6. Elements of Gauss Reduction Theory via geometric continued fractions
(10 May 2011)

Let S be a ring (here we are mostly interested in cases $S=\mathbb{C},\mathbb{R},\mathbb{Z}$). We say that two matrices A and B with coefficients in S are $GL(2,S)$-*conjugate* if there exists a $GL(2,S)$-matrix C such that $B=C^{-1}AC$.

A classical approach to study conjugacy classes of matrices is based on finding a nice section of the decomposition of $Mat(n,S)$ or certain its subsets into the conjugacy classes. This means that one finds a set of *normal forms*, such that each class has exactly one normal form. Each of such forms represents the corresponding conjugacy class. Technically this approach is very useful. Suppose that one intended to prove some $GL(2,S)$-invariant statement. Then it is enough to prove the statement only for normal forms. In addition knowing normal forms of two operators it is immediate if they are conjugate or not: congruence of operators is equivalent to the coincidence of their normal forms. For this reason it is important to know a fast algorithm that finds the normal form for a given operator.

In this lecture we describe conjugacy classes in $SL(2,\mathbb{Z})$ in terms of continued fractions, this is exactly the subject of Gauss reduction theory.

6.1. **Normal forms.**

In this subsection we show several different normal forms for matrices. We start with the Jordan normal forms, that are broadly used for the case of algebraically closed fields. Then we study cases of matrices with real coefficients: the matrices with complex eigenvalues are expressed here via rotation matrices multiplied by a scalar. Further we show Frobenius normal forms for matrices with rational elements. Finally we discuss the case of matrices with integer elements.

6.1.1. *Algebraically closed fields.* If the set S is an algebraically closed field, then it is usual to consider Jordan normal forms. For the case of $GL(2,\mathbb{C})$ the complete list of Jordan normal forms is as follows.

$$\begin{pmatrix}\lambda_1 & 0\\ 0 & \lambda_2\end{pmatrix} \quad \text{for } \lambda_1,\lambda_2\in\mathbb{C}, \qquad \begin{pmatrix}\lambda & 1\\ 0 & \lambda\end{pmatrix} \quad \text{for } \lambda\in\mathbb{C}.$$

6.1.2. *Case of real numbers.* For $GL(2,\mathbb{R})$-conjugation of matrices with real coefficients the situation is a little more complicated, since an operator may have to complex conjugate roots. Such operators correspond to composition of a rotation and a homothety. The list of normal forms here is as follows:

$$\begin{pmatrix}\lambda_1 & 0\\ 0 & \lambda_2\end{pmatrix} \lambda_1,\lambda_2\in\mathbb{R}, \quad \begin{pmatrix}r\cos\varphi & -r\sin\varphi\\ r\sin\varphi & r\cos\varphi\end{pmatrix} r>0,\ 0\le\phi<2\pi, \quad \begin{pmatrix}\lambda & 1\\ 0 & \lambda\end{pmatrix} \lambda\in\mathbb{R}.$$

6.1.3. *Case of rational numbers.* For the case of non-algebraically closed field (like in case of $\mathbb{Q}$) it is better to use *Frobenius normal forms* of matrices that are constructed from

Frobenius blocks at the diagonals:

$$\begin{pmatrix} 0 & 0 & \cdots & 0 & -c_0 \\ 1 & 0 & \cdots & 0 & -c_1 \\ 0 & 1 & \cdots & 0 & -c_2 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & 1 & -c_{n-1} \end{pmatrix}$$

It is interesting to notice that the coefficients c_i of the above Frobenius block are exactly the coefficients of its characteristic polynomial:

$$x^n + c_{n+1}x^{n-1} + \cdots + c_0.$$

So, any rational two-dimensional matrix after an appropriate rational change of coordinates coincides with exactly one of the following matrices:

$$\begin{pmatrix} 0 & a \\ 1 & b \end{pmatrix} \ a, b \in \mathbb{Q}, \quad \begin{pmatrix} \lambda & 0 \\ 0 & \lambda \end{pmatrix} \ \lambda \in \mathbb{Q}.$$

Actually Frobenius normal forms with real or complex parameters can be also used as normal forms for the real or complex situation.

6.1.4. *Case of integer numbers.* Now we show the $GL(2, \mathbb{Z})$-conjugacy classes for $SL(2, \mathbb{Z})$-matrices. This case contains three distinct subcases depending on eigenvalues: *complex case* of matrices with complex eigenvectors, *totally real case* of matrices with two distinct real eigenvalues, and *degenerate case* of matrices with multiple eigenvalues.

