \ldots, A_k$. The polygon or broken line $nA_0 \ldots nA_k$ is called *n-multiple* (or *multiple*) to $A_0, \ldots, A_k$.

Let p_i (for $i = 1, \ldots, k$) be rational numbers and $[a_{1,i} : a_{2,i}; \ldots; a_{n_i,i}]$ be their odd continued fractions. Define

$$]p_1, \ldots, p_k[= [a_{1,1}:a_{2,1}; \ldots ;a_{n_1,1}; a_{1,2};a_{2,2}; \ldots ;a_{n_2,2}; \ldots ; a_{1,k};a_{2,k}; \ldots ;a_{n_k,k}].$$

Now we are ready to formulate the generalization of Euclidean theorem on sum of angles in triangles.

Theorem 6.9. On sum of integer tangents of angles in integer triangles.
a). *Let $(\alpha_1, \alpha_2, \alpha_3)$ be an ordered triple of angles. There exists a triangle with consecutive angles integer congruent to α_1, α_2, and α_3 if and only if there exists $i \in \{1, 2, 3\}$ such that the angles $\alpha = \alpha_i$, $\beta = \alpha_{i+1(\mathrm{mod}\,3)}$, $\gamma = \alpha_{i+2(\mathrm{mod}\,3)}$ satisfy the following conditions:*
i) for $\xi =] \operatorname{ltan}\alpha, -1, \operatorname{ltan}\beta[$ the following holds $\xi < 0$, or $\xi > \operatorname{ltan}\alpha$, or $\xi = \infty$;
ii) $] \operatorname{ltan}\alpha, -1, \operatorname{ltan}\beta, -1, \operatorname{ltan}\gamma[= 0$.
b). *Let α, β, and γ be the consecutive angles of some integer triangle. Then this triangle is multiple to the triangle with vertices $A_0 = (0, 0)$, $B_0 = (\lambda_2 \operatorname{lcos}\alpha, \lambda_2 \operatorname{lsin}\alpha)$, and $C_0 = (\lambda_1, 0)$, where*

$$\lambda_1 = \frac{\operatorname{lcm}(\operatorname{lsin}\alpha, \operatorname{lsin}\beta, \operatorname{lsin}\gamma)}{\gcd(\operatorname{lsin}\alpha, \operatorname{lsin}\gamma)}, \quad \textit{and} \quad \lambda_2 = \frac{\operatorname{lcm}(\operatorname{lsin}\alpha, \operatorname{lsin}\beta, \operatorname{lsin}\gamma)}{\gcd(\operatorname{lsin}\alpha, \operatorname{lsin}\beta)}.$$

We are not ready to prove the first statement of this Theorem now, we do it later in Section ???. We give the second statement of the theorem after a small remark.

Remark 6.10. The statement of Theorem 6.9(a) does not necessarily holds for even continued fractions for the tangents. For instance, consider an integer triangle with the integer area equals 7 and all angles integer congruent to larctan 7/3. For the odd continued fractions $7/3 = [2; 2 : 1]$ of all angles we have

$$[2; 2 : 1 : -1 : 2 : 2 : 1 : -1 : 2 : 2 : 1] = 0.$$

If instead we take the even continued fractions $7/3 = [2; 3]$, then we have

$$[2; 3 : -1 : 2 : 3 : -1 : 2 : 3] = \frac{35}{13} \neq 0.$$

Proof of the second statement of Theorem 6.9. Consider a triangle $\triangle ABC$ with rational angles α, β, and γ (at vertices at A, B, and C respectively). Suppose that for any $k > 1$ and any integer triangle $\triangle KLM$ the triangle $\triangle ABC$ is not integer congruent to the k-multiple of $\triangle KLM$. In other words, we have

$$\gcd\big(\mathrm{l}\ell(AB), \mathrm{l}\ell(BC), \mathrm{l}\ell(CA)\big) = 1.$$

Suppose that S is the integer area of $\triangle ABC$. Then by the definition of integer sine the following holds

$$\begin{cases} \mathrm{l}\ell(AB)\,\mathrm{l}\ell(AC) & = & S/\operatorname{lsin}\alpha \\ \mathrm{l}\ell(BC)\,\mathrm{l}\ell(BA) & = & S/\operatorname{lsin}\beta \\ \mathrm{l}\ell(CA)\,\mathrm{l}\ell(CB) & = & S/\operatorname{lsin}\gamma \end{cases}.$$

Since $\gcd(\mathrm{l}\ell(AB), \mathrm{l}\ell(BC), \mathrm{l}\ell(CA)) = 1$, we have $\mathrm{l}\ell(AB) = \lambda_1$ and $\mathrm{l}\ell(AC) = \lambda_2$.

