In this section we study a geometrical interpretation of Gauss-Kuzmin distribution of integers as an elements of continued fractions. It turns out that the frequency of a positive integer \( k \) in a continued fraction almost everywhere equals to
\[
\frac{1}{\ln 2} \ln \left( 1 + \frac{1}{k(k+2)} \right).
\]
This mean for a general real \( x \) we have 42% of '1', 17% of '2', 9% of 3, etc. So we show theorems on Gauss-Kuzmin distribution and show how the statistics relates to Möbius geometry.

### 7.1. Some information from ergodic theory

Let \( X \) be a set, \( \Sigma \) be a \( \sigma \)-algebra on \( X \) and \( \mu \) be a measure on the elements of \( \Sigma \). The collection \( (X, \Sigma, \mu) \) is called a measure space.

Let \( T \) be a transformation of a set \( X \) then for any \( \mu \)-integrable function \( f \) on \( X \) one can define two averages. The time average for \( f \) at point \( x \) is
\[
\lim_{n \to \infty} \frac{1}{n} \sum_{k=0}^{n-1} f(T^k x).
\]
The space average \( I_f \) is
\[
I_f = \frac{1}{\mu(X)} \int f \, d\mu.
\]
The space average always exists. The time average does not exist for all \( x \), nevertheless in the case we are interested it exists for almost all \( x \). We formulate the related theorem after one important definition.

#### Definition 7.1
Let \( (X, \Sigma, \mu) \) be a measure set. A transformation \( T : X \to X \) is measure preserving if and only if it is measurable and
\[
\mu(T^{-1}(A)) = \mu(A)
\]
for any set \( A \) of \( \Sigma \).

For measure preserving transformations we have.

#### Theorem 7.2. (Birkhoff Pointwise Ergodic Theorem.)
Consider a measure space \( (X, \Sigma, \mu) \) and a measure preserving transformation \( T \). Let \( f \) be \( \mu \)-integrable function on \( X \). Then the time average converges almost everywhere to an invariant function \( \overline{f} \).

#### Definition 7.3
Consider a probability measure space \( (X, \Sigma, \mu) \). Let \( T \) be a measure preserving transformation on the set \( X \). Then \( T \) is ergodic if for any \( X' \in \Sigma \) satisfying \( T^{-1}(X') = X' \) either \( \mu(X') = 0 \) or \( \mu(X') = 1 \).
Theorem 7.4. (Birkhoff-Khinchin Ergodic Theorem.) Consider a probability measure space \((X, \Sigma, \mu)\) and a measure preserving transformation \(T\). Suppose that \(T\) is ergodic. Then the values of time average function are equivalent to the space average (i.e., \(\overline{f}(x) = I_f\)) almost everywhere. \(\square\)

7.2. The measure space related to continued fractions. In this subsection we define a measure space that is closely related to distributions of the elements of continued fractions. For this measure we formulate a statement on density points for measurable subsets, which we essentially use in the proofs below.

7.2.1. Definition of the measure space related to continued fractions. Consider the measure space of a segment \(I = \{x|0 \leq x < 1\}\) with the Borel \(\sigma\)-algebra \(\Sigma\) and a measure \(\hat{\mu}\) defined on a measurable set \(S\) as follows
\[\hat{\mu}(S) = \frac{1}{\ln 2} \int_S \frac{dx}{1 + x}.\]
The coefficient \(1/\ln 2\) is taken such that the measure of the segment \(I\) equals to 1.

7.2.2. Theorems on density points of measurable subsets. We start from a classical theorem on Lebesgue measure space. Denote by \(B(x, \varepsilon)\) the standard ball of radius \(\varepsilon\) centered at \(x\).

Theorem 7.5. (Lebesgue density.) Let \(\lambda\) be the \(n\)-dimensional Lebesgue measure on \(\mathbb{R}^n\). If \(A \subset \mathbb{R}^n\) is a Borel measurable set, then almost every point \(x \in A\) is a Lebesgue density point:
\[\lim_{\varepsilon \to 0} \frac{\lambda(A \cap B(x, \varepsilon))}{\lambda(B(x, \varepsilon))} = 1.\]
\(\square\)

Here ”almost every point” means ”except for a zero measure subset”.