For the case of complex conjugate roots we have exactly three conjugacy classes, they are represented by matrices:

$$\begin{pmatrix} 1 & 1 \\ -1 & 0 \end{pmatrix}; \quad \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}; \quad \begin{pmatrix} 0 & 1 \\ -1 & -1 \end{pmatrix}.$$

All matrices of $SL(2, \mathbb{Z})$ with coinciding real eigenvalues are

$$\pm \begin{pmatrix} 1 & n \\ 0 & 1 \end{pmatrix} \quad \text{for } n \in \mathbb{Z}.$$

Finally, we discuss the most complicated case of totally real matrices.

Definition 6.1. A matrix

$$\pm \begin{pmatrix} a & c \\ b & d \end{pmatrix}.$$

in $SL(2, \mathbb{Z})$ is said to be *reduced*, if the following holds: $d > b > a \geq 0$.

If both eigenvalues are positive then '+' sign is taken, otherwise we take '$-$' sign.

Theorem 6.2. *(***Reduction to reduced matrices.)**

(*i*) *Each conjugacy class of $SL(2, \mathbb{Z})$ contains at least one reduced matrix.*

(*ii*) *Each conjugacy class of $SL(2, \mathbb{Z})$ contains finitely many reduced matrices.*

(iii) *The number of reduced matrices in a conjugacy class of A is the length of the minimal period of a geometric continued fraction related to A (see Corollary 6.14 and the discussions in Subsection 6.7).*

Remark. The definition of a reduced operator is slightly different to a classical one: *an operator in $SL(2,\mathbb{Z})$ is reduced if and only if it has non-negative entries which are non-decreasing downwards and to the right.* Still it is more convenient to leave our definition for the generalizations to multidimensional case.

Let us have a closer look on the set of conjugacy classes. For this reason we introduce the following filtration on the space of all matrices.

Definition 6.3. A *Hessenberg complexity* of the matrix A is the area of the parallelogram generated by vectors $(1,0)$ and $A(1,0)$. We denote it by $\varsigma(A)$.

Geometrically Hessenberg complexity mean how far with respect to integer geometry the operator A sends the first basis vector. It is clear that

$$\varsigma\begin{pmatrix} a & c \\ b & d \end{pmatrix} = |b|.$$

In the following table we show all the reduced matrices with small Hessenberg complexity (all the parameters n in the table are positive integers).

$\chi(A)$	1	2	3	4
Reduced operators	$\pm\begin{pmatrix} 0 & -1 \\ 1 & 1+n \end{pmatrix}$	$\pm\begin{pmatrix} 1 & n \\ 2 & 1+2n \end{pmatrix}$	$\pm\begin{pmatrix} 1 & n \\ 3 & 1+3n \end{pmatrix}$ $\pm\begin{pmatrix} 2 & 1+2n \\ 3 & 2+3n \end{pmatrix}$	$\pm\begin{pmatrix} 1 & n \\ 4 & 1+4n \end{pmatrix}$ $\pm\begin{pmatrix} 3 & 2+3n \\ 4 & 3+4n \end{pmatrix}$

In total the number of families of complexity n is exactly twice the value of Euler function $\phi(n)$ (i.e., the number of relatively prime with n positive numbers not smaller than n).

6.2. **Geometric continued fraction associated to a totally real operator.** In this section we construct a complete invariant of an $SL(2,\mathbb{Z})$-conjugacy class of $SL(2,\mathbb{Z})$ totally real matrices (with distinct real eigenvalues) in terms of the corresponding continued fractions.

Definition 6.4. Consider a totally real operator A. The operator A has exactly two distinct eigen straight lines. The geometric continued fraction (introduced in Lecture 4 on page ???) defined by these two straight line is said to be *associated* to A.

From Corollary 4.12 it immediately follows.

Proposition 6.5. *A geometric continued fraction associated to a totally real matrix A is an invariant of A with respect to the $GL(2,\mathbb{Z})$-conjugations.* □

Let us now study in more details the relation between the LLS-sequences for the sails in the same continued fraction.

