

# ON THE FLEXIBILITY OF KOKOTSAKIS MESHES.

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ABSTRACT. In this paper we study geometric, algebraic, and computational aspects of flexibility and infinitesimal flexibility of Kokotsakis meshes. A Kokotsakis mesh is a mesh that consists of a face in the middle and a certain band of faces attached to the middle face by its perimeter. In particular any  $(3 \times 3)$ -mesh made of quadrangles is a Kokotsakis mesh. We express the infinitesimal flexibility condition in terms of Ceva and Menelaus theorems. Further we study semi-algebraic properties of the set of flexible meshes and give equations describing it. For  $(3 \times 3)$ -meshes we obtain flexibility conditions in terms of face angles.

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## INTRODUCTION

Consider a mesh having one special  $n$ -vertex face and a “strip” of faces attached to the special face along the perimeter, such that any vertex of a special face is a vertex of exactly three faces in the strip. This mesh is called a *Kokotsakis mesh*. Kokotsakis meshes with polygonal faces are known in the literature as *polyhedral surfaces of Kokotsakis type*.

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In this paper we study conditions of flexibility and infinitesimal flexibility for Kokotsakis meshes.

The first steps in studying such meshes were made in 1932 by A. Kokotsakis in his paper [5], where he found conditions of infinitesimal flexibility. Recently Kokotsakis meshes for 4-gons (such meshes are also called  $(3 \times 3)$ -meshes) were used in the study of discrete conjugate nets flexibility. A. I. Bobenko, T. Hoffmann, W. K. Schief proved in [3] that a nondegenerate discrete conjugate net is isometrically deformable if and only if all its  $3 \times 3$  subcomplexes are isometrically deformable  $(3 \times 3)$ -meshes. In [3] they also showed that the conjugate nets admit infinitesimal second-order deformations if and only if their reciprocal-parallel surfaces constitute discrete Bianchi surfaces (see also [2]).

In this paper we give a geometrical interpretation of infinitesimal flexibility in terms of Ceva and Menelaus configurations. Further we work with equations that define the set of flexible Kokotsakis meshes, in particular we show semi-algebraicity of this set. The technique developed in this paper allows us to calculate linear independence of the gradients of the first 6 flexibility conditions for a chosen flexible quadrangular Kokotsakis mesh. This means that we can check if the codimension of a stratum of meshes is smaller than or equal to 6. The computational complexity of the problem does not allow us to proceed further. We remind the reader that the codimension of the set of flexible Voss meshes as well as that of symmetric flexible meshes is 8, for information on Voss surfaces we refer to [3]. So it is most probable that the codimensions of irreducible components coincide, and equal 6, 7, or 8. If we are lucky and these codimensions are exactly 8, than the geometric solution of the problem can be obtained by enumeration of all possible components.

To check the conditions at a single point we introduce more simple “flexibility vector calculus” conditions. In the case of  $(3 \times 3)$ -meshes we now able to calculate the 8 first such conditions.

Finally, for the case of quadrangular Kokotsakis  $(3 \times 3)$ -meshes we show that the property of the mesh to be flexible depends only on the shapes of faces but not on the geometry of the mesh in  $\mathbb{R}^3$ . We propose an algorithm to write these formulae explicitly.

**Organization of the paper.** We start in Section 1 with showing an infinitesimal flexibility condition in algebraic and geometric terms. Section 2 studies properties of the set of flexible meshes. We describe equations defining this set. In Section 3 we introduce a technique to write flexibility conditions in terms of angles of the faces of meshes. We conclude the paper in Section 4 with some open questions for further study.

**General notation.** To avoid the annoying description of multiple cases we consider the index  $i$  belonging to the cyclic group  $\mathbb{Z}/n\mathbb{Z}$ . We also choose a partial *ordering*  $1 < 2 < 3 < \dots < n$  for this group.

We denote the vertices of the special face in the middle of the mesh by  $A_i$ ,  $i = 1, \dots, n$ , preserving their order at the boundary. Let  $a_i = A_{i+1} - A_i$ . Notice that by definition we have  $\sum_{i=1}^n a_i = 0$ .