The measure \(\hat{\mu}\) is equivalent to the one-dimensional Lebesgue measure \(\lambda\) on the segment \([0, 1]\) (for more information on measure theory see [Geometry of Sets and Measures in Euclidean Spaces: Fractals and Rectifiability by Pertti Mattila]), hence we have the similar statement in the case of measure space \((X, \Sigma, \hat{\mu})\).

Corollary 7.6. (\(\hat{\mu}\)-density.) Let \(X = [0, 1]\) and \(\hat{\mu}\) be as above. If \(A \subset X\) is a \(\hat{\mu}\)-measurable set with positive measure \(\hat{\mu}(A)\), then almost every point in \(A\) satisfies
\[\lim_{\varepsilon \to 0} \frac{\hat{\mu}(A \cap B(x, \varepsilon))}{\hat{\mu}(B(x, \varepsilon))} = 1.\]
\(\square\)

7.3. On Gauss map.
Figure 1. Gauss map.

7.3.1. Gauss map and corresponding invariant measure. We consider the measure space $(X, \Sigma, \hat{\mu})$ defined in the previous subsection. Define a Gauss map $T$ of a segment $[0, 1]$ to itself as follows

$$T(x) = \{1/x\},$$

where $\{r\}$ denotes the quotient part $r - [r]$.

**Proposition 7.7.** The Gauss map $T$ is a measure preserving for the above $(X, \Sigma, \hat{\mu})$.

We start with the following lemma.

**Lemma 7.8.** Let $x = [0; a_1 : a_2 : \ldots]$. Then

$$T^{-1}(x) = \{[0; k : a_1 : a_2 : \ldots]|k \in \mathbb{Z}_+\} = \left\{\frac{1}{x+k} \big| k \in \mathbb{Z}_+\right\}.$$

**Proof.** The first equality follows directly from the fact that

$$T([0; b_1 : b_2 : \ldots]) = [0; b_2 : b_3 \ldots]$$

and the fact that any real number has the unique ordinary continued fraction with the last element not equal to 1.

The second equality is straightforward. \(\square\)

**Proof of Proposition 7.7.** Consider a measurable set $S$. From Lemma 7.8 it follows that

$$\hat{\mu}(T^{-1}(S)) = \frac{1}{\ln 2} \int_{T^{-1}(S)} \frac{dx}{1+x} = \frac{1}{\ln 2} \sum_{k=1}^{\infty} \left( \int_{T^{-1}(S) \cap [\frac{1}{k+1}, \frac{1}{k}]} \frac{dx}{1+x} \right).$$

Notice that on each open segment $[1/k, 1/(k+1)]$ the operator $T$ is one-to one with the open segment $[0, 1]$. Let us denote the inverse function to $T$ on the segment $[1/k, 1/(k+1)]$ by $T_{(k)}^{-1}$. Therefore,

$$T \left( T^{-1}(S) \cap \left[ \frac{1}{k+1}, \frac{1}{k} \right] \right) = T(T_{(k)}^{-1}(S)) = S,$$
and we can apply the rule of differentiation of a composite function. From Lemma 7.8 we know
\[ T^{-1}_{(k)}(x) = \frac{1}{x + k}. \]

Then we have
\[
\int_{T^{-1}(S) \cap [\frac{1}{x + k}, \frac{1}{x + k}]} \frac{dx}{1 + x} = \int_{T^{-1}_{(k)}(S)} \frac{dx}{1 + x} = \int_{T^{(k)}(S)} \frac{dT^{-1}_{(k)}(x)}{1 + T^{-1}_{(k)}(x)} = \int_{S} \frac{d\left(\frac{1}{x + k}\right)}{1 + \frac{1}{x + k}}
\]

(the minus sign is taken, since the map \( T^{-1}_{(k)} : x \to \frac{1}{x + k} \) changes the orientation). So we have
\[ \hat{\mu}(T^{-1}(S)) = \frac{1}{\ln 2} \sum_{k=1}^{\infty} \int_{S} \frac{dx}{(x + k)(x + k + 1)}. \]