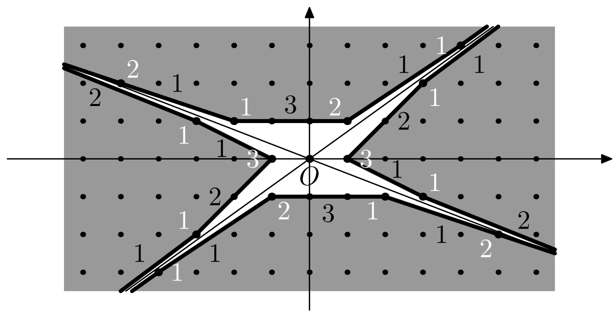

FIGURE 1. Geometric continued fraction of the operator
$$\begin{pmatrix} 7 & 18 \\ 5 & 13 \end{pmatrix}.$$

6.3. **LLS-sequences for dual sails. LLS-sequences for matrices.** Two sails are called *dual* with respect to each other if the LLS-sequence of one sail coincide (up to an index shift in the case of LR-irrational angles) with the inverted LLS-sequence of the second sail.

Proposition 6.6. *Let A be a totally real operator with no integer eigenvectors. Then the sails of the opposite angles are congruent. The sails of the adjacent angles are dual.*

On Figure 1 we show the geometric continued fraction for the operator
$$\begin{pmatrix} 7 & 18 \\ 5 & 13 \end{pmatrix}.$$
Integer lengths of edges are denoted by black digits, and integer sines — by white. The LLS-sequences of all four sails are equivalent to or inverse to $(\ldots, 2, 1, 1, 3, 2, 1, 1, \ldots)$.

Proof. The sails of opposite angles are congruent since they are taken one to another by the symmetry about the origin.

Let us prove the duality. Let one of the sails for the geometric continued fraction be a broken line (A_i). Without loss of generality we may fix coordinates such that $A_0 = (1, 0)$ and $A_1 = (1, a_0)$.

Denote $B_0 = (0, 1)$. Notice that B_0 is on the dual sail (since the sail is defined by lines $y = \alpha x$ and $y = \beta x$ where $a_0 < \alpha < a_0+1$ and $-1 < \beta < 0$ in the chosen coordinates). Denote the rest points of the dual sail (B_i) starting from B_0.

Let (a_i) and (b_i) be the LLS-sequences for the sails (A_i) and (B_i).

In Section 3 we have already shown the edge-angle duality in the orthant of points with positive coordinates. In other words, we know that $a_i = b_{-i-1}$ for $i \geq 1$.

Let B_i' be the point symmetric to B_i with respect to the origin ($i \in \mathbb{Z}$). Recall that
$$B_{-1} = (a_1, a_0 a_1 + 1) \quad \text{and} \quad A_2 = (a_1 a_2 + 1, a_0 a_1 a_2 + a_0 + a_2).$$

Consider a linear transformation taking A_2 to $(1,0)$ and A_1 to $(1,a_2)$. It is

$$\begin{pmatrix} a_0a_1+1 & -a_1 \\ a_0a_1a_2+a_0+a_2 & -a_1a_2-1 \end{pmatrix}.$$

This transformation takes B'_{-1} to $(0,1)$. Now we have edge-angle duality of $A_2A_1A_0\ldots$ and $B'_{-1}B'_0\ldots$. Therefore, $a_i = b_{-i-1}$ for $i \leq 1$.

Hence the LLS-sequences are inverse to each other. □

Let us define LLS-sequence for operators.

Definition 6.7. The *LLS-sequence* of an operator A in $GL(2,\mathbb{R})$ is the LLS-sequence for any of its sails up to the choice of a direction of a sequence and zero element.

In the next proposition we show some basic properties of LLS-sequences for totally real operators.

Proposition 6.8. (i). *For any infinite in two sides sequence of integers there exists a totally real operator whose LLS-sequence coincides with the given.*

(ii). *Let totally real operators A and B have the same LLS-sequence. Then there exists an operator C commuting with A and conjugate to B.* □

Remark 6.9. Recall that two totally real operators commute if and only if they are diagonalizable at the same basis.

6.4. Complete invariants of $SL(2,\mathbb{Z})$ conjugacy classes and reduced operators.

6.4.1. *Periodicity of algebraic sails.* Consider now the case of totally real operators in $SL(2,\mathbb{Z})$. It turns out that all such operators have irreducible over rational numbers characteristic polynomial. The sails of such operators are called *algebraic*.

Let A be such operator. Denote by $\Xi(A)$ the group of all $SL(2,\mathbb{Z})$-operators commuting with A and having positive real eigenvalues. By Dirichlet unity theorem the group $\Xi(A)$ is isomorphic to $\mathbb{Z}$. Any sail of the operator A is invariant under the action of the group $\Xi(A)$, moreover the operators of the group $\Xi(A)$ act on the sails by shifting the edges of the broken line along thy broken line. Therefore, the LLS-sequences of algebraic sails are periodic. The converse is also true (see Corollary 6.13 below).