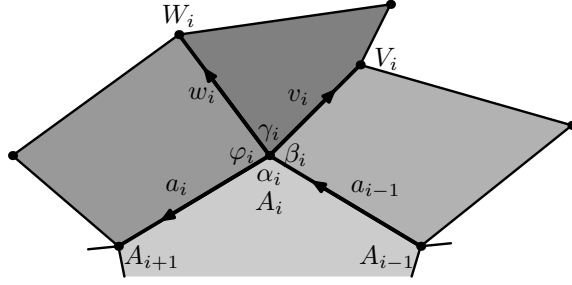


FIGURE 1. Notation.

Consider a vertex  $A_i$  ( $i = 1, \dots, n$ ). We denote vertices  $V_i$  and  $W_i$  such that the four faces meeting at  $A_i$  contain angles  $A_{i+1}A_iA_{i-1}$ ,  $A_{i-1}A_iV_i$ ,  $V_iA_iW_i$ , and  $W_iA_iA_{i+1}$ . We denote these angles by  $\alpha_i$ ,  $\beta_i$ ,  $\gamma_i$ , and  $\varphi_i$ , respectively. Let  $v_i = V_i - A_i$  and  $w_i = W_i - A_i$ .

*Remark.* Without loss of generality we assume that the special face  $A_1A_2 \dots A_n$  is fixed, while the other  $2n$  faces may move.

Denote by  $\langle u_1, u_2 \rangle$  and by  $[u_1, u_2]$  the scalar and vector products, respectively.

## 1. INFINITESIMAL FLEXIBILITY OF KOKOTSAKIS MESHES

**1.1. Algebraic conditions.** In this subsection we recall some results on infinitesimal flexibility from the work [5] by A. Kokotsakis. We rewrite his results in a slightly modified way more convenient for us.

Consider a  $(2 \times 2)$ -mesh with a common vertex  $A$ . Let us fix a face  $BAC$ . Let  $BAV$ ,  $VAW$ , and  $WAC$  be the other three faces. Denote by  $b$ ,  $c$ ,  $v$ , and  $w$  the vectors  $B - A$ ,  $C - A$ ,  $V - A$ , and  $W - A$  respectively. This mesh is flexible in general position and has one degree of freedom.

**Proposition 1.1.** *Suppose that the vectors  $v$  and  $w$  are not collinear and the vectors  $b$  and  $c$  are not in the plane spanned by the vectors  $v$  and  $w$ . Then for any infinitesimal motion  $(\dot{v}, \dot{w})$  of the described  $(2 \times 2)$ -mesh there exists a non-zero real  $\lambda$  such that*

$$\begin{cases} \dot{v} &= \lambda \frac{[b, v]}{(b, v, w)} \\ \dot{w} &= \lambda \frac{[c, w]}{(c, v, w)} \end{cases} .$$

*Proof.* Any infinitesimal motion is a solution of the following linear system with 6 variables (coordinates of  $\dot{v}$  and  $\dot{w}$ ) and 5 equations:

$$\begin{cases} \langle v, \dot{v} \rangle = 0 \\ \langle w, \dot{w} \rangle = 0 \\ \langle \dot{v}, w \rangle + \langle v, \dot{w} \rangle = 0 \\ \langle b, \dot{v} \rangle = 0 \\ \langle c, \dot{w} \rangle = 0 \end{cases} .$$

On one hand the solution mentioned in the statement of the proposition satisfies this system. On the other hand, if the conditions on vectors  $b$ ,  $c$ ,  $v$ , and  $w$  hold then the rank of the linear system is 1 (this follows from direct calculations in coordinates).  $\square$

From Proposition 1.1 we immediately have the following statement (this statement is another formulation of the result in [5] by A. Kokotsakis). For a Kokotsakis mesh  $M$  put by definition

$$\chi(M) = \prod_{i=1}^n \frac{(a_{i-1}, v_i, w_i)}{(a_i, v_i, w_i)}.$$

**Corollary 1.2.** *A Kokotsakis mesh  $M$  is infinitesimally flexible if and only if  $\chi(M)=1$ .*  $\square$

### 1.2. Geometric infinitesimal condition for a Kokotsakis mesh with planar faces.

Consider points  $A_1, \dots, A_n$  in a plane. We denote this plane by  $\pi$ . Let  $l_i$  denote the intersection of the plane containing the points  $A_i$ ,  $V_i$ , and  $W_i$  with the plane  $\pi$ . We suppose that the lines  $l_i$  and  $l_{i+1}$  are not parallel, and denote their intersection point by  $B_i$ .

