9. Continued fractions and the second Kepler law (31 May 2011)

9.1. Continued fractions with arbitrary coefficients. In this section we generalize geometry of ordinary continued fractions to the case of continued fractions with arbitrary elements. We show a relation between odd or infinite continued fractions and broken lines in the plane having a selected point (say, the origin). We conclude this section with a few words about conditions for a broken line to be closed in terms of elements of the corresponding continued fraction.

Further we use the following notation. For a couple of vectors v and w denote by $|v \times w|$ the oriented volume of the parallelogram spanned by the vectors v and w.

9.1.1. *Construction of broken lines from the elements of continued fractions.* In this subsection we give a natural geometric interpretation of an odd or infinite continued fraction with arbitrary elements. It would be a broken line defined by the positions of the first vertex and the selected point O, direction of the first edge, and the continued fraction.

So consider a continued fraction $[a_0; a_1 : \ldots : a_{2n}]$. We are also given by the vertex A_0, selected point O, and the direction v of the first edge. We construct all the rest vertices A_k inductively in k.

Base of induction. For the second vertex we take

$$A_1 = A_0 + \lambda v,$$

where λ is defined from the equation $|OA_0 \times OA_1| = a_0$.

Step of induction. Suppose now we have the points $A_0, \ldots, A_k$, for $k \geq 1$, Let us get A_{k+1}. Consider a point

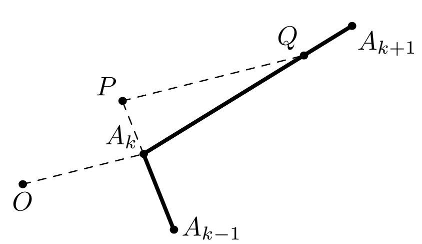
$$P = A_k + \frac{1}{a_{2k-2}} A_{k-1}A_k.$$

In other words P is a point in the line $A_{k-1}A_k$ such that the area OA_kP equals 1. Let

$$Q = P + a_{2k-1}OA_k.$$

Finally the point A_{k+1} is defined as follows (see on Figure 1)

$$A_{k+1} = A_k + a_{2k}A_kQ.$$



$$PQ = a_{2k}OA_k;$$
$$A_kA_{k+1} = a_{2k+1}A_kQ.$$

Figure 1. Construction of A_{k+1}.

Let us now explain a geometric meaning of the elements of continued fractions in terms of characteristics of the corresponding broken line.

Proposition 9.1. *The following holds*

$$a_{2k} = |OA_k \times OA_{k+1}|, \quad k = 0, \dots, n,$$
$$a_{2k-1} = \frac{|A_kA_{k-1} \times A_kA_{k+1}|}{a_{2k-2}a_{2k}}, \quad k = 1, \dots, n.$$

Proof. We prove this statement by induction in k.

Base of induction. From the definition of the point A_1 we get

$$|OA_0 \times OA_1| = a_0.$$

Step of induction. Let the statement be true for $k-1$, we prove it for k.

First, we verify the formula for a_{2k}:

$$|OA_k \times OA_{k+1}| = a_{2k}|OA_k \times OQ| = a_{2k}|OA_k \times OP| = \frac{a_{2k}}{a_{2k-2}}|OA_{k-1} \times OA_k| = a_{2k}.$$

The last equality holds by induction.

Second, for a_{2k-1} we have

$$\frac{|A_kA_{k-1} \times A_kA_{k+1}|}{a_{2k-2}a_{2k}} = \frac{|A_kA_{k-1} \times A_kQ|}{a_{2k-2}} = |PA_k \times PQ| = a_{2k-1}|OA_k \times OP| =$$
$$\frac{a_{2k-1}}{a_{2k-2}}|OA_{k-1} \times OA_k| = a_{2k-1}.$$

The step of induction is completed. □

Example 9.2. Let us construct a broken line having the first vector $A_0 = (1, 0)$, the direction $v = (0, 1)$, and the continued fraction $[a_0; a_1 : a_2]$, here we take an origin as a selected point O. Then we have

$$A_1 = (1, a_0).$$

Further we find the corresponding points P and Q:

$$P = (1, 1 + a_0), \qquad Q = (1 + a_1, 1 + a_0 + a_0a_1).$$

Finally we get

$$A_2 = (1 + a_1a_2, a_0 + a_2 + a_0a_1a_2).$$

9.1.2. *Inverse problem.* Now suppose we have a point O and a broken line $A_0 \dots A_n$ such that for any k the points O, A_k, and A_{k+1} are not in a line. Let us extend the definition of the LLS-sequence for this data.