Since the integrated functions under the sign of integration are nonnegative, we can change the order of the sum and the integration operations. We get
\[
\hat{\mu}(T^{-1}(S)) = \frac{1}{\ln 2} \int_{S} \left( \sum_{k=1}^{\infty} \left(\frac{1}{x + k} - \frac{1}{x + k + 1}\right) \right) dx
\]
\[ = \frac{1}{\ln 2} \int_{S} \frac{dx}{x + 1} = \hat{\mu}(S). \]

So for any measurable set \( S \) we have
\[ \hat{\mu}(T^{-1}(S)) = \hat{\mu}(S). \]

Therefore, the Gauss map \( T \) preserves the measure \( \hat{\mu} \). \( \square \)

7.3.2. An example of an invariant set for Gauss map. Let us consider one example of a measurable set which is invariant under the Gauss map.

Denote by \( \Psi \) the set of all irrational numbers in the segment \([0,1]\) whose continued fractions contain only finitely many ’1’. It is clear that
\[ T^{-1}(\Psi) = \Psi, \]

since operation \( T^{-1} \) shifts elements of continued fractions by one and inserts the first element.

**Proposition 7.9.** The set \( \Psi \) is measurable (i.e., \( \Psi \in \Sigma \)).

*Proof.* Denote by \( \Upsilon_n \) the set of all irrational numbers that contain an element ’1’ exactly at place \( n \). Notice that
\[ \Upsilon_1 = [1/2, 1], \]
and, therefore, it is measurable. Hence for any \( n \) the set
\[
\Upsilon_{n+1} = T^n(\Upsilon_1)
\]
is measurable.

Denote by \( \Psi_0 \) the set of all irrational numbers that do not contain an element ‘1’. Since
\[
\Psi_0 = X \setminus \bigcup_{n=1}^\infty \Upsilon_n,
\]
the set \( \Psi_0 \) is also measurable. Then
\[
T^{-n}(\Psi_0)
\]
is measurable for any positive integer \( n \). Hence
\[
\Psi = \bigcup_{n=1}^\infty T^{-n}(\Psi_0)
\]
is measurable. \( \square \)

We will prove later that the Gauss map is ergodic, and, therefore, \( \Psi \) is either of zero measure or full measure in \( X \).

7.3.3. Ergodicity of Gauss map.

**Proposition 7.10.** The Gauss map is ergodic.

Before to prove Proposition 7.10 we introduce a supplementary notation and prove two lemmas.

For a sequence of positive integers \( (a_1, \ldots, a_n) \) denote by \( I_{(a_1, \ldots, a_n)} \) the segment with endpoints \([0; a_1 : \ldots : a_{n-1} : a_n]\) and \([0; a_1 : \ldots : a_{n-1} : a_n + 1]\). It is clear that the map
\[
T^n : I_{(a_1, \ldots, a_n)} \rightarrow [0, 1]
\]
is one-to-one on the segment \( I_{(a_1, \ldots, a_n)} \) and the inverse to \( T^n \) is
\[
T_{(a_1, \ldots, a_n)}^{-1} : x \rightarrow [0; a_1 : \ldots : a_n : 1/x].
\]
In terms of \( k \)-convergents \( p_k/q_k = [0; a_1 : \ldots : a_k] \) the expression for \( T_{(a_1, \ldots, a_n)}^{-1}(x) \) is as follows (see Proposition 1.11):
\[
T_{(a_1, \ldots, a_n)}^{-1}(x) = \frac{p_n/x + p_{n-1}}{q_n/x + q_{n-1}} = \frac{p_n + p_{n-1}x}{q_n + q_{n-1}x}.
\]

**Lemma 7.11.** The measure of a segment \( I_{(a_1, \ldots, a_n)} \) satisfy the following
\[
\hat{\mu}(I_{(a_1, \ldots, a_n)}) < \frac{1}{\ln 2(q_n + q_{n-1})(p_n + q_n)}.
\]
Proof. We have
\[
\hat{\mu}(I_{a_1 \ldots a_n}) = \frac{1}{\ln 2} \int_{I_{a_1 \ldots a_n}} \frac{dx}{1 + x} = \frac{1}{\ln 2} \ln \left( \frac{1 + p_n + p_{n-1}}{q_n + q_{n-1}} \right).
\]