Therefore, the triangle $\triangle ABC$ is integer congruent to the triangle $\triangle A_0B_0C_0$ of the theorem. □

6.5. **Examples of integer triangles.** Let us define certain types of triangles occurring in integer geometry. Since dual angles are not necessary congruent we have more different types than in Euclidean case.

Definition 6.11. The integer triangle $\triangle ACB$ is called *dual* to the triangle $\triangle ABC$.
The integer triangle is said to be *self-dual* if it is integer congruent to the dual triangle.
The integer triangle is said to be *pseudo-isosceles* if it has at least two integer congruent angles.
The integer triangle is said to be *integer isosceles* if it is pseudo-isosceles and self-dual.
The integer triangle is said to be *pseudo-regular* if all its angles and all its edges are integer congruent.
The integer triangle is said to be *integer regular* if it is pseudo-regular and self-dual.

By the first criterion of integer congruence for integer triangles the number of all integer congruence classes for integer triangles with bounded integer area is always finite. On Figure 5 we show the complete list of 33 triangles representing all integer congruence classes of integer triangles with integer areas not greater than 10. We enumerate the vertices of the triangle in the clockwise way. Near each vertex of any triangle we write the tangent of the corresponding rational angle. Inside any triangle we write its area. We draw dual triangles on the same light gray area (if they are not self-dual). Integer regular triangles are colored in dark grey, integer isosceles but not integer regular triangles are white, and the others are light grey.

Integer triangles of small area. The above criterions allows us to enumerate all integer triangles of small integer area up to the integer equivalence. In the following table we write down the numbers $N(d)$ of noncongruent integer triangles of integer area d for $d \leq 20$ (here dual noncongruent triangles are counted as two).

d	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20
$N(d)$	1	1	2	3	2	4	4	5	5	6	4	10	6	8	8	11	6	13	8	14

Let us prove an easy statement on statistics $N(d)$.

Proposition 6.12. *We have*

$$\frac{d}{3} \leq N(d) \leq \frac{d(d+1)}{2}.$$

Proof. First, let us show that $N(d) \leq d(d+1)/2$. From Theorem 6.9(b) we know that any integer triangle is equivalent to some $\triangle A_0B_0C_0$ where

$$A_0 = (0,0), \quad B_0 = (\lambda_2 q, \lambda_2 p), \quad C_0 = (\lambda_1, 0),$$

where $0 < q < p$. The area of such triangle is exactly $\lambda_1\lambda_2 q$. Hence $\lambda_2 q \leq d$. There are exactly $d(d+1)/2$ integer points satisfying all listed conditions. Each of such points can be chosen to construct B_0. Then C_0 is constructed in a unique way $C_0 = (d/(\lambda_2 q), 0)$, if d is divisible by $\lambda_2 q$ we have an integer triangle. Hence there is at most $d(d+1)/2$ triangles of area d.