We use equalities of Proposition 9.1 to define the elements:

$$a_{2k} = |OA_k \times OA_{k+1}|, \quad k = 0, \dots, n;$$
$$a_{2k-1} = \frac{|A_kA_{k-1} \times A_kA_{k+1}|}{a_{2k-2}a_{2k}}, \quad k = 1, \dots, n.$$

We call the sequence $(a_0, \dots, a_{2n})$ the *LLS-sequence* of the broken line with respect to the point O, and $[a_0; \dots : a_{2n}]$ — the *corresponding continued fraction.*

Proposition 9.3. *Let $A_0 \dots A_n$ and $B_0 \dots B_n$ be two broken lines with LLS-sequences $(a_0, \dots, a_{2n})$ and $(b_0, \dots, b_{2n})$ respectively. Suppose the first broken line is taken to the second by some operator in $SL(2, \mathbb{R})$ with determinant equals λ. Then we have:*

$$\begin{cases} a_{2k} = \lambda b_{2k}, & k = 0, \dots, n \\ a_{2k-1} = \frac{1}{\lambda} b_{2k-1}, & k = 1, \dots, n \end{cases}.$$

Proof. The volume of any parallelogram is multiplied by λ, then the statement follows directly from formulas of Proposition 9.1. □

9.1.3. *On geometric meaning of corresponding continued fractions.* For this subsection we fix the point O to be at the origin.

Consider a continued fraction $[a_0; a_1 : \dots : a_k]$ as a rational function in variables $a_0, \dots, a_k$. This rational function is a ratio of two relatively prime polynomials with non-negative integer coefficients, denote them by P_k and Q_k. Actually the polynomials P_k and Q_k are uniquely defined (up to a symmetry) by the condition

$$\frac{P_k(x_0, \dots, x_k)}{Q_k(x_0, \dots, x_k)} = [x_0; x_1 : \dots : x_k], \quad \text{and} \quad P_k(0, \dots, 0) + Q_k(0, \dots, 0) = 1.$$

(see more detailed in Lecture 1).

Theorem 9.4. *Let $A_0 \dots A_n$ be a broken line such that $A_0 = (1, 0)$, and $A_1 = (1, a_0)$. Suppose its LLS-sequence is $(a_0, a_1, \dots, a_{2n})$. Then*

$$A_n = \big(Q_{2n+1}(a_0, a_1, \dots, a_{2n}), P_{2n+1}(a_0, a_1, \dots, a_{2n})\big).$$

Proof. We prove this statement by induction in n.

Base of induction. If the broken line is a segment A_0A_1 with LLS-sequence (a_0) then $A_1 = (1, a_0)$.

Step of induction. Suppose the statement holds for all broken lines with k vertices, let us prove it for an arbitrary broken line with $k+1$ vertex.

Consider a broken line $A_0 \dots A_k$ with LLS-sequence $(a_0, \dots, a_{2k})$. Let us apply a linear transformation with unit determinant taking A_1 to $(1, 0)$ and the line A_2A_1 to the line $x = 1$. This transformation is uniquely defined by all these conditions, it is

$$T = \begin{pmatrix} a_0a_1 + 1 & -a_1 \\ -a_0 & 1 \end{pmatrix}.$$

Denote the resulting broken line by $B_0B_1 \dots B_k$. By Proposition 9.3 all the elements of the LLS-sequence for $B_0B_1 \dots B_k$ are the same. By the assumption of induction we have

$$B_k = \big(Q_{2k-1}(a_2, \dots, a_{2k}), P_{2k-1}(a_2, \dots, a_{2k})\big).$$

Denote the coordinates of B_k by q and p respectively. Then we have

$$A_k = T^{-1}(B_k) = \big(p + a_1q, a_0p + (a_0a_1 + 1)q\big).$$

The polynomials satisfy

$$\frac{a_0p+(a_0a_1+1)q}{p+a_1q}=a_0+\frac{1}{a_1+\frac{p}{q}}=\frac{P_{2k+1}(a_0,a_1,\ldots,a_{2k})}{Q_{2k+1}(a_0,a_1,\ldots,a_{2k})}.$$