Remark. On Figure 1 we show the sails of totally real algebraic operator with a period of the LLS-sequence equals $(2,1,1,3)$.

Theorem 6.10. (On complete invariant of Dirichlet groups.) *A geometric continued fraction is a complete invariant of Dirichlet groups.* □

Theorem 6.11. (On complete invariant of conjugacy classes.) *Suppose A is an $SL(2,\mathbb{Z})$ operator with positive eigenvalues. A period of LLS-sequence of geometric continued fraction corresponding to A defined by a shift corresponding to the action of A is a complete invariant of a $SL(2,\mathbb{Z})$-conjugacy class of A.* □

6.4.2. *Reduced algebraic operators and their LLS-sequences.* We calculate periods of LLS-sequences of of the reduced operators. It turns out that we have all possible sequences, and hence any operator is conjugate to some reduced operator. Later we write the algorithm to calculate a reduced operator conjugate to the given one.

Actually, for a reduced operator there exist only finitely many other reduced operators conjugate to it. We will say about it later. So reduced operators are a good replacement for Jordan normal forms for the case of $SL(2,\mathbb{Z})$.

6.5. **LLS-sequence of reduced operators.** In next theorem we write periods of LLS-sequences for all reduced algebraic operators.

Theorem 6.12. *Consider an $SL(2,\mathbb{Z})$-operator*

$$\begin{pmatrix} a & c+\lambda a \\ b & d+\lambda b \end{pmatrix}.$$

Then one of the periods of the LLS-sequence is as follows.
(*i*) *Let $b > a \geq 1$, $0 < d \leq b$, $\lambda \geq 1$, and the odd ordinary continued fraction for b/a equal $[a_0{:}a_1;\ldots;a_{2n}]$. Then a period is*

$$(a_0, a_1, \ldots, a_{2n}, \lambda).$$

(*ii*) *Let $a = 0$, $b = 1$, $d = 1$, and $\lambda \geq 2$. Then a period is*

$$(1, \lambda - 1).$$

Notice that for any couple of relatively prime integers (a, b) where $b > a \geq 0$ there exists a couple of integers (c, d), satisfying $0 < d \leq b$ and $ad - bc = 1$.

Remark. For negative values of λ in the case $a = 0$ the periods are $(1, |\lambda| - 3)$. In the case $a > 0$ the periods equal

$$(a'_0, a'_1, \ldots, a'_{2m}, |\lambda| - 2),$$

where $[a'_0{:}a'_1;\ldots;a'_{2m}]$ — is the odd ordinary continued fraction for $b/(b-a)$.

Proof. The discriminant of the characteristic polynomial of the operator A equals $((a + \lambda b + d)^2 - 4$. Since $\lambda \geq 1$, $b \geq 1$, $d \geq 1$, and $a \geq 0$, the discriminant is nonnegative. Besides it equals zero in the exceptional case $a = 0$, $b = 1$, $\lambda = 1$, $d = 1$. Therefore, any reduced operator is totally real in all cases. Since $t^2 - 4$ for integer $t > 2$ is not a square of some integer, the sails of the operator A are algebraic.

Let us now construct a period for the LLS-sequence. Note that both eigenvalues of the operator A:

$$\frac{a + \lambda b + d \pm \sqrt{((a + \lambda b + d)^2 - 4}}{2}$$

are positive, and thus the operator A takes each sail to itself. Consider the sail S, whose convex hull contains the point $P = (1, 0)$.

Suppose the operator A is of Item (ii). Then the set of the vertices for one of its sails coincides with the set of points $A^n(1,0)$ with an integer parameter n. Simple calculations lead to the result of the theorem.

Let now an operator A be an operator of Item (i), i. e. $b > a > 0$.

Denote by α the closed convex angle with vertex at the origin and edges passing through the points P and $A(P)$. Consider the boundary of the convex hull of all integer points inside α except the origin. The boundary consists of two rays and a finite broken line (the essential part of the sail).Denote the finite broken line in the boundary by S_α.

