7. Möbius geometry and Gauss-Kuzmin distribution (24 May 2011)

In this section we study a geometrical interpretation of Gauss-Kuzmin distribution of integers as an elements of continued fractions. It turns out that the frequency of a positive integer k in a continued fraction almost everywhere equals to

$$\frac{1}{\ln 2}\ln\left(1+\frac{1}{k(k+2)}\right).$$

This mean for a general real x we have 42% of '1', 17% of '2', 9% of 3, etc. So we show theorems on Gauss-Kuzmin distribution and show how the statistics relates to Möbius geometry.

7.1. **Some information from ergodic theory.** Let X be a set, Σ be a σ-algebra on X and μ be a measure on the elements of Σ. The collection (X,Σ,μ) is called a *measure space.*

Let T be a transformation of a set X then for any μ-integrable function f on X one can define two averages. The *time average* for f at point x is

$$\lim_{n\to\infty}\frac{1}{n}\sum_{k=0}^{n-1} f(T^k x).$$

The *space average* I_f is

$$I_f=\frac{1}{\mu(X)}\int f d\mu.$$

The space average always exists. The time average does not exist for all x, nevertheless in the case we are interested it exists for almost all x. We formulate the related theorem after one important definition.

Definition 7.1. Let (X,Σ,μ) be a measure set. A transformation $T: X\to X$ is *measure preserving* if and only if it is measurable and

$$\mu(T^{-1}(A))=\mu(A)$$

for any set A of Σ.

For measure preserving transformations we have.

Theorem 7.2. (Birkhoff Pointwise Ergodic Theorem.) *Consider a measure space (X,Σ,μ) and a measure preserving transformation T. Let f be μ-integrable function on X. Then the time average converges almost everywhere to an invariant function $\overline{f}$.*

Definition 7.3. Consider a probability measure space (X,Σ,μ). Let T be a measure preserving transformation on the set X. Then T is *ergodic* if for any $X'\in\Sigma$ satisfying $T^{-1}(X')=X'$ either $\mu(X')=0$ or $\mu(X')=1$.

Theorem 7.4. (Birkhoff-Khinchin Ergodic Theorem.) *Consider a probability measure space* (X, Σ, μ) *and a measure preserving transformation* T*. Suppose that* T *is ergodic. Then the values of time average function are equivalent to the space average (i.e.,* $\overline{f}(x) = I_f$*) almost everywhere.* □

7.2. **The measure space related to continued fractions.** In this subsection we define a measure space that is closely related to distributions of the elements of continued fractions. For this measure we formulate a statement on density points for measurable subsets, which we essentially use in the proofs below.

7.2.1. *Definition of the measure space related to continued fractions.* Consider the measure space of a segment $I = \{x | 0 \le x < 1\}$ with the Borel σ-algebra Σ and a measure $\hat{\mu}$ defined on a measurable set S as follows

$$\hat{\mu}(S) = \frac{1}{\ln 2} \int\limits_S \frac{dx}{1+x}.$$

The coefficient $1/\ln 2$ is taken such that the measure of the segment I equals to 1.

7.2.2. *Theorems on density points of measurable subsets.* We start from a classical theorem on Lebesgue measure space. Denote by $B(x, \varepsilon)$ the standard ball of radius ε centered at x.

Theorem 7.5. (Lebesgue density.) *Let* λ *be the n-dimensional Lebesgue measure on* $\mathbb{R}^n$*. If* $A \subset \mathbb{R}^n$ *is a Borel measurable set, then almost every point* $x \in A$ *is a Lebesgue density point:*

$$\lim_{\varepsilon \to 0} \frac{\lambda\big(A \cap B(x, \varepsilon)\big)}{\lambda\big(B(x, \varepsilon)\big)} = 1.$$

□

Here "almost every point" means "except for a zero measure subset".

The measure $\hat{\mu}$ is equivalent to the one-dimensional Lebesgue measure λ on the segment $[0, 1]$ (for more information on measure theory see [*Geometry of Sets and Measures in Euclidean Spaces: Fractals and Rectifiability* by Pertti Mattila]), hence we have the similar statement in the case of measure space $(X, \Sigma, \hat{\mu})$.