Notice that the polynomial $a_0p+(a_0a_1+1)q$ has a unit coefficient in the monomial $a_0a_1\cdot\ldots\cdot a_{2k}$ coming from a_0a_1q. Therefore (see Remark **??**), $a_0p+(a_0a_1+1)q$ coincides with $P_{2k+1}(a_0,a_1,\ldots,a_{2k})$ and $(p+a_1q)$ coincides with $Q_{2k+1}(a_0,a_1,\ldots,a_{2k})$. So we are done with the step of induction. This concludes the proof of the theorem. □

In particular we get the following corollary. In the classical case it forms the basis of geometry of ordinary continued fractions.

Corollary 9.5. *Let $A_0\ldots A_n$ be a broken line such that $A_0=(1,0)$, and $A_1=(1,a_0)$. Suppose that the corresponding continued fraction is $\alpha=[a_0;a_1:\ldots:a_{2n}]$ and $A_n=(x,y)$. Then*

$$\frac{y}{x}=\alpha.$$

(*If the corresponding continued fraction has an infinite value, then $x/y=0$.*) □

Remark 9.6. If the numbers $a_0,a_1,\ldots,a_{2n}$ in Corollary 9.5 are positive integers, then the broken line $A_0\ldots A_n$ coincides with the sail of C_α. In addition the LLS-sequence of the sail for A_0OA_n coincides with the LLS-sequence broken line $A_0\ldots A_n$.

Corollary 9.5 implies the following statement.

Corollary 9.7. *Let $A_0\ldots A_n$ and $B_0\ldots B_m$ be two broken lines with $B_0=A_0$, such that the vectors A_0A_1 and B_0B_1 either have the same direction if $a_0/b_0>0$ or opposite otherwise. Suppose the corresponding continued fractions coincide:*

$$[a_0;\ldots:a_{2n}]=[b_0;\ldots:b_{2m}].$$

Then the points A_n, B_m, and the origin O are in a line.

Proof. Consider an $SL(2,\mathbb{R})$-operator taking A_0 to $(1,0)$ and A_1 to the line $x=1$. By Proposition 9.3 the continued fractions for both broken lines are not changed. Hence by Corollary 9.4 the points A_n, B_m, and the origin are in a line. □

Remark 9.8. (*On closed broken line.*) How to find that a certain continued fraction defines a closed broken line? From Theorem 9.4 we see that a broken line defined by an LLS-sequence $(a_0,a_1,\ldots,a_{2n})$ with $A_0=(1,0)$ and A_0A_1 being collinear to $(0,1)$ is closed if and only if

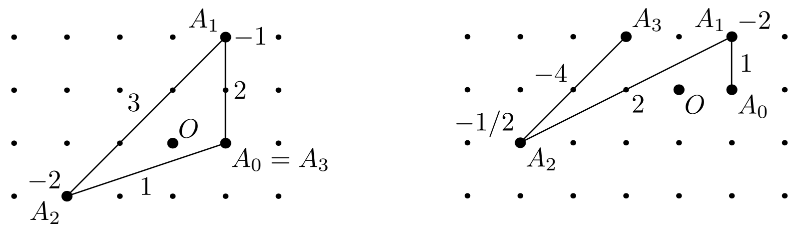
$$Q_{2n+1}(a_0,a_1,\ldots,a_{2n})=1 \quad \text{and} \quad P_{2n+1}(a_0,a_1,\ldots,a_{2n})=0.$$

So, *these two polynomial conditions on the elements of the LLS-sequence are necessary and sufficient conditions for the broken line to be closed.*

Notice that the condition $P_{2n+1}=0$ can be rewritten in the following nice form

$$[a_0;a_1:\ldots:a_{2n}]=0.$$

The condition $P_{2n+1}=0$ was introduced in [**?**] for certain broken-lines with integer vertices.



$[2; -1 : 3 : -2 : 1] = \frac{0}{1}$ $\qquad$ $[1; -2 : 2 : -1/2 : -4] = \frac{-1}{1}$

FIGURE 2. Examples of broken lines and their continued fractions.