Now we show that the essential part of the sail of the angle α is completely contained in the convex hull of a sail of the operator A. Denote by S_α^∞ the following infinite broken line:

$$\bigcup_{i=-\infty}^{+\infty} \Big(A^i(S_\alpha) \Big).$$

By the construction the convex hull of S_α^∞ coincides with the convex hull of the sail for A. It remains to verify if S_α^∞ coincides with the boundary of its convex hull. The broken line S_α^∞ is the boundary of the convex hull if the convex angles generated by adjacent edges of the broken line do not contain the origin. To check this it is sufficient to study all the angles of one of the periods of the broken line S_α^∞, for instance all the angles with vertices at vertices of S_α except the point $A(P)$. All convex angles generated by couples of adjacent edges at inner vertices of the sail S_α do not contain the origin by definition. It remains to check the angle with vertex at $P = (1,0)$.

Since $b/a > 1$, the first edge is parallel to the vector $(0,1)$. Consider the second edge of the angle, let the first integer point at it be Q. Since the triangle OPQ does not contain integer points distinct to the integer points of the segment PQ and the vertex O, the segment PQ contains the point with coordinates $(x,-1)$. By the convexity of finite broken line $A^{-1}(S_\alpha)$ the value of x is determined by the eigen direction:

$$\left(\frac{-a+\lambda b+d+\sqrt{((a+\lambda b+d)^2-4}}{2b}, -1 \right),$$

namely,

$$x = \left\lfloor \frac{-a+\lambda b+d+\sqrt{((a+\lambda b+d)^2-4}}{2b} \right\rfloor + 1,$$

where $\lfloor t \rfloor$ denotes the maximal integer that does not exceed t. From condition $b \geq d > 0$ we have

$$\begin{array}{l} \lfloor x \rfloor \leq \left\lfloor \dfrac{2\lambda b + 2d}{2b} \right\rfloor + 1 = \left\lfloor \lambda + \dfrac{d}{b} \right\rfloor + 1 = \lambda + 1 \\ \lfloor x \rfloor \geq \left\lfloor \dfrac{2\lambda b + 2d - 2}{2b} \right\rfloor + 1 = \left\lfloor \lambda + \dfrac{d-1}{b} \right\rfloor + 1 = \lambda + 1 \end{array},$$

and, therefore,

$$x = \lambda + 1.$$

Hence for $\lambda > 1$ the convex angle at vertex P does not contain the origin. Therefore, the broken line S_α^∞ coincides with the sail.

In Lecture 3 it was shown that the sail S_α consists of $n+1$ segments. The integer lengths of the consecutive segments equal $a_0, a_2, \ldots, a_{2n}$, and the integer sines of the corresponding angles equal $a_1, a_3, \ldots, a_{2n-1}$ respectively. Now note, that from the explicit value of x it follows that the integer sine for the angle at point P equals λ. Hence the LLS-sequence of the sail S has a period

$$(a_0, a_1, \ldots, a_{2n}, \lambda).$$

Therefore the LLS-sequence of the operator A has the prescribed period. □

Corollary 6.13. *A sail with the periodic LLS-sequence is algebraic* (*i. e. a sail of some algebraic totally real operator*)*.*

Proof. In Theorem 6.12 we constructed the algebraic operators for all finite sequences as periods. Then in Proposition 6.8 we showed that the sails with equivalent LLS-sequences are either equivalent or dual. Therefore any sail with periodic LLS-sequence is algebraic. □

Corollary 6.14. *Consider an operator $A \in SL(2,\mathbb{Z})$. Let a minimal period of LLS-sequence A consist of n elements. Then there are exactly $2n$ reduced matrices congruent to A (n with positive eigenvectors, and n with negative).*

Remark 6.15. **(On matrices of $GL(2,\mathbb{Z}) \setminus SL(2,\mathbb{Z})$.)** Consider some sail with periodic LLS-sequence. Let a minimal period of LLS-sequence is even and consists of $2n$ elements. Then there exists an $SL(2,\mathbb{Z})$-operator A with positive eigenvalues, that makes an n-edge shift of the sail along the sail. Precisely this operator generates the group $\Xi(A)$ of the sail shifts (see above). Let a minimal period of LLS-sequence is odd and consists of $2n+1$ elements (in particular this implies that the sail is equivalent to any dual sail). Then there exists a $GL(2,\mathbb{Z})$-operator B with negative discriminant, whose square makes an $(2n+1)$-edge shift of the sail along the sail. Moreover, the operator B^2 generates the group $\Xi(T)$.

6.6. **An algorithm to find a period of the LLS-sequence.** The main idea of the calculation of the period is to find a reduced operator with the sails equivalent to the sails of the given one. Then it remains to calculate the period of the reduced operator by Theorem 6.12.