For a  $(3 \times 3)$ -mesh constructed with planar faces the infinitesimal flexibility conditions are shown in the following proposition. For more information see the work [6] by H. Pottmann, J. Wallner and the book [7] by R. Sauer.

**Proposition 1.3.** *Consider a mesh in general position. Then it is infinitesimally flexible if and only if the lines  $B_1B_3$ ,  $A_2, A_3$ , and  $A_4A_1$  intersect in one point (i.e. the points  $A_i, B_i, i = 1, 2, 3, 4$  are aligned in a Desargues configuration).*

Let us generalize this statement for an arbitrary Kokotsakis mesh with planar polygons as faces. For  $i = 1, \dots, n$  we define real numbers  $t_i$  from the following affine equations:

$$A_i = t_i B_{i-1} + (1 - t_i) B_i.$$

**Theorem 1.4.** *The closed broken line  $A_1B_1A_2B_2 \dots A_nB_nA_1$  satisfies the generalized Ceva-Menelaus condition, i.e.*

$$\prod_{i=1}^n \frac{1 - t_i}{t_i} = (-1)^n.$$

*Remark 1.5.* If there is one point  $B_i$  “at infinity” one should replace

$$\frac{1 - t_i}{t_i} \cdot \frac{1 - t_{i+1}}{t_{i+1}} \quad \text{by} \quad (-1)^\varepsilon \frac{1 - t_{i+1}}{t_i}$$

in the product. Here  $t_i$  and  $1 - t_{i+1}$  denotes the absolute values of the vectors  $A_i - B_{i-1}$  and  $B_{i+1} - A_{i+1}$  respectively. We take  $\varepsilon = 0$  if the directions of these two vectors are opposite and  $\varepsilon = 1$  if they coincide.

We leave for the reader the other cases of some of the vertices  $B_i$  being “at infinity” as an exercise.

*Proof.* Choose some orientation on the plane  $\pi$ . For an ordered couple  $e_1, e_2$  of vectors denote

$$\operatorname{sgn}(e_1, e_2) = \begin{cases} 1 & \text{if } (e_1, e_2) \text{ is a positively oriented basis of the plane} \\ 0 & \text{if } e_1 \text{ and } e_2 \text{ are collinear} \\ -1 & \text{otherwise} \end{cases}.$$

Denote by  $h_{P,QR}$  the (non-oriented) distance from the point  $P$  to the line  $QR$ . Then for any  $i \in \{1, \dots, n\}$  we have:

$$\frac{(v_i, w_i, a_i)}{(v_i, w_i, a_{i-1})} = \frac{\operatorname{sgn}(B_{i-1}B_i, A_iA_{i+1})h_{A_{i+1},B_{i-1}B_i}}{\operatorname{sgn}(B_{i-1}B_i, A_{i-1}A_i)h_{A_{i+1},B_{i-1}B_i}} = \frac{\operatorname{sgn}(B_{i-1}B_i, A_iA_{i+1})}{\operatorname{sgn}(B_{i-1}B_i, A_{i-1}A_i)} \cdot \frac{|A_iA_{i+1}|/\sin(A_{i+1}A_iB_i)}{|A_{i-1}A_i|/\sin(A_{i-1}A_iB_{i-1})}.$$

Therefore,

$$\begin{aligned} \prod_{i=1}^n \frac{1-t_i}{t_i} &= \prod_{i=1}^n \frac{\operatorname{sgn}(B_{i-2}B_{i-1}, A_{i-1}A_i)|B_{i-1}A_i|}{-\operatorname{sgn}(B_iB_{i+1}, A_iA_{i+1})|A_iB_i|} = \\ (-1)^n \cdot \prod_{i=1}^n \left( \frac{\operatorname{sgn}(B_{i-1}B_i, A_iA_{i+1})}{\operatorname{sgn}(B_{i-1}B_i, A_{i-1}A_i)} \cdot \frac{|A_iA_{i+1}|}{|A_{i-1}A_i|} \cdot \frac{\sin(A_{i-1}A_iB_{i-1})}{\sin(A_{i+1}A_iB_i)} \right) &= \\ (-1)^n \cdot \prod_{i=1}^n \frac{(v_i, w_i, a_i)}{(v_i, w_i, a_{i-1})} &= (-1)^n. \end{aligned}$$

The second equality holds by the law of sines for the triangles  $A_iA_{i+1}B_i$  for  $i = 1, \dots, n$ .  $\square$

Denote by  $[A, B; C, D]$  the point of intersection of the lines passing through  $A, B$  and  $C, D$  respectively.