Example 9.9. Let us study an example of broken lines consisting of three edges. These curves are defined by continued fractions of type $[a_0; a_1 : a_2 : a_3 : a_4]$. Then the conditions for a broken line to form a triangle are as follows:

$$\left\{ \begin{array}{rcl} a_0a_1a_2a_3a_4 + a_0a_1a_2 + a_0a_1a_4 + a_0a_3a_4 + a_2a_3a_4 + a_0 + a_2 + a_4 & = & 0 \\ a_1a_2a_3a_4 + a_1a_2 + a_1a_4 + a_3a_4 + 1 & = & 1 \end{array} \right. .$$

(See on Figure 2.)

There is one problem which is interesting in the frames of this section. Suppose we have a broken line and two distinct points O_1 and O_2. Then we have two LLS-sequences for the same curve with respect to O_1 and O_2. *Study the conditions on the initial data* (*i.e., LLS-sequences, positions of the first points of the broken lines, and direction of the first vector*) *that define congruent broken lines.*

9.2. **Differentiable curves.** Now let us study what happens if we consider a curve as a broken line with infinitesimally small segments. It turns out that the LLS-sequence "splits" to a couple of functions which we call *areal and angular densities*. We introduce the necessary notions and discuss basic properties of these functions. In particular we show that the areal density is inverse to a velocity of a point defined by the second Kepler law.

In this section we suppose that the curves has a natural (unit length) parametrization.

9.2.1. *Definition of areal and angular densities.* Consider a curve γ of class C^2 with an arc-length parameter t. Let us define the areal and the angular elements at a point similar to the discrete case.

Definition 9.10. The *areal density* and the *angular density* at t are respectively

$$A(t) = \lim_{\varepsilon \to 0} \frac{|O\gamma(t) \times O\gamma(t+\varepsilon)|}{\varepsilon} = |O\gamma(t) \times \dot{\gamma}(t)|$$

and

$$B(t) = \lim_{\varepsilon \to 0} \frac{|\gamma(t)\gamma(t-\varepsilon) \times \gamma(t)\gamma(t+\varepsilon)|}{\varepsilon |O\gamma(t-\varepsilon) \times O\gamma(t)||O\gamma(t) \times O\gamma(t+\varepsilon)|}.$$

Let us give geometric interpretations for the functions A and B. We start with A.

Proposition 9.11. (Relation with the second Kepler law.) *Suppose that a body moves by a trajectory of a curve γ with velocity $1/A$. Then the sector area velocity of a body is constant and equals 1.*

Proof. The proof follows directly from the definition. $\square$

Instead of giving a geometrical interpretation of B, we prove the following formula for A^2B. For a given curve γ denote by $\kappa(t)$ the signed curvature at point t.

Proposition 9.12. *Consider a point $\gamma(t)$ of a curve γ. Let the vectors $O\gamma(t)$ and $\dot\gamma(t)$ be non-collinear. Then the following holds.*

$$A^2(t)B(t) = \kappa(t).$$

Proof. We have the following

$$\begin{aligned} A^2(t)B(t) &= \lim_{\varepsilon\to 0}\left(\left(\frac{|O\gamma(t)\times O\gamma(t+\varepsilon)|}{\varepsilon}\right)^2 \frac{|\gamma(t)\gamma(t-\varepsilon)\times\gamma(t)\gamma(t+\varepsilon)|}{\varepsilon|O\gamma(t-\varepsilon)\times O\gamma(t)||O\gamma(t)\times O\gamma(t+\varepsilon)|}\right) \\ &= \lim_{\varepsilon\to 0}\frac{|\gamma(t)\gamma(t-\varepsilon)\times\gamma(t)\gamma(t+\varepsilon)|}{\varepsilon^3}. \end{aligned}$$

Write our curve as

$$\gamma(t) = (x(t), y(t)).$$

Since we are using the arclength parametrization, we know that

$$|\gamma'(t)| = \sqrt{(x'(t))^2 + (y'(t))^2} = 1.$$

From multivariate calculus, the signed curvature of the curve at $\gamma(t)$ is

$$\kappa(t) = x'y'' - y'x''.$$

Via Taylor series expansions, we know that

$$\begin{aligned} \gamma(t+\varepsilon) &= \gamma(t) + \varepsilon\gamma'(t) + \frac{\varepsilon^2}{2}\gamma''(t) + \text{higher order terms}, \\ \gamma(t-\varepsilon) &= \gamma(t) - \varepsilon\gamma'(t) + \frac{\varepsilon^2}{2}\gamma''(t) + \text{higher order terms}. \end{aligned}$$