The last inequality follows from the convexity of \(\ln\) function. \(\square\)

**Lemma 7.12.** For any invariant set \(S\) and an interval \(I_{a_1 \ldots a_n}\) it holds
\[
\hat{\mu}(S \cap I_{a_1 \ldots a_n}) > \frac{1}{2} \hat{\mu}(S) \hat{\mu}(I_{a_1 \ldots a_n}).
\]

**Proof.** Since the map \(T\) is surjective, we also have
\(T(S) = S\).

Let \(\hat{\mu}(S) = c > 0\). Let us prove that \(c = 1\) then. Notice that
\[
\frac{1}{\ln 2} \int_{S \setminus I_{a_1 \ldots a_n}} \frac{dx}{1 + x} = \frac{1}{\ln 2} \int_S \frac{d\left(\frac{p_n + p_{n-1}x}{q_n + q_{n-1}x}\right)}{1 + \frac{p_n + p_{n-1}x}{q_n + q_{n-1}x}}
\]
\[
= \frac{1}{\ln 2} \int_S \frac{dx}{(q_n + q_{n-1}x)(q_n + q_{n-1}x + p_n + p_{n-1}x)}
\]
\[
\geq \frac{1}{\ln 2 \cdot q_n(q_n + q_{n-1} + p_n + p_{n-1})} \int_S \frac{dx}{1 + x}
\]
\[
= \frac{1}{\ln 2 \cdot q_n(q_n + q_{n-1} + p_n + p_{n-1})} \hat{\mu}(S)
\]
\[
> \frac{1}{\ln 2(q_n + q_{n-1})(2p_n + 2q_n)} \hat{\mu}(S) > \frac{1}{2} \hat{\mu}(S) \hat{\mu}(I_{a_1 \ldots a_n}).
\]

The last inequality follows from Lemma 7.11. \(\square\)

**Proof of Proposition 7.10.** Let \(S\) be a measurable subset \(S\) such that \(T^{-1}(S) = S\). Suppose also \(\hat{\mu}(S) > 0\).

For any irrational number \(y = [0 : a_1; a_2; \ldots]\) from Lemma 7.12 we have
\[
\frac{{\hat{\mu}(X \setminus S) \cap B(y, \hat{\mu}(I_{a_1 \ldots a_n}))}}{{\hat{\mu}(B(y, \hat{\mu}(I_{a_1 \ldots a_n}))}) < 1 - \frac{1}{2} \frac{{\hat{\mu}(S) \hat{\mu}(I_{a_1 \ldots a_n})}}{{\hat{\mu}(B(y, \hat{\mu}(I_{a_1 \ldots a_n}))}} = 1 - \frac{1}{4} \hat{\mu}(S).
\]

Hence \(y\) is not a \(\hat{\mu}\)-density point of \(X \setminus S\). Therefore, by Corollary 7.6 almost every point of \([0, 1] \setminus \mathbb{Q}\) is not in \(X \setminus S\), and, therefore, it is in \(S\). Hence
\[\hat{\mu}(S) \geq \hat{\mu}([0, 1] \setminus \mathbb{Q}) = 1,\]
Hence $\mu(S) = 1$, this concludes the proof of ergodicity of $T$. \hfill \Box

7.4. **Pointwise Gauss-Kuzmin theorem.** Consider $x$ in the segment $[0, 1]$. Let the ordinary continued fraction for $x$ be $[0; a_1 : \ldots : a_n]$ (odd or infinite). For a positive integer $k$ denote

$$\hat{P}_{n,k}(x) = \frac{\#(k,n)}{n},$$

where $\#(k,n)$ is the number of integer elements $a_i$ equal to $k$ for $i = 1, \ldots, n$. Denote

$$\hat{P}_k(x) = \lim_{n \to \infty} \hat{P}_{n,k}(x).$$

**Theorem 7.13.** For any positive integer $k$ and almost every $x$ (i.e., in the complement to a zero measure set) the following holds

$$\hat{P}_k(x) = \frac{1}{\ln 2} \ln \left( 1 + \frac{1}{k(k+2)} \right).$$

We think of this theorem as of the *pointwise Gauss-Kuzmin theorem*. To prove pointwise Gauss-Kuzmin theorem we use Birkhoff Ergodic theorems.