Denote  $P_0 = B_1$ . Let  $O_i = [B_n, A_{i+1}; B_{i+1}, P_{i-1}]$ , and  $P_i = [B_{i+1}, B_n; B_i, O_i]$ , for  $i = 1, \dots, n-2$ .

**Proposition 1.6. Geometric condition I.** *The condition*

$$\prod_{i=1}^n \frac{1-t_i}{t_i} = 1$$

*is equivalent to the condition that the points  $A_n$  and  $P_{n-2}$  coincide (see Figure 2.I).*

In the case  $n = 3$  we have: *the lines  $B_1A_3$ ,  $B_2A_1$ , and  $B_3A_2$  intersect at one point (Ceva configuration).* In the case  $n = 4$  we have: *the lines  $B_1B_3$ ,  $A_2, A_3$ , and  $A_4A_1$  intersect at one point (Desargues configuration).*

*Proof.* We prove the proposition by induction on  $n$ .

By Ceva's theorem and Theorem 1.4 the statement holds for all configurations with  $n = 3$ . For pairwise distinct collinear points  $A, B, C$  denote

$$\sigma(ABC) = \begin{cases} 1 & \text{if the point } B \text{ is in the segment } [A, C] \\ -1 & \text{otherwise} \end{cases}.$$

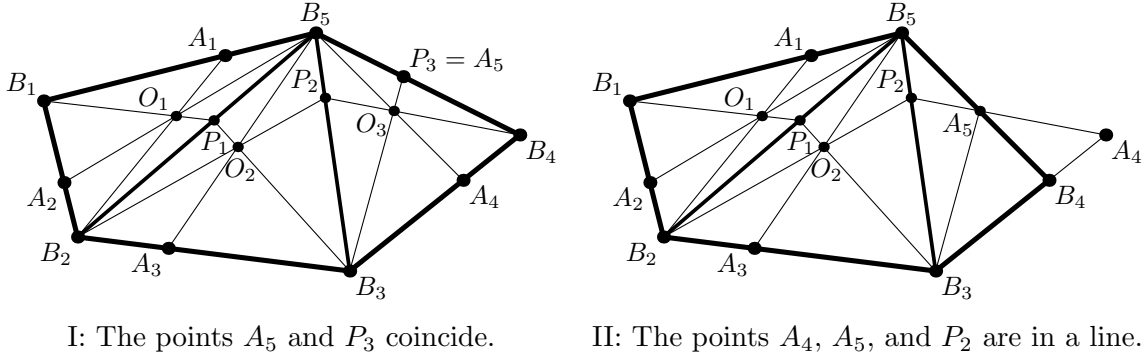


FIGURE 2. Geometric conditions I and II for a pentagon.

Suppose the statement holds for all configurations with  $n = 4$ . By Ceva's Theorem in the triangle  $B_n B_1 B_2$  we have:

$$\prod_{i=1}^n \sigma(B_{i-1} A_i B_i) \frac{B_{i-1} A_i}{A_i B_i} = \sigma(B_n P_1 B_2) \frac{B_n P_1}{P_1 B_2} \prod_{i=3}^n \sigma(B_{i-1} A_i B_i) \frac{B_{i-1} A_i}{A_i B_i}.$$

Then by induction the statement is true for the broken line  $P_1 B_2 A_3 B_3 \dots A_n B_n P_1$ .  $\square$

**Proposition 1.7. Geometric condition II.** *The condition*

$$\prod_{i=1}^n \frac{1 - t_i}{t_i} = -1$$

*is equivalent to the condition that the points  $P_{n-3}$ ,  $A_{n-1}$ , and  $A_n$  are collinear (see Figure 2.II).*

In the case  $n = 3$  we have: *the points  $A_1$ ,  $A_2$ , and  $A_3$  are contained in one line* (Menelaus configuration).