The vector from $\gamma(t)$ to $\gamma(t+\varepsilon)$ is

$$\left(\varepsilon x' + \frac{\varepsilon^2}{2}x'', \varepsilon y' + \frac{\varepsilon^2}{2}y''\right) + \text{higher order terms}$$

and the vector from $\gamma(t)$ to $\gamma(t-\varepsilon)$ is

$$\left(-\varepsilon x' + \frac{\varepsilon^2}{2}x'', -\varepsilon y' + \frac{\varepsilon^2}{2}y''\right) + \text{higher order terms}.$$

By direct calculation, we have

$$\lim_{\varepsilon\to 0}\frac{|\gamma(t)\gamma(t-\varepsilon)\times\gamma(t)\gamma(t+\varepsilon)|}{\varepsilon^3}=\lim_{\varepsilon\to 0}\frac{\varepsilon^3(x'y''-y'x'')+\text{4th order terms}}{\varepsilon^3}$$
$$=\lim_{\varepsilon\to 0}\frac{\varepsilon^3\kappa(t)+\text{4th order terms}}{\varepsilon^3}$$
$$=\kappa(t).$$

Hence, $A^2(t)B(t)=\kappa(t)$. □

Now we prove the theorem on finite reconstruction of an arclength parameterized curve (i.e., in some small neighborhood) knowing the areal density and a starting point. This is analogous to the algorithm that finds a broken line by the elements of the corresponding continued fraction described in Subsection 9.1.1. The significant difference to the discrete case is that we do not need to know the angular distribution function.

Theorem 9.13. *Suppose we are given by the points O and $\gamma(t_0)$ and the areal density $A(t)$, smoothly depending on t in a neighborhood of t_0.*
— If $|A(t_0)|>|O\gamma(t_0)|$, then there is no finite curve with the given data.
— If $|O\gamma(t_0)|>|A(t_0)|>0$, then the curve is uniquely defined in some neighborhood of the point $\gamma(t_0)$.

Remark 9.14. Notice that $A^2(t)B(t)$ defines the oriented curvature. Therefore, if one knows the functions A and B then the curve is uniquely reconstructed until the time t_0 where the vectors $O\gamma(t_0)$ and $\dot\gamma(t_0)$ are collinear, or in other words where $|A(t_0)|=0$.

Proof. Consider a system of polar coordinates (r,φ) with the origin at the point O. To get the curve we should solve the system of differential equations:

$$\begin{cases} r^2\dot\varphi = A \\ \dot r^2 + r^2\dot\varphi^2 = 1 \end{cases}.$$

This system is equivalent to the union of the following two systems:

$$\begin{cases} \dot\varphi = \frac{A}{r^2} \\ \dot r = \sqrt{1-\frac{A^2}{r^2}} \end{cases} \quad\text{and}\quad \begin{cases} \dot\varphi = \frac{A}{r^2} \\ \dot r = -\sqrt{1-\frac{A^2}{r^2}} \end{cases}.$$

By the main theorem of theory of ordinary differential equations (see for instance in [**?**]) this system has a finite solution if $|r|>|A|>0$. This concludes the proof. □

Let us say a few words about density functions and their broken line approximations. Let $\gamma(t)$ be a curve with arclength parameter $t\in[0,T]$ and densities $A(t)$ and $B(t)$ with respect to some point O in the complement to the curve. Let also the vectors $O\gamma(t)$ and $\dot\gamma(t)$ are linearly independent for all admissible t. For an integer n consider a broken line $\gamma_n=A_{0,n}\ldots A_{n,n}$ such that $A_{i,n}=\gamma\big(\frac{i}{n}T\big)$. Let the corresponding LLS-sequence be $(a_{0,n},\ldots,a_{2n,n})$. Denote by A_n and B_n the following functions

$$A_n(t)=a_{2\lfloor nt/T\rfloor+1,n},\quad\text{and}\quad B_n(t)=a_{2\lfloor nt/T\rfloor,n}.$$

Theorem 9.15. *Let γ be in C^2. Then the sequences of functions (A_n) and (B_n) pointwise converge to the functions A and B respectively.*

Proof. This follows directly from the definition of density functions and Proposition 9.1. □

9.2.2. *Example of curves and their continued fractions.* In this subsection we calculate the areal and angular densities for straight lines, ellipses, and logarithmic spirals.