Corollary 7.6. ($\hat{\mu}$-density.) *Let* $X = [0, 1]$ *and* $\hat{\mu}$ *be as above. If* $A \subset X$ *is a* $\hat{\mu}$*-measurable set with positive measure* $\hat{\mu}(A)$*, then almost every point in* A *satisfies*

$$\lim_{\varepsilon \to 0} \frac{\hat{\mu}\big(A \cap B(x, \varepsilon)\big)}{\hat{\mu}\big(B(x, \varepsilon)\big)} = 1.$$

□

7.3. **On Gauss map.**

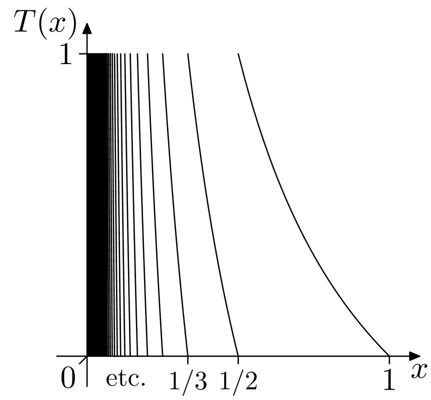

FIGURE 1. Gauss map.

7.3.1. *Gauss map and corresponding invariant measure.* We consider the measure space $(X, \Sigma, \hat{\mu})$ defined in the previous subsection. Define a *Gauss map* T of a segment $[0, 1]$ to itself as follows

$$T(x) = \{1/x\},$$

where $\{r\}$ denotes the quotient part $r - \lfloor r \rfloor$.

Proposition 7.7. *The Gauss map T is a measure preserving for the above $(X, \Sigma, \hat{\mu})$.*

We start with the following lemma.

Lemma 7.8. *Let $x = [0; a_1 : a_2 : \ldots]$. Then*

$$T^{-1}(x) = \{[0; k : a_1 : a_2 : \ldots] | k \in \mathbb{Z}_+\} = \left\{ \frac{1}{x+k} \middle| k \in \mathbb{Z}_+ \right\}.$$

Proof. The first equality follows directly from the fact that

$$T([0; b_1 : b_2 : \ldots]) = [0; b_2 : b_3 \ldots]$$

and the fact that any real number has the unique ordinary continued fraction with the last element not equal to 1.

The second equality is straightforward. □

Proof of Proposition 7.7. Consider a measurable set S. From Lemma 7.8 it follows that

$$\hat{\mu}(T^{-1}(S)) = \frac{1}{\ln 2} \int\limits_{T^{-1}(S)} \frac{dx}{1+x} = \frac{1}{\ln 2} \sum_{k=1}^{\infty} \left(\int\limits_{T^{-1}(S) \cap [\frac{1}{k+1}, \frac{1}{k}]} \frac{dx}{1+x} \right).$$

Notice that on each open segment $[1/k, 1/(k+1)]$ the operator T is one-to one with the open segment $[0, 1]$. Let us denote the inverse function to T on the segment $[1/k, 1/(k+1)]$ by $T^{-1}_{(k)}$. Therefore,

$$T\Big(T^{-1}(S) \cap \Big[\frac{1}{k+1}, \frac{1}{k}\Big]\Big) = T(T^{-1}_{(k)}(S)) = S,$$

and we can apply the rule of differentiation of a composite function. From Lemma 7.8 we know

$$T_{(k)}^{-1}(x) = \frac{1}{x+k}.$$

Then we have

$$\int\limits_{T^{-1}(S)\cap[\frac{1}{k+1},\frac{1}{k}]} \frac{dx}{1+x} = \int\limits_{T_{(k)}^{-1}(S)} \frac{dx}{1+x} = \int\limits_{T(T_{(k)}^{-1}(S))} \frac{dT_{(k)}^{-1}(x)}{1+T_{(k)}^{-1}(x)} = \int\limits_{S} \frac{-d\big(\frac{1}{x+k}\big)}{1+\frac{1}{x+k}}$$
$$= \int\limits_{S} \frac{dx}{(x+k)(x+k+1)}$$