**Proof.** Consider a subset $S \in \mathcal{I}$. Let $\chi_S$ be the characteristic function of $S$, i.e.,

$$\chi_S(x) = \begin{cases} 1, & \text{if } x \in S, \\ 0, & \text{otherwise}. \end{cases}$$

Then

$$\hat{P}_{n,k}(x) = \frac{1}{n} \sum_{s=0}^{n-1} \chi_{[\frac{1}{k+1}, \frac{1}{k})}(T^s x).$$

Hence by Birkhoff Pointwise Ergodic Theorem the limit $\hat{P}_k(x)$ exists almost everywhere. Since the transformation $T$ is ergodic we apply Birkhoff-Khinchin Ergodic Theorem and get

$$\hat{P}_k(x) = \int_0^1 \chi_{[\frac{1}{k+1}, \frac{1}{k})} d\hat{\mu} = \frac{1}{\ln 2} \int_{1/(k+1)}^{1/k} \frac{dx}{1+x} = \frac{1}{\ln 2} \ln \left( 1 + \frac{1}{k(k+2)} \right).$$

\hfill \Box

7.5. **Original Gauss-Kuzmin theorem.** Let $\alpha$ be some irrational between zero and unity, and let $[0 : a_1; a_2; a_3; \ldots]$ be its ordinary continued fraction.

Let $m_n(x)$ denote the measure of the set of reals $\alpha$ contained in the segment $[0; 1]$, such that $T^n(\alpha) < x$ (here $T$ is Gauss map). In his letters to P. S. Laplace K. F. Gauss formulated without proofs the following theorem. It was further proved by R. O. Kuzmin [?], and then proved one more time by P. Lévy [?].

**Theorem 7.14.** **Gauss-Kuzmin.** For $0 \leq x \leq 1$ the following holds:

$$\lim_{n \to \infty} m_n(x) = \frac{\ln(1 + x)}{\ln 2}.$$
This theorem is technically more complicated, for the proof we refer to the original manuscript of R. O. Kuzmin [?]. (see also in A. Ya. Hinchin [?]).

Denote by $P_n(k)$ for an arbitrary integer $k > 0$ the measure of the set of all reals $\alpha$ of the segment $[0; 1]$, such that each of them has the number $k$ at $n$-th position. A limit $\lim_{n \to \infty} P_n(k)$ is called a frequency of $k$ for ordinary continued fractions and denoted by $P(k)$.

**Corollary 7.15.** For any positive integer $k$ the following holds

$$P(k) = \frac{1}{\ln 2} \ln \left(1 + \frac{1}{k(k+2)}\right).$$

**Proof.** Notice, that $P_n(k) = m_n(\frac{1}{k}) - m_n(\frac{1}{k+1})$. Now the statement of the corollary follows from Gauss-Kuzmin theorem. \qed

7.6. **Cross-ratio in projective geometry.**

7.6.1. **Projective linear group.** The projective linear group is the quotient group

$$\text{PGL}(\mathbb{R}, n) = \text{GL}(\mathbb{R}, n)/\text{Z}(\mathbb{R}, n),$$

where $\text{Z}(\mathbb{R}, n)$ is the one-dimensional subgroup of all nonzero scalar transformations of $\mathbb{R}^n$. One can say that the group $\text{PGL}(\mathbb{R}, n)$ acts on the equivalence classes with of vectors in $\mathbb{R}^n$ with respect to $\text{Z}(\mathbb{R}, n)$, which is

$$\mathbb{R}^n/\text{Z}(\mathbb{R}, n) = \mathbb{RP}^{n-1}.$$

Consider an affine part $\mathbb{R}^{n-1} \subset \mathbb{RP}^{n-1}$. The stabilizer for the affine part is exactly the group $\text{Aff}(\mathbb{R}, n-1)$.