Example 9.16. Lines. Let us study the case of lines. Without lose of generality we consider the point O to be at the origin and take the line $x = a$. Then the corresponding densities are

$$A(t) = a \qquad \text{and} \qquad B(t) = 0.$$

Example 9.17. Ellipses and their centers. Consider an ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ with $a \geq b > 0$. Let O be at the symmetry center of the ellipse i.e. at the origin. Then the areal and angular densities are as follows

$$A(t) = \frac{ab}{\sqrt{a^2 \sin^2 t + b^2 \cos^2 t}} \qquad \text{and} \qquad B(t) = \frac{1}{ab\sqrt{a^2 \sin^2 t + b^2 \cos^2 t}}.$$

Notice that here we get the constant function for the ratio:

$$\frac{A(t)}{B(t)} = a^2 b^2.$$

Example 9.18. Ellipses and their foci. As in the previous example we consider an ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ with $a \geq b > 0$. Let now O be at one of the foci, for instance at $(-\sqrt{a^2 - b^2}, 0)$. Then the densities are as follows

$$A(t) = \frac{ab + b\sqrt{a^2 - b^2} \cos t}{\sqrt{a^2 \sin^2 t + b^2 \cos^2 t}} \quad \text{and} \quad B(t) = \frac{a}{b\sqrt{a^2 \sin^2 t + b^2 \cos^2 t}\big(a + \cos t\sqrt{a^2 - b^2}\big)^2}.$$

Remark on the Kepler planetary motion. If we put the Sun at the chosen focus and consider a planet whose orbit is the ellipse, then according to three Kepler laws the planet will move with velocity $\lambda/A(t)$ at any t. Here the constant λ is defined from the third Kepler law: *the square of the orbital period of a planet is directly proportional to the cube of the semi-major axis of its orbit,* or in other words

$$\frac{T^2}{a^3} = \frac{T_e^2}{a_e^3},$$

where T is the period for our orbit, and T_e and a_e are respectively the period and the semi-major axis for the Earth. Denote by L the length of the ellipse, i.e.,

$$L = 4a \int_0^{\pi/2} \sqrt{1 - \left(1 - \frac{b^2}{a^2}\right) \cos^2 t} \; dt$$

Since $T = |\lambda| \int_0^L |1/A(t)|dt$, we get

$$\lambda = \pm \frac{T_e}{\int_0^L |1/A(t)|dt} \left(\frac{a}{a_e}\right)^{\frac{3}{2}}.$$

We skip a description for parabolas and hyperbolas, they are similar to the case of ellipses.

Example 9.19. Logarithmic spirals. Consider a logarithmic spiral

$$\Big\{ \big(ae^{bt}\cos t, ae^{bt}\sin t\big) \Big| t \in \mathbb{R} \Big\}.$$

Then the densities for this spiral are as follows

$$A(t) = \frac{ae^{bt}}{\sqrt{b^2+1}} \quad \text{and} \quad B(t) = \frac{e^{-3bt}\sqrt{b^2+1}}{a^3}.$$

It is interesting to notice that for the spirals we have

$$A^3(t)B(t) = \frac{1}{b^2+1},$$

i.e., the function A^3B is constant for any logarithmic spiral.

Notice that if A^2B is a constant function, then the curvature is constant, and hence we get circles. What do we have if AB (or A) is constant?

9.2.3. *Open problems.* We conclude this section with two open problems concerning the density functions. We start with a question on convergency that is in some sense the inverse problem to Theorem 9.15.

Problem 1. What properties should have the LLS-sequences of broken lines if their sequence converges to certain curve.

The second problem comes from Remark 9.8 on closed broken lines.

Problem 2. What are the conditions on the functions $A(t)$ and $B(t)$ for the resulting curve γ to be closed?

Exercises.

[1] Calculate the areal and angular densities for straight lines.
[2] Calculate the areal and angular densities for ellipses.
[3] Calculate the areal and angular densities for logarithmic spirals.

E-mail address, Oleg Karpenkov: `karpenkov@tugraz.at`

TU Graz /Kopernikusgasse 24, A 8010 Graz, Austria/