(the minus sign is taken, since the map $T_{(k)}^{-1} : x \to \frac{1}{x+k}$ changes the orientation). So we have

$$\hat{\mu}(T^{-1}(S)) = \frac{1}{\ln 2} \sum_{k=1}^{\infty} \int\limits_{S} \frac{dx}{(x+k)(x+k+1)}.$$

Since the integrated functions under the sign of integration are nonnegative, we can change the order of the sum and the integration operations. We get

$$\hat{\mu}(T^{-1}(S)) = \frac{1}{\ln 2} \int\limits_{S} \Big(\sum_{k=1}^{\infty} \frac{1}{(x+k)(x+k+1)} \Big) dx = \frac{1}{\ln 2} \int\limits_{S} \Big(\sum_{k=1}^{\infty} \Big(\frac{1}{x+k} - \frac{1}{x+k+1} \Big) \Big) dx$$
$$= \frac{1}{\ln 2} \int\limits_{S} \frac{dx}{x+1} = \hat{\mu}(S).$$

So for any measurable set S we have

$$\hat{\mu}(T^{-1}(S)) = \hat{\mu}(S).$$

Therefore, the Gauss map T preserves the measure $\hat{\mu}$. □

7.3.2. *An example of an invariant set for Gauss map.* Let us consider one example of a measurable set which is invariant under the Gauss map.

Denote by Ψ the set of all irrational numbers in the segment $[0, 1]$ whose continued fractions contain only finitely many '1'. It is clear that

$$T^{-1}(\Psi) = \Psi,$$

since operation T^{-1} shifts elements of continued fractions by one and inserts the first element.

Proposition 7.9. *The set* Ψ *is measurable* (*i.e.,* $\Psi \in \Sigma$).

Proof. Denote by Υ_n the set of all irrational numbers that contain an element '1' exactly at place n. Notice that

$$\Upsilon_1 = [1/2, 1],$$

and, therefore, it is measurable. Hence for any n the set

$$\Upsilon_{n+1} = T^n(\Upsilon_1)$$

is measurable.

Denote by Ψ_0 the set of all irrational numbers that do not contain an element '1'. Since

$$\Psi_0 = X \setminus \bigcup_{n=1}^{\infty} \Upsilon_n,$$

the set Ψ_0 is also measurable. Then

$$T^{-n}(\Psi_0)$$

is measurable for any positive integer n. Hence

$$\Psi = \bigcup_{n=1}^{\infty} T^{-n}(\Psi_0)$$

is measurable. □

We will prove later that the Gauss map is ergodic, and, therefore, Ψ is either of zero measure or full measure in X.

7.3.3. *Ergodicity of Gauss map.*

Proposition 7.10. *The Gauss map is ergodic.*

Before to prove Proposition 7.10 we introduce a supplementary notation and prove two lemmas.

For a sequence of positive integers $(a_1, \ldots, a_n)$ denote by $I_{(a_1,\ldots,a_n)}$ the segment with endpoints $[0; a_1 : \ldots : a_{n-1} : a_n]$ and $[0; a_1 : \ldots : a_{n-1} : a_n + 1]$. It is clear that the map

$$T^n : I_{(a_1,\ldots,a_n)} \to [0, 1]$$

is one-to-one on the segment $I_{(a_1,\ldots,a_n)}$ and the inverse to T^n is

$$T^{-1}_{(a_1,\ldots,a_n)} : x \to [0; a_1 : \ldots : a_n : 1/x].$$

In terms of k-convergents $p_k/q_k = [0; a_1 : \ldots : a_k]$ the expression for $T^{-1}_{(a_1,\ldots,a_n)}(x)$ is as follows (see Proposition 1.11):

$$T^{-1}_{(a_1,\ldots,a_n)}(x) = \frac{p_n/x + p_{n-1}}{q_n/x + q_{n-1}} = \frac{p_n + p_{n-1}x}{q_n + q_{n-1}x}.$$

Lemma 7.11. *The measure of a segment $I_{(a_1,\ldots,a_n)}$ satisfy the following*

$$\hat{\mu}(I_{(a_1,\ldots,a_n)}) < \frac{1}{\ln 2 (q_n + q_{n-1})(p_n + q_n)}.$$