7.6.2. **Cross-ratio, infinitesimal cross ratio.** Consider a line in $\mathbb{R}^2$ with a Euclidean coordinate on it.

**Definition 7.16.** Consider a 4-tuple of points in a line with coordinates $z_1, z_2, z_3,$ and $z_4$. The value

$$\frac{(z_1 - z_3)(z_2 - z_4)}{(z_2 - z_3)(z_1 - z_4)}$$

is called the cross ratio of the 4-tuple.

Cross-ratio of four points is an invariant of projective transformations. It also do not depend on the choice of Euclidean coordinate on the line. Notice that the group $\text{Aff}(\mathbb{R}, 2)$ is a subgroup of projective transformations (it is stabilizer of $\infty$-point), hence cross-ratio is $\text{Aff}(\mathbb{R}, 2)$-invariant.

We are also interested in the infinitesimal cross-ratio: here two vectors are infinitesimally small and therefore the other two are the same:

$$\frac{dx dy}{(x - y)^2}.$$  

Since an infinitesimal cross-ration is in some sense limit point of the cross-rations of 4-tuples of points:

$$x, y, x + \varepsilon dx, y + \varepsilon dy$$
(for ε tending to 0), it is also projective invariant.

7.7. Smooth manifold of continued fractions. Denote the set of all geometric continued fractions by $CF_1$. Consider an arbitrary element of $CF_1$, it is a continued fraction defined by an (unordered) pair of nonparallel lines $(\ell_1, \ell_2)$ passing through an integer points.

Denote the sets of all ordered collections of two independent and dependent straight lines by $FCF_1$ and $\Delta_1$ respectively. We say that $FCF_1$ is a space geometric framed continued fractions. We have:

$$FCF_1 = (\mathbb{RP}^1 \times \mathbb{RP}^1) \setminus \Delta_1 = T^2 \setminus \Delta_1$$

and

$$CF_1 = FCF_1 / \mathbb{Z}_2,$$

where $\mathbb{Z}/2\mathbb{Z}$ is the group transposing the lines in geometric continued fractions. Note, that $FCF_1$ is a 2-fold covering of $CF_1$. We call the map of “forgetting” of the order in the ordered collections the natural projection of the manifold $FCF_1$ to the manifold $CF_1$ and denote it by $p$, $(p : FCF_1 \to CF_1)$.

7.8. Möbius measure on the manifolds of continued fractions. A group $PGL(2, \mathbb{R})$ of transformations of $\mathbb{RP}^1$ takes the set of all straight lines passing through the origin in the plane into itself. Hence, $PGL(2, \mathbb{R})$ naturally acts on $CF_1$ and $FCF_1$. It is clear that the action of $PGL(2, \mathbb{R})$ is transitive, i.e., it takes any (framed) continued fraction to any other. Notice that a stabilizer of any geometric continued fraction is one dimensional.

Definition 7.17. A form on the manifold $CF_1$ (respectively $FCF_1$) is said to be a Möbius form if it is invariant under the action of $PGL(2, \mathbb{R})$.

Proposition 7.18. All Möbius forms of the manifolds $CF_1$ and $FCF_1$ are proportional.

Proof. Transitivity of the action of $PGL(2, \mathbb{R})$ implies that all Möbius forms of the manifolds $CF_1$ and $FCF_1$ are proportional.

Let $\omega$ be some volume form of the manifold $M$. Denote by $\mu_\omega$ a measure of the manifold $M$ that at any open measurable set $S$ contained at the same piece-wise connected component of $M$ is defined by an equality:

$$\mu_\omega(S) = \left\| \int_S \omega \right\|.$$

Definition 7.19. A measure $\mu$ of the manifold $CF_1$ ($FCF_1$) is said to be a Möbius measure if there exists a Möbius form $\omega$ of $CF_1$ ($FCF_1$) such that $\mu = \mu_\omega$.

From Proposition 7.18 we have the following.

Corollary 7.20. Any two Möbius measures are proportional.