For the rest part of this Section we denote by $[[a,c][b,d]]$ the operator

$$\begin{pmatrix} a & c \\ b & d \end{pmatrix}.$$

Data. Suppose we know the integer entries of an operator $[[a,c][b,d]]$ with unit determinant and positive discriminant. Let also the characteristic polynomial does not has roots ± 1. From the listed conditions it follows that the operator $[[a,c][b,d]]$ is totally real operator in $SL(2,\mathbb{Z})$ with irreducible characteristic polynomial.

It is requested to construct one of the periods of the LLS-sequence of the totally real algebraic operator $[[a, c][b, d]]$.

Description of the algorithm. We start from a matrix A. The algorithm works iteratively, at some of the iterations we get a reduced operator congruent to A, or to $-A$. Then we calculate a period of the LLS-sequence according to Theorem 6.12. The algorithm stops since each iteration reduces the Hessenberg complexity of the operator under consideration.

Step 1. If $b < 0$, then we multiply the operator $[[a, c][b, d]]$ by $[[-1, 0][0, -1]]$. The LLS-sequence does not change at that (and the shift corresponding to the matrix is also preserved).

Step 2. So, now the element b is positive. Conjugate the operator $[[a, c][b, d]]$ by the operator $[[1, \lfloor a/b \rfloor][0, 1]]$. We obtain the operator $[[a', b][c', d']]$, where $0 \le a' < b$. The Hessenberg complexity (i.e., b) does not changed.

Step 3.1. Suppose $b = 1$, then $a' = 0$, $c' = -1$. Moreover we have $|d'| > 2$, since otherwise the original operator is not algebraic. Therefore a period of the LLS-sequence equals $(1, |d'| - 2)$.

Step 3.2.1. Suppose, $b > 1$. If $d' > b$, then we have found a reduced operator, now we go to Step 4.

Step 3.2.2. Suppose, $b > 1$. If $d' < -b$, then we conjugate by the operator $[[-1, 1][0, 1]]$ and multiply by $[[-1, 0][0, -1]]$. We get the operator $[[a'', c''][b'', d'']]$ with $b'' = b$, $a'' = b - a'$, and $d'' = -b - d' > 0$ (with the same LLS-sequence and the same shift of LLS-sequence), further we should go to Step 3.2.1, or to Step 3.2.3. The Hessenberg complexity (i.e., b) does not changed.

Step 3.2.3. Suppose, $b > 1$. The case $|d'| \le |b|$. Note that the absolute values of b and d' do not coincide since the determinant of the operator does not have divisors distinct to the unity. Therefore, it remains the case $|d'| < |b|$. We have:

$$|c'| = \left| \frac{a'd' - 1}{b} \right| \le \frac{(b-1)^2 + 1}{b} = b - 1.$$

We conjugate the operator $[[a', c'][b, d']]$ with the operator $[[0, -1][-1, 0]]$ and obtain $[[d', b][c', a']]$, where $|c'| < |b|$. Now we return back to Step 2 with the obtained operator $[[d', b][c', a']]$. The Hessenberg complexity (i.e., c') is reduced at least by 1.

Step 4. We obtained a reduced operator $[[\tilde{a}, \tilde{c}][\tilde{b}, \tilde{d}]]$, $|\tilde{b}| > 1$ congruent either to the original operator A, or to $-A$. By Theorem 6.12 to construct one of the periods of the LLS-sequences of the reduced operator we should construct the odd ordinary continued fraction for $|\tilde{b}/\tilde{a}|$ and find the integer $\lfloor |(\tilde{d} - 1)/\tilde{b}| \rfloor$.

Corollary 6.16. *For any totally-real operator A in $SL(2, \mathbb{Z})$ either A or $-A$ is conjugate to a reduced operator.*

6.7. **A question on Hessenberg complexity of the minimal period.** Note that for any operator there exist finitely many reduced operators with the same trace and LLS-sequence. If we study the reduced operators that make shifts of sails on a minimal possible period, then the number of such operators coincides with the length of the minimal period. Recall that for a reduced operator $[[a,c][b,d]]$ the Hessenberg complexity is $|b|$.

Problem 1. Study the minimal Hessenberg complexity for reduced operators with LLS-sequence having a length n period $(a_1,\ldots,a_n)$.

Remark. The minimal Hessenberg complexity coincides with the minimal positive value of the integer sine of the angles POQ, where O is the origin, $P=(x,y)$ is an arbitrary integer point distinct to O, and $Q=A(P)$. Therefore, the minimal Hessenberg complexity, considered as the minimal possible integer sine, is well defined for all operators and it is invariant under conjugations.