Proof. We have

$$\hat{\mu}(I_{(a_1,\ldots,a_n)}) = \frac{1}{\ln 2}\int\limits_{I_{(a_1,\ldots,a_n)}} \frac{dx}{1+x} = \frac{1}{\ln 2}\left|\int\limits_{[0:a_1;\ldots;a_n]}^{[0:a_1;\ldots;a_n;1]} \frac{dx}{1+x}\right| = \frac{1}{\ln 2}\left|\ln\left(\frac{1+\frac{p_n+p_{n-1}}{q_n+q_{n-1}}}{1+\frac{p_n}{q_n}}\right)\right|$$
$$= \frac{1}{\ln 2}\left|\ln\left(1+\frac{1}{(q_n+q_{n-1})(p_n+q_n)}\right)\right| < \frac{1}{\ln 2(q_n+q_{n-1})(p_n+q_n)}.$$

The last inequality follows from the convexity of ln function. □

Lemma 7.12. *For any invariant set S and an interval $I_{(a_1,\ldots,a_n)}$ it holds*

$$\hat{\mu}(S\cap I_{(a_1,\ldots,a_n)}) > \frac{1}{2}\hat{\mu}(S)\hat{\mu}(I_{(a_1,\ldots,a_n)}).$$

Proof. Since the map T is surjective, we also have

$$T(S) = S.$$

Let $\hat{\mu}(S) = c > 0$. Let us prove that $c = 1$ then. Notice that

$$\frac{1}{\ln 2}\int\limits_{S\cap I_{(a_1,\ldots,a_n)}} \frac{dx}{1+x} = \frac{1}{\ln 2}\int\limits_S \frac{d\left(\frac{p_n+p_{n-1}x}{q_n+q_{n-1}x}\right)}{1+\frac{p_n+p_{n-1}x}{q_n+q_{n-1}x}}$$
$$= \frac{1}{\ln 2}\int\limits_S \frac{dx}{(q_n+q_{n-1}x)(q_n+q_{n-1}x+p_n+p_{n-1}x)}$$
$$\geq \frac{1}{\ln 2\cdot q_n(q_n+q_{n-1}+p_n+p_{n-1})}\int\limits_S \frac{dx}{1+x}$$
$$= \frac{1}{\ln 2\cdot q_n(q_n+q_{n-1}+p_n+p_{n-1})}\hat{\mu}(S)$$
$$> \frac{1}{\ln 2(q_n+q_{n-1})(2p_n+2q_n)}\hat{\mu}(S) > \frac{1}{2}\hat{\mu}(S)\hat{\mu}(I_{(a_1,\ldots,a_n)}).$$

The last inequality follows from Lemma 7.11. □

Proof of Proposition 7.10. Let S be a measurable subset S such that $T^{-1}(S) = S$. Suppose also $\hat{\mu}(S) > 0$.

For any irrational number $y = [0 : a_1; a_2; \ldots]$ from Lemma 7.12 we have

$$\frac{\hat{\mu}(X\setminus S)\cap B(y,\hat{\mu}(I_{(a_1,\ldots,a_n)}))}{\hat{\mu}(B(y,\hat{\mu}(I_{(a_1,\ldots,a_n)})))} < 1-\frac{1}{2}\frac{\hat{\mu}(S)\hat{\mu}(I_{(a_1,\ldots,a_n)})}{\hat{\mu}(B(y,\hat{\mu}(I_{(a_1,\ldots,a_n)})))} = 1-\frac{1}{4}\hat{\mu}(S).$$