*Proof.* We prove the proposition by induction on  $n$ . By Menelaus' theorem and Theorem 1.4 the statement holds for all configurations with  $n = 3$ . The induction step is the same as in the proof of Proposition 1.7, so we omit it here.  $\square$

*Remark 1.8.* The condition of Corollary 1.2 can be rewritten as Geometric condition I in the case of even  $n$ , and as Geometric condition II in the case of odd  $n$ .

**Corollary 1.9.** *Any Kokotsakis mesh with a triangle in the middle, in general position, is not infinitesimal flexible (and therefore it is not flexible).*  $\square$

In particular we have the following conclusion. Consider a closed mesh in general position (i.e. there are no 3 vertices lying in one line) whose edge-graph is 4-valent. Then the mesh is not flexible if at least one of its faces is a triangle. Actually this means that the Euler characteristic of the surface is non-positive. So a flexible mesh in general position is not homeomorphic to the sphere or to the projective plane.

## 2. FLEXIBILITY CONDITIONS

**2.1. Semialgebraicity of the set of flexible Kokotsakis meshes.** First we define a configuration space of all Kokotsakis meshes whose central face is an  $n$ -gon. Any Kokotsakis mesh contains  $3n$  points. So the configuration space of all possible vertex configurations is  $(\mathbb{R}^3)^{3n}$ . We will not consider configurations for which some face has three consequent vertices in a line, or which has faces with coinciding vertices. Denote this set by  $\Sigma_n$ . Finally, the *configuration space of all Kokotsakis meshes* is by definition  $(\mathbb{R}^3)^{3n} \setminus \Sigma_n$ , denote it by  $\Theta_n$ .

Notice that we are not restricted to the case of planar faces in this section. Consider the vector field:

$$\begin{aligned} \dot{v}_i &= \left( \prod_{j=1}^{i-1} \frac{(a_{j-1}, v_j, w_j)}{(a_j, v_j, w_j)} \right) [a_{i-1}, v_i] \quad \text{and} \\ \dot{w}_i &= \left( \prod_{j=1}^i \frac{(a_{j-1}, v_j, w_j)}{(a_j, v_j, w_j)} \right) [a_i, w_i] \quad \text{for } i = 1, \dots, n. \end{aligned}$$

and denote it by  $\xi_n$ . This vector field is clearly everywhere defined in  $\Theta_n$ .

Further when, in this section, we speak of a motion we assume that the special face in the middle is fixed while the others are moving.

**Theorem 2.1.** *Consider a flexible mesh in  $\Theta_n$ . The motion trajectory of the mesh in some small neighborhood is contained in some integral curve of the vector field  $\xi_n$ .*

*Proof.* By Proposition 1.1 a tangent vector for the trajectory of a flexible mesh  $M$  in  $\Theta_n$  is proportional to the vector  $\xi(M)$ . This implies the statement of the theorem.  $\square$

Denote the set of all flexible meshes by  $F_{\Theta_n}$ . Let us consider an infinite sequence  $f_i$  of algebraic functions on  $(\mathbb{R}^3)^{3n}$ , defined inductively:

- i). Take  $f_1$  to be the numerator of the rational function  $\chi(M) - 1$ .
- ii). Take  $f_n$  to be the numerator of the rational function  $\langle \text{grad}(f_{n-1}), \xi_n \rangle$ .

*Remark 2.2.* We ignore denominators since they are nonzero algebraic functions on  $\Theta_n$ .

**Theorem 2.3.** *The closure of  $F_{\Theta_n}$  in  $\Theta_n$  is a semialgebraic set coinciding with*

$$\bigcap_{i=1}^{+\infty} \{f_i = 0\} \cap \Theta_n.$$

*Proof.* Notice that the vector field  $\xi(M)$  is non-zero at all the points of  $\Theta_n$ .

By Proposition 1.1, for a flexible mesh  $M$  not contained in  $\Sigma_n$  the velocity vector is proportional to the vector  $\xi(M)$ , and therefore satisfies all the conditions  $f_n = 0$ .