If n is even, then the problem is equivalent to finding the minimal numerator among the numerators of the rationals:

$$[a_1{:}\ldots;a_{n-1}],\quad [a_2{:}\ldots;a_n],\quad [a_3{:}\ldots;a_n;a_1],\quad \ldots\quad ,[a_n{:}a_1;\ldots;a_{n-2}].$$

Example 6.17. Let the period contains two elements: (a,b), where $a<b$, then the minimal Hessenberg complexity equals a.

Example 6.18. Let the period contains four elements: (a,b,c,d). Let d is not smaller than the other elements of the period, let also $d>a$ except for the case $a=b=c=d$. Then the rational with the minimal numerator can be found from the following table.

(a,b,c,d) $d\ge a,b,c$	Rational with the minimal numerator
$d>a,b,c$	$[a{:}b;c]$
$d=c; b<a<d$	$[d{:}a;b]$
$d=c; a<b<d$	$[a{:}b;c]$
$d=c; a=b<d$	$[a{:}b;c]$ and $[d{:}a;b]$
$d=b; d>a,c$	$[a{:}b;c]$ and $[c{:}d;a]$
$d=c=b; d>a$	$[a{:}b;c]$ and $[c{:}d;a]$
$d=c=b=a$	all

If n is odd, then the problem is equivalent to finding the minimal numerator among the numerators of the rationals (we define $a_{n+k}=a_k$):

$$[a_1{:}\ldots;a_{2n-1}],\quad [a_2{:}\ldots;a_{2n}],\quad [a_3{:}\ldots;a_{2n};a_1],\quad \ldots\quad ,[a_{2n}{:}a_1;\ldots;a_{2n-2}].$$

Example 6.19. Let the period consists of three elements: (a,b,c), where $c\ge a,b$. Then the fraction $[a{:}b;c;a;b]$ has the minimal numerator (or of one of some equivalent minimal numerators in the case of $a=c$ or $b=c$).

One can suppose that we should skip one of the maximal elements of the period, but that is not true for the six element sequence: $(1,4,5,4,1,4)$. The minimum of the numerators is attained at the fraction $[1:4;5;4;1]$, and not at the fraction $[4:1;4;1;4]$.

6.8. **On frequencies of occurrences of the reduced operators.** First we describe a proper probabilistic space. Let $P=(a_1,a_2,\ldots,a_{2n-1},a_{2n})$ be some period. Denote by $\#_N(P)$ the quantity of all operators satisfying the following conditions:

i). The absolute value of any entry of the operator does not exceed N.
ii). The sequence P is one of the periods of SL-sequence for the operator.
iii). Starting from the operator, the algorithm of the previous section constructs the reduced operator $[[a,b][c,d]]$, where $(a,b)=(0,1)$ for the case $P=(1,a_2)$; and

$$b/a = [a_1 : a_2; \ldots ; a_{2n-1}]$$

in the remaining cases.

Then one studies the relative statistics of $\#_N(P)$ while N tends to infinity. The following questions are of interest.

Problem 2. a). Which reduced operator with a given trace (or with a fixed LLS-sequence) is the most frequent as a result of the algorithm of the previous section?
b). What is the probability of that?
c). Is it true that the maximal possible probability is attained at reduced operators with minimal Hessenberg complexity?

In Table 1 we give some results of calculation of $\#_{25000}(P)$ for the operators with small absolute value of the trace. We remind that the minimal absolute value of the trace of totally real $SL(2,\mathbb{Z})$-operator equals 3.

It is interesting to note that $SL(2,\mathbb{Z})$-operators corresponding to $P=(1,2)$ are more frequent than the $SL(2,\mathbb{Z})$-operators corresponding to $P=(1,1)$. This occurs since the sails whose LLS-sequences has the period $(1,1)$ are equivalent to their duals. If we enumerate the operators with multiplicities equivalent to the number of equivalent sails for the operators then we get:

$$4\#_{25000}(1,1) > 2\#_{25000}(1,2) + 2\#_{25000}(2,1).$$

In conclusion we formulate the following question. Denote by $GK(P)$ the probability of the sequence $P=(a_1,a_2,\ldots,a_{2n-1})$ in the sense of Gauss-Kuzmin:

$$GK(P) = \frac{1}{\ln(2)} \ln\left(\frac{(\alpha_1+1)\alpha_2}{\alpha_1(\alpha_2+1)}\right),$$

where $\alpha_1 = [a_1{:}a_2;\ldots;a_{2n-2};a_{2n-1}]$, $\alpha_2 = [a_1{:}a_2;\ldots;a_{2n-2};a_{2n-1}{+}1]$.