Hence y is not a $\hat{\mu}$-density point of $X\setminus S$. Therefore, by Corollary 7.6 almost every point of $[0,1]\setminus\mathbb{Q}$ is not in $X\setminus S$, and, therefore, it is in S. Hence

$$\hat{\mu}(S) \geq \hat{\mu}([0,1]\setminus\mathbb{Q}) = 1,$$

Hence $\hat{\mu}(S) = 1$, this concludes the proof of ergodicity of T. □

7.4. **Pointwise Gauss-Kuzmin theorem.** Consider x in the segment $[0, 1]$. Let the ordinary continued fraction for x be $[0; a_1 : \ldots : a_n]$ (odd or infinite). For a positive integer k denote

$$\hat{P}_{n,k}(x) = \frac{\#(k, n)}{n},$$

where $\#(k, n)$ is the number of integer elements a_i equal to k for $i = 1, \ldots, n$. Denote

$$\hat{P}_k(x) = \lim_{n\to\infty} \hat{P}_{n,k}(x).$$

Theorem 7.13. *For any positive integer k and almost every x (i.e., in the complement to a zero measure set) the following holds*

$$\hat{P}_k(x) = \frac{1}{\ln 2} \ln\left(1 + \frac{1}{k(k+2)}\right).$$

We think of this theorem as of the *pointwise Gauss-Kuzmin theorem.* To prove pointwise Gauss-Kuzmin theorem we use Birkhoff Ergodic theorems.

Proof. Consider a subset $S \in I$. Let χ_S be the *characteristic function* of S, i.e.,

$$\chi_S(x) = \begin{cases} 1, & \text{if } x \in S, \\ 0, & \text{otherwise.} \end{cases}$$

Then

$$\hat{P}n, k(x) = \frac{1}{n} \sum_{s=0}^{n-1} \chi_{[\frac{1}{k+1}, \frac{1}{k}]}(T^s x).$$

Hence by Birkhoff Pointwise Ergodic Theorem the limit $\hat{P}_k(x)$ exists almost everywhere. Since the transformation T is ergodic we apply Birkhoff-Khinchin Ergodic Theorem and get

$$\hat{P}_k(x) = \int_0^1 \chi_{[\frac{1}{k+1}, \frac{1}{k}]} d\hat{\mu} = \frac{1}{\ln 2} \int_{1/(k+1)}^{1/k} \frac{dx}{1+x} = \frac{1}{\ln 2} \ln\left(1 + \frac{1}{k(k+2)}\right).$$

□

7.5. **Original Gauss-Kuzmin theorem.** Let α be some irrational between zero and unity, and let $[0 : a_1; a_2; a_3; \ldots]$ be its ordinary continued fraction.

Let $m_n(x)$ denote the measure of the set of reals α contained in the segment $[0; 1]$, such that $T^n(\alpha) < x$ (here T is Gauss map). In his letters to P. S. Laplace K. F. Gauss formulated without proofs the following theorem. It was further proved by R. O. Kuzmin [**?**], and then proved one more time by P. Lévy [**?**].

Theorem 7.14. Gauss-Kuzmin. *For $0 \le x \le 1$ the following holds:*

$$\lim_{n\to\infty} m_n(x) = \frac{\ln(1 + x)}{\ln 2}.$$

□

This theorem is technically more complicated, for the proof we refer to the original manuscript of R. O. Kuzmin [**?**]. (see also in A. Ya. Hinchin [**?**]).

Denote by $P_n(k)$ for an arbitrary integer $k > 0$ the measure of the set of all reals α of the segment $[0;1]$, such that each of them has the number k at n-th position. A limit $\lim\limits_{n\to\infty} P_n(k)$ is called a *frequency of* k for ordinary continued fractions and denoted by $P(k)$.

Corollary 7.15. *For any positive integer k the following holds*

$$P(k) = \frac{1}{\ln 2} \ln\left(1 + \frac{1}{k(k+2)}\right).$$

Proof. Notice, that $P_n(k) = m_n(\frac{1}{k}) - m_n(\frac{1}{k+1})$. Now the statement of the corollary follows from Gauss-Kuzmin theorem. □

7.6. **Cross-ration in projective geometry.**

7.6.1. *Projective linear group.* The *projective linear group* is the quotient group

$$PGL(\mathbb{R}, n) = GL(\mathbb{R}, n)/Z(\mathbb{R}, n),$$

where $Z(\mathbb{R}, n)$ is the one-dimensional subgroup of all nonzero scalar transformations of $\mathbb{R}^n$. One can say that the group $PGL(\mathbb{R}, n)$ acts on the equivalence classes with of vectors in $\mathbb{R}^n$ with respect to $Z(\mathbb{R}, n)$, which is

$$\mathbb{R}^n/Z(\mathbb{R}, n) = \mathbb{R}P^{n-1}.$$

Consider an affine part $\mathbb{R}^{n-1} \subset \mathbb{R}P^{n-1}$. The stabilizer for the affine part is exactly the group $\mathrm{Aff}(\mathbb{R}, n-1)$.