On the other hand all functions  $f_n$  are algebraic and therefore their common locus is an algebraic variety (by Hilbert's basis theorem it is sufficient to take only a finite number of such equations). Therefore in a nonsingular point of this variety we have all the existence conditions for a finite motion along the field  $\xi(M)$  (see in [1]). Since the set of singularities

of an algebraic variety is nowhere dense in this variety, the closure of the set of nonsingular points of a variety is the variety itself.  $\square$

**2.2. Vector calculus of flexibility conditions.** In practice it is quite hard to calculate the above  $f_n$  explicitly. To check some given configuration we recommend a slightly different technique.

Since  $\chi(M) = 0$  along any trajectory, we immediately have many conditions of flexibility:  $\chi^{(n)}(M) = 0$ , where the derivatives are taken with respect of the time parameter of the vector field. We recommend to simplify the obtained function after each round of differentiation using previous ones.

Let us introduce basic ingredients to calculate derivatives. We use a prime to indicate differentiation.

$$\begin{aligned} \langle a_i, v_i \rangle' &= -(a_{i-1}, a_i, v_i) \prod_{j=1}^{i-1} \frac{(a_{j-1}, v_j, w_j)}{(a_j, v_j, w_j)}; \\ \langle a_{i-1}, w_i \rangle' &= (a_{i-1}, a_i, w_i) \prod_{j=1}^i \frac{(a_{j-1}, v_j, w_j)}{(a_j, v_j, w_j)}; \\ (a_{i-1}, a_i, v_i)' &= \left( \langle a_{i-1}, a_{i-1} \rangle \langle a_i, v_i \rangle - \langle a_{i-1}, a_i \rangle \langle a_{i-1}, v_i \rangle \right) \prod_{j=1}^{i-1} \frac{(a_{j-1}, v_j, w_j)}{(a_j, v_j, w_j)}; \\ (a_{i-1}, a_i, w_i)' &= \left( \langle a_{i-1}, a_i \rangle \langle a_i, w_i \rangle - \langle a_i, a_i \rangle \langle a_{i-1}, w_i \rangle \right) \prod_{j=1}^i \frac{(a_{j-1}, v_j, w_j)}{(a_j, v_j, w_j)}; \\ (a_k, v_i, w_i)' &= \left( \langle a_k, v_i \rangle \langle a_{i-1}, w_i \rangle - \langle a_{i-1}, a_k \rangle \langle v_i, w_i \rangle \right) \prod_{j=1}^{i-1} \frac{(a_{j-1}, v_j, w_j)}{(a_j, v_j, w_j)} + \\ &\quad \left( \langle a_i, a_k \rangle \langle v_i, w_i \rangle - \langle a_i, v_i \rangle \langle a_k, w_i \rangle \right) \prod_{j=1}^i \frac{(a_{j-1}, v_j, w_j)}{(a_j, v_j, w_j)}. \end{aligned}$$

In fact, to calculate  $\chi^{(n)}(M)$ , one only needs the cases  $k = i-1$  and  $k = i$  for the last formula.

**Example 2.4.** The second condition  $(\chi(M))' = 0$  is equivalent to

$$\begin{aligned} &\sum_{i=1}^n \left( \frac{\langle a_i, v_i \rangle \langle a_{i-1}, w_i \rangle - \langle a_{i-1}, a_i \rangle \langle v_i, w_i \rangle}{(a_i, v_i, w_i)} \prod_{j=1}^{i-1} \frac{(a_{j-1}, v_j, w_j)}{(a_j, v_j, w_j)} \right) + \\ &\sum_{i=1}^n \left( \frac{\langle a_i, a_i \rangle \langle v_i, w_i \rangle - \langle a_i, v_i \rangle \langle a_i, w_i \rangle}{(a_i, v_i, w_i)} \prod_{j=1}^i \frac{(a_{j-1}, v_j, w_j)}{(a_j, v_j, w_j)} \right) - \\ &\sum_{i=1}^n \left( \frac{\langle a_{i-1}, v_i \rangle \langle a_{i-1}, w_i \rangle - \langle a_{i-1}, a_{i-1} \rangle \langle v_i, w_i \rangle}{(a_{i-1}, v_i, w_i)} \prod_{j=1}^{i-1} \frac{(a_{j-1}, v_j, w_j)}{(a_j, v_j, w_j)} \right) - \\ &\sum_{i=1}^n \left( \frac{\langle a_{i-1}, a_i \rangle \langle v