Remark 7.21. The projection $p$ takes the Möbius measures of the manifold $FCF_1$ to the Möbius measures of the manifold $CF_1$. That establishes an isomorphism between the spaces of Möbius measures for $CF_1$ and $FCF_1$. Since the manifold of framed continued fractions possesses simpler chart system, all formulae of the work are given for the case of framed continued fractions manifold. To calculate a measure of some set $F$ of the unframed
continued fractions manifold one should: take $p^{-1}(F)$; calculate Möbius measure of the obtained set of the manifold of framed continued fractions; divide the result by 2.

### 7.9. Explicit formulae for the Möbius form

Let us write down Möbius forms of the framed one-dimensional continued fractions manifold $FCF_1$ explicitly in special charts.

Consider a vector space $\mathbb{R}^2$ equipped with standard metrics on it. Let $l$ be an arbitrary straight line in $\mathbb{R}^2$ that does not pass through the origin, let us choose some Euclidean coordinates $O_lX_l$ on it. Denote by $FCF_{1,l}$ a chart of the manifold $FCF_1$ that consists of all ordered pairs of straight lines both intersecting $l$. Let us associate to any point of $FCF_{1,l}$ (i.e. to a collection of two straight lines) coordinates $(x_l, y_l)$, where $x_l$ and $y_l$ are the coordinates on $l$ for the intersections of $l$ with the first and the second straight lines of the collection respectively. Denote by $|v|_l$ the Euclidean length of a vector $v$ in the coordinates $O_lX_lY_l$ of the chart $FCF_{1,l}$. Note that the chart $FCF_{1,l}$ is a space $\mathbb{R} \times \mathbb{R}$ minus its diagonal.

Consider the following form in the chart $FCF_{1,l}$:

$$\omega_l(x_l, y_l) = \frac{dx_l \wedge dy_l}{|x_l - y_l|^2}.$$ 

**Proposition 7.22.** The measure $\mu_{\omega_l}$ coincides with the restriction of some Möbius measures to $FCF_{1,l}$.

**Proof.** Any transformation of the group $PGL(2, \mathbb{R})$ is in the one-to-one correspondence with the set of all projective transformations of the straight line $l$ projectivization. Note that the expression

$$\frac{\Delta x_l \Delta y_l}{|x_l - y_l|^2}$$

is an infinitesimal cross-ratio of four point with coordinates $x_l$, $y_l$, $x_l + \Delta x_l$ and $y_l + \Delta y_l$. Hence the form $\omega_l(x_l, y_l)$ is invariant for the action of transformations (of the everywhere dense set) of the chart $FCF_{1,l}$, that are induced by projective transformations of $l$. Therefore, the measure $\mu_{\omega_l}$ coincides with the restriction of some Möbius measures to $FCF_{1,l}$. \[\square\]

**Corollary 7.23.** A restriction of an arbitrary Möbius measure to the chart $FCF_{1,l}$ is proportional to $\mu_{\omega_l}$.

**Proof.** The statement follows from the proportionality of any two Möbius measures. \[\square\]

Consider now the manifold $FCF_1$ as a set of ordered pairs of distinct points on a circle $\mathbb{R}/\pi\mathbb{Z}$ (this circle is a one-dimensional projective space obtained from unit circle by identifying antipodal points). The doubled angular coordinate $\varphi$ of the circle $\mathbb{R}/\pi\mathbb{Z}$ inducing by the coordinate $x$ of straight line $\mathbb{R}$ naturally defines the coordinates $(\varphi_1, \varphi_2)$ of the manifold $FCF_1$. 
Proposition 7.24. The form \( \omega_l(x_l, y_l) \) is extendable to some form \( \omega_1 \) of \( FCF_1 \). In coordinates \((\varphi_1, \varphi_2)\) the form \( \omega_1 \) can be written as follows:

\[
\omega_1 = \frac{1}{4} \cot^2 \left( \frac{\varphi_1 - \varphi_2}{2} \right) d\varphi_1 \wedge d\varphi_2.
\]

We leave a proof of Proposition 7.24 as an exercise for the reader.

7.10. Relative frequencies of faces of one-dimensional continued fractions. Without loose of generality in this subsection we consider only Möbius form \( \omega_1 \) of Proposition 7.24. Denote the natural projection of the form \( \mu_{\omega_1} \) to the manifold of one-dimensional continued fractions \( CF_1 \) by \( \mu_1 \).