7.6.2. *Cross-ratio, infinitesimal cross ratio.* Consider a line in $\mathbb{R}^2$ with a Euclidean coordinate on it.

Definition 7.16. Consider a 4-tuple of points in a line with coordinates z_1, z_2, z_3, and z_4. The value

$$\frac{(z_1 - z_3)(z_2 - z_4)}{(z_2 - z_3)(z_1 - z_4)}$$

is called the *cross ratio* of the 4-tuple.

Cross-ration of four points is an invariant of projective transformations. It also do not depend on the choice of Euclidean coordinate on the line. Notice that the group $\mathrm{Aff}(2, \mathbb{R})$ is a subgroup of projective transformations (it is stabilizer of ∞-point), hence cross-ration is $\mathrm{Aff}(2, \mathbb{R})$-invariant.

We are also interested in the *infinitesimal cross-ratio*: here two vectors are infinitesimally small and therefore the other two are the same:

$$\frac{dxdy}{(x-y)^2}.$$

Since an infinitesimal cross-ration is in some sense limit point of the cross-rations of 4-tuples of points:

$$x, \quad ,y, \quad x + \varepsilon dx, \quad y + \varepsilon dy$$

(for ε tending to 0), it is also projective invariant.

7.7. **Smooth manifold of continued fractions.** Denote the set of all geometric continued fractions by CF_1. Consider an arbitrary element of CF_1, it is a continued fraction defined by an (unordered) pair of nonparallel lines (ℓ_1, ℓ_2) passing through an integer points.

Denote the sets of all ordered collections of two independent and dependent straight lines by FCF_1 and Δ_1 respectively. We say that FCF_1 is a space geometric *framed continued fractions*. We have:

$$FCF_1 = \big(\mathbb{R}P^1 \times \mathbb{R}P^1\big) \setminus \Delta_1 = T^2 \setminus \Delta_1 \qquad \text{and} \qquad CF_1 = FCF_1/\mathbb{Z}_2,$$

where $\mathbb{Z}/2\mathbb{Z}$ is the group transposing the lines in geometric continued fractions. Note, that FCF_1 is a 2-fold covering of CF_1. We call the map of "forgetting" of the order in the ordered collections the *natural projection* of the manifold FCF_1 to the manifold CF_1 and denote it by p, $(p : FCF_1 \to CF_1)$.

7.8. **Möbius measure on the manifolds of continued fractions.** A group $PGL(2,\mathbb{R})$ of transformations of $\mathbb{R}P^1$ takes the set of all straight lines passing through the origin in the plane into itself. Hence, $PGL(2,\mathbb{R})$ naturally acts on CF_1 and FCF_1. It is clear that the action of $PGL(2,\mathbb{R})$ is transitive, i. e., it takes any (framed) continued fraction to any other. Notice that a stabilizer of any geometric continued fraction is one dimensional.

Definition 7.17. A form on the manifold CF_1 (respectively FCF_1) is said to be a *Möbius form* if it is invariant under the action of $PGL(2,\mathbb{R})$.

Proposition 7.18. *All Möbius forms of the manifolds CF_1 and FCF_1 are proportional.*

Proof. Transitivity of the action of $PGL(2,\mathbb{R})$ implies that all Möbius forms of the manifolds CF_1 and FCF_1 are proportional. □

Let ω be some volume form of the manifold M. Denote by μ_ω a measure of the manifold M that at any open measurable set S contained at the same piece-wise connected component of M is defined by an equality:

$$\mu_\omega(S) = \left| \int_S \omega \right|.$$

Definition 7.19. A measure μ of the manifold CF_1 (FCF_1) is said to be a *Möbius measure* if there exists a Möbius form ω of CF_1 (FCF_1) such that $\mu = \mu_\omega$.

From Proposition 7.18 we have the following.