Problem 3. Let

$$\begin{array}{ll} P_1 = (a_1,a_2,\ldots a_{2n-1},a_{2n}), & P_1' = (a_1,a_2,\ldots a_{2n-1}), \\ P_2 = (a_2,a_3,\ldots a_{2n},a_1), & P_2' = (a_2,a_3,\ldots a_{2n}). \end{array}$$

Is the following true:

$$\lim_{n\to\infty} \frac{\#_n(P_1)}{\#_n(P_2)} = \frac{GK(P_1')}{GK(P_2')}\ ?$$

Exercises.

Absolute value of the trace	Notation for classes of equivalent operators	Period P	Operator $[[a,b][c,d]]$	Value of $\#_{25000}(P)$
3	L_3	$(1,1)$	$[[0,1][-1,3]]$	663160
4	L_4	$(1,2)$	$[[0,1][-1,4]]$	834328
		$(2,1)$	$[[1,2][1,3]]$	304776
5	L_5	$(1,3)$	$[[0,1][-1,5]]$	818200
		$(3,1)$	$[[1,3][1,4]]$	194528
6	$L_{6,1}$	$(1,4)$	$[[0,1][-1,6]]$	777128
		$(4,1)$	$[[1,4][1,5]]$	141784
	$L_{6,2}$	$(2,2)$	$[[1,2][2,5]]$	446432
7	$L_{7,1}$	$(1,5)$	$[[0,1][-1,7]]$	734904
		$(5,1)$	$[[1,5][1,6]]$	110848
	$L_{7,2}$	$(1,1,1,1)$	$[[2,3][3,5]]$	201744
8	$L_{8,1}$	$(1,6)$	$[[0,1][-1,8]]$	695560
		$(6,1)$	$[[1,6][1,7]]$	90688
	$L_{8,2}$	$(2,3)$	$[[1,2][3,7]]$	435472
		$(3,2)$	$[[1,3][2,7]]$	310872
9	L_9	$(1,7)$	$[[0,1][-1,9]]$	660984
		$(7,1)$	$[[1,7][1,8]]$	76552
10	$L_{10,1}$	$(1,8)$	$[[0,1][-1,10]]$	630592
		$(8,1)$	$[[1,8][1,9]]$	66064
	$L_{10,2}$	$(2,4)$	$[[1,2][4,9]]$	408216
		$(4,2)$	$[[1,4][2,9]]$	239712
	$L_{10,3}$	$(1,1,1,2)$	$[[2,3][5,8]]$	260872
		$(2,1,1,1)$	$[[2,5][3,8]]$	114084
		$(1,2,1,1)$	$[[3,4][5,7]]$	149832
		$(1,1,2,1)$	$[[3,5][4,7]]$	114084

TABLE 1. Values of $\#_{25000}(P)$ for the operators with small absolute values of the traces.

[1] Let A be a complex matrix and

$$f(x)=\sum_{n=0}^{\infty} c_n x^n$$

be an analytic function. Denote

$$f(A)=\sum_{n=0}^{\infty} c_n A^n.$$

Use Jordan normal forms to find the expressions for the elements of $f(A)$.

[2] Prove that the matrices of $SL(2,\mathbb{R})$ shown in Subsubsection 6.1.2 are normal forms.

[3] Is it true that two $SL(2,\mathbb{R})$-operators are integer-conjugate if and only if their associated continued fractions are equivalent?

[4] Consider a totally real $SL(2,\mathbb{Z})$-matrix. Let its eigenlines be $y = \alpha_{1,2}x$. Then α_1 and α_2 are conjugate quadratic irrationalities.

[5] Two totally real operators commute if and only if they are diagonalizable at the same basis.

[6] Let A be a totally real matrix in $SL(2,\mathbb{Z})$. Prove that $\Xi(A)$ is isomorphic to $\mathbb{Z}$.

[7] Is the statement of Proposition 6.6 true for rational angles. R-angles or L-angles? Find a correct analog in these cases.

[8] Prove Proposition 6.8.

[9] Classify the conjugacy classes in non totally real case.

[10] Find a reduced form for the matrix

$$\begin{pmatrix} 103 & 69 \\ 100 & 67 \end{pmatrix}.$$

E-mail address, Oleg Karpenkov: `karpenkov@tugraz.at`

TU Graz /Kopernikusgasse 24, A 8010 Graz, Austria/