Consider an arbitrary segment \( F \) with vertices at integer points. Denote by \( CF_1(F) \) the set of continued fractions that contain the segment \( F \) as a face.

Definition 7.25. The quantity \( \mu_1(CF_1(F)) \) is called relative frequency of the face \( F \).

Note that the relative frequencies of faces of the same integer-linear type are equivalent. Any face of one-dimensional continued fraction is at unit integer distance from the origin. Thus, integer-linear type of a face is defined by its integer length (the number of inner integer points plus unity). Denote the relative frequency of the edge of integer length \( k \) by \( \mu_1(\ell''k'') \).

Proposition 7.26. For any positive integer \( k \) the following holds:

\[
\mu_1(\ell''k'') = \ln \left( 1 + \frac{1}{k(k+2)} \right).
\]

Proof. Consider a particular representative of an integer-linear type of the length \( k \) segment: the segment with vertices \((0,1)\) and \((k,1)\). One-dimensional continued fraction contains the segment as a face iff one of the straight lines defining the fraction intersects the interval with vertices \((-1,1)\) and \((0,1)\) while the other straight line intersects the interval with vertices \((k,1)\) and \((k+1,1)\), see on Figure 2.

For the straight line \( l \) defined by the equation \( y = 1 \) we calculate the Möbius measure of Cartesian product of the described couple of intervals. By the last subsection it follows...
that this quantity coincides with relative frequency $\mu_1(k'')$. So,

$$
\mu_1(k'') = \int_{-1}^{0} \int_{k}^{k+1} \frac{d\xi d\eta}{(\xi - \eta)^2} = \int_{k}^{k+1} \left( \frac{1}{\eta} - \frac{1}{\eta + 1} \right) d\eta = \\
\ln \left( \frac{1}{k(k+2)} \right) = \ln \left( 1 + \frac{1}{k(k+2)} \right). 
$$

This proves the proposition. \qed

**Remark 7.27.** Note that the argument of the logarithm $\frac{(k+1)(k+1)}{k(k+2)}$ is a cross-ratio of points $(-1, 1), (0, 1), (k, 1),$ and $(k+1, 1)$.

**Corollary 7.28.** Relative frequency $\mu_1(k'')$ up to the factor

$$
\ln 2 = \int_{1}^{0} \int_{-1}^{0} \frac{d\xi d\eta}{(\xi - \eta)^2}
$$

coincides with Gauss-Kuzmin frequency $P(k)$ for $k$ to be an element of continued fraction. \qed

**Exercises.**

[1] (a) Prove that the measure

$$
\mu(S) = \frac{1}{\ln 2} \int_{S} \frac{dx}{1 + x}
$$

is a probability measure on a segment $[0, 1]$, i.e., $\mu([0, 1]) = 1$.
(b) Find $\mu([a, b])$ for $0 \leq a < b \leq 1$, where $\mu$ is as above.

[2] **Ergodicity of the doubling map.** Consider a space $(S^1, \Sigma, \lambda)$, where $X$ is a unite circle, $\Sigma$ is a Borel $\sigma$-algebra, and $\lambda$ is a Lebesgue measure. Consider the doubling map $T : S^1 \to S^1$ such that

$$
T(\varphi) = 2\varphi.
$$

Prove that $T$ is measure preserving and ergodic.

[3] Define the frequencies of subsequences in continued fractions. What is the frequency of the sequence $(1, 2, 3)$.

[4] Prove $\hat{\mu}$-density theorem from Lebesgue density theorem.

[5] Recall that $\Psi_0$ is a subset irrational numbers in $[0, 1]$ whose continued fractions do not contain '1' element. Prove elementary (without using ergodic theorems) that

$$
\hat{\mu}(\Psi_0) = 0.
$$

[6] Prove the projective invariance of cross-ratio and infinitesimal cross-ratio.

[7] Prove that $\mathbb{RP}^1$ is homeomorphic to a circle. Prove that $CF_1$ is homeomorphic to Möbius band.


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