Corollary 7.20. *Any two Möbius measures are proportional.* □

Remark 7.21. The projection p takes the Möbius measures of the manifold FCF_1 to the Möbius measures of the manifold CF_1. That establishes an isomorphism between the spaces of Möbius measures for CF_1 and FCF_1. Since the manifold of framed continued fractions possesses simpler chart system, all formulae of the work are given for the case of framed continued fractions manifold. To calculate a measure of some set F of the unframed continued fractions manifold one should: take $p^{-1}(F)$; calculate Möbius measure of the obtained set of the manifold of framed continued fractions; divide the result by 2.

7.9. **Explicit formulae for the Möbius form.** Let us write down Möbius forms of the framed one-dimensional continued fractions manifold FCF_1 explicitly in special charts.

Consider a vector space $\mathbb{R}^2$ equipped with standard metrics on it. Let l be an arbitrary straight line in $\mathbb{R}^2$ that does not pass through the origin, let us choose some Euclidean coordinates O_lX_l on it. Denote by $FCF_{1,l}$ a chart of the manifold FCF_1 that consists of all ordered pairs of straight lines both intersecting l. Let us associate to any point of $FCF_{1,l}$ (i. e. to a collection of two straight lines) coordinates (x_l, y_l), where x_l and y_l are the coordinates on l for the intersections of l with the first and the second straight lines of the collection respectively. Denote by $|\overline{v}|_l$ the Euclidean length of a vector $\overline{v}$ in the coordinates $O_lX_lY_l$ of the chart $FCF_{1,l}$. Note that the chart $FCF_{1,l}$ is a space $\mathbb{R}\times\mathbb{R}$ minus its diagonal.

Consider the following form in the chart $FCF_{1,l}$:

$$\omega_l(x_l, y_l) = \frac{dx_l \wedge dy_l}{|x_l - y_l|_l^2}.$$

Proposition 7.22. *The measure μ_{ω_l} coincides with the restriction of some Möbius measures to $FCF_{1,l}$.*

Proof. Any transformation of the group $PGL(2,\mathbb{R})$ is in the one-to-one correspondence with the set of all projective transformations of the straight line l projectivization. Note that the expression

$$\frac{\Delta x_l \Delta y_l}{|x_l - y_l|_l^2}$$

is an infinitesimal cross-ratio of four point with coordinates x_l, y_l, $x_l{+}\Delta x_l$ and $y_l{+}\Delta y_l$. Hence the form $\omega_l(x_l, y_l)$ is invariant for the action of transformations (of the everywhere dense set) of the chart $FCF_{1,l}$, that are induced by projective transformations of l. Therefore, the measure μ_{ω_l} coincides with the restriction of some Möbius measures to $FCF_{1,l}$. □

Corollary 7.23. *A restriction of an arbitrary Möbius measure to the chart $FCF_{1,l}$ is proportional to μ_{ω_l}.*

Proof. The statement follows from the proportionality of any two Möbius measures. □

Consider now the manifold FCF_1 as a set of ordered pairs of distinct points on a circle $\mathbb{R}/\pi\mathbb{Z}$ (this circle is a one-dimensional projective space obtained from unit circle by identifying antipodal points). The doubled angular coordinate φ of the circle $\mathbb{R}/\pi\mathbb{Z}$ inducing by the coordinate x of straight line $\mathbb{R}$ naturally defines the coordinates (φ_1, φ_2) of the manifold FCF_1.

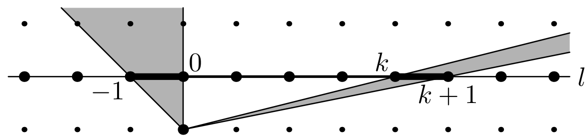

Figure 2. Rays defining a continued fraction should lie in the domain colored in gray.

Proposition 7.24. *The form $\omega_l(x_l, y_l)$ is extendable to some form ω_1 of FCF_1. In coordinates (φ_1, φ_2) the form ω_1 can be written as follows:*

$$\omega_1 = \frac{1}{4}\cot^2\left(\frac{\varphi_1 - \varphi_2}{2}\right) d\varphi_1 \wedge d\varphi_2.$$

We leave a proof of Proposition 7.24 as an exercise for the reader.