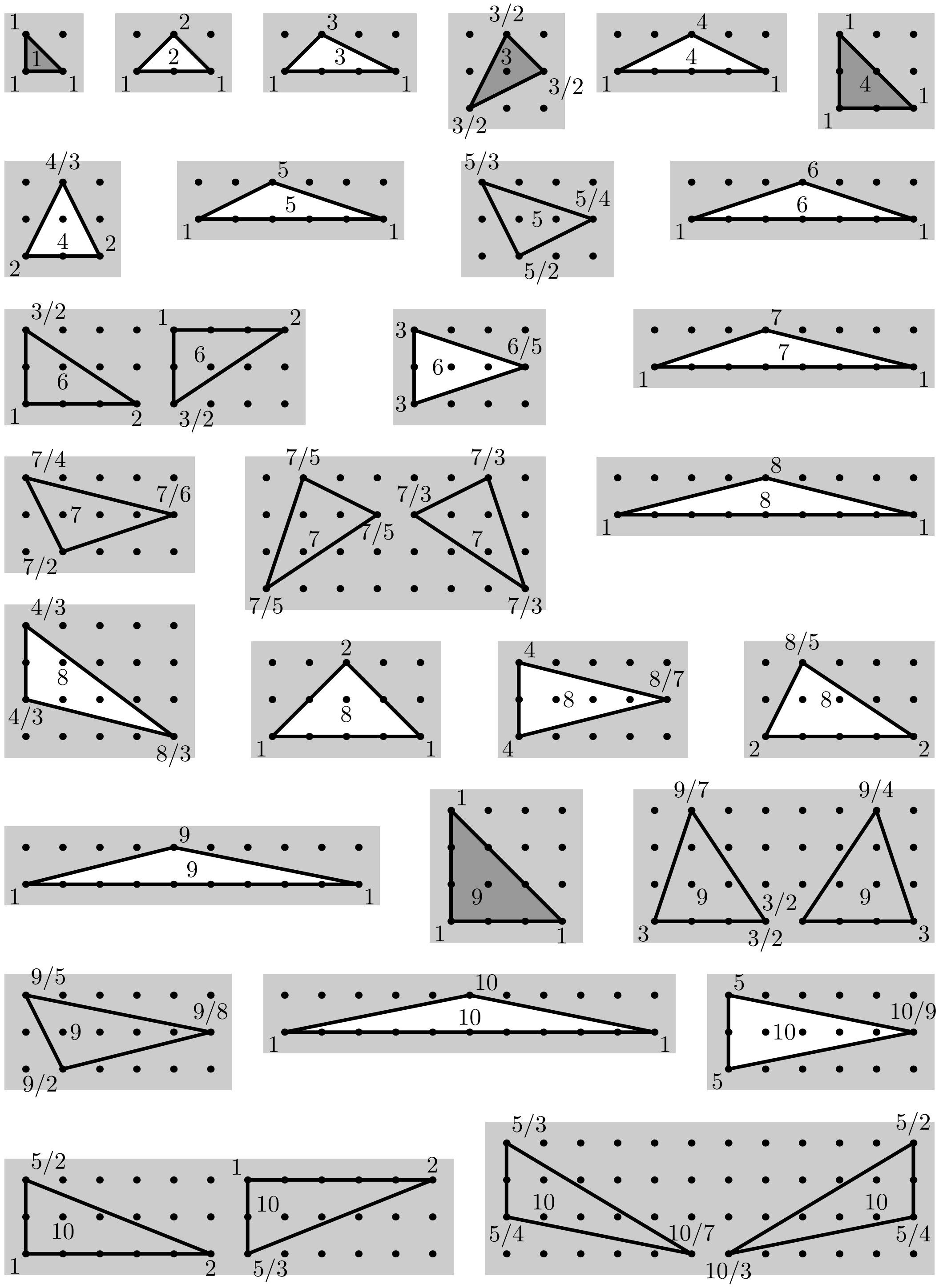


FIGURE 5. List of integer triangles of integer volume less than or equal 10.

Let us show, that $N(d) \geq d/3$. There are exactly d non-congruent angles that appear in triangles of area d:

$$\mathrm{larctan}\,\frac{k}{d}, \qquad k = 1, 2, \ldots, d.$$

Each triangle contains at most three non-congruent angles. Hence $N(d) \geq d/3$. □

It seems that the growth rate is linear. For instance, for a prime number d we always have $N(d) \leq d$. Still in some exceptional cases when d has many divisors, it can happen that $N(d) > d$. For instance, $N(240) = 248$.

Exercises.

[1] Find all the triangles of area 11.
[2] Let $\mathrm{ltan}(ABC) = 7/5$; $\mathrm{l}\ell(AB) = 3$; $\mathrm{l}\ell(BC) = 5$. Find the rest angles and the edge.
[3] A triangle with angles α, β, and γ exists if and only if

$$\begin{cases} \tan(\alpha+\beta+\gamma) = 0 \\ \tan(\alpha+\beta) \notin [0; \tan\alpha] \end{cases}$$

(here without loss of generality we suppose that α is acute).
[4] Is there a triangle with angles congruent to larctan(24/7), larctan(24/13), and larctan 4.
[5] Suppose q has n divisors. Find the upper estimate on $N(q)$ linear in d and n.

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