7.10. **Relative frequencies of faces of one-dimensional continued fractions.** Without loose of generality in this subsection we consider only Möbius form ω_1 of Proposition 7.24. Denote the natural projection of the form μ_{ω_1} to the manifold of one-dimensional continued fractions CF_1 by μ_1.

Consider an arbitrary segment F with vertices at integer points. Denote by $CF_1(F)$ the set of continued fractions that contain the segment F as a face.

Definition 7.25. The quantity $\mu_1(CF_1(F))$ is called *relative frequency* of the face F.

Note that the relative frequencies of faces of the same integer-linear type are equivalent. Any face of one-dimensional continued fraction is at unit integer distance from the origin. Thus, integer-linear type of a face is defined by its integer length (the number of inner integer points plus unity). Denote the relative frequency of the edge of integer length k by $\mu_1(''k'')$.

Proposition 7.26. *For any positive integer k the following holds:*

$$\mu_1(''k'') = \ln\left(1 + \frac{1}{k(k+2)}\right).$$

Proof. Consider a particular representative of an integer-linear type of the length k segment: the segment with vertices $(0,1)$ and $(k,1)$. One-dimensional continued fraction contains the segment as a face iff one of the straight lines defining the fraction intersects the interval with vertices $(-1,1)$ and $(0,1)$ while the other straight line intersects the interval with vertices $(k,1)$ and $(k{+}1,1)$, see on Figure 2.

For the straight line l defined by the equation $y = 1$ we calculate the Möbius measure of Cartesian product of the described couple of intervals. By the last subsection it follows that this quantity coincides with relative frequency $\mu_1(''k'')$. So,

$$\mu_1(''k'') = \int_{-1}^{0}\int_{k}^{k+1} \frac{dx_l dy_l}{(x_l-y_l)^2} = \int_{k}^{k+1}\left(\frac{1}{y_l}-\frac{1}{y_l+1}\right)dy_l =$$
$$\ln\left(\frac{(k+1)(k+1)}{k(k+2)}\right) = \ln\left(1+\frac{1}{k(k+2)}\right).$$

This proves the proposition. □

Remark 7.27. Note that the argument of the logarithm $\frac{(k+1)(k+1)}{k(k+2)}$ is a cross-ratio of points $(-1,1)$, $(0,1)$, $(k,1)$, and $(k+1,1)$.

Corollary 7.28. *Relative frequency $\mu_1(''k'')$ up to the factor*

$$\ln 2 = \int_{-1}^{0}\int_{1}^{+\infty} \frac{dx_l dy_l}{(x_l-y_l)^2}$$

coincides with Gauss-Kuzmin frequency $P(k)$ for k to be an element of continued fraction. □

Exercises.

[1] **(a)** Prove that the measure
$$\mu(S) = \frac{1}{\ln 2}\int_S \frac{dx}{1+x}$$
is a probability measure on a segment $[0,1]$, i.e., $\mu([0,1]) = 1$.
(b) Find $\mu([a,b])$ for $0 \le a < b \le 1$, where μ is as above.

[2] **Ergodicity of the doubling map.** Consider a space (S^1, Σ, λ), where X is a unite circle, Σ is a Borel σ-algebra, and λ is a Lebesgue measure. Consider the doubling map $T : S^1 \to S^1$ such that
$$T(\varphi) = 2\varphi.$$
Prove that T is measure preserving and ergodic.

[3] Define the frequencies of subsequences in continued fractions. What is the frequency of the sequence $(1,2,3)$.

[4] Prove $\hat{\mu}$-density theorem from Lebesgue density theorem.

[5] Recall that Ψ_0 is a subset irrational numbers in $[0,1]$ whose continued fractions do not contain '1' element. Prove elementary (without using ergodic theorems) that
$$\hat{\mu}(\Psi_0) = 0.$$

[6] Prove the projective invariance of cross-ratio and infinitesimal cross-ratio.

[7] Prove that $\mathbb{R}P^1$ is homeomorphic to a circle. Prove that CF_1 is homeomorphic to Möbius band.

[8] Prove Proposition 7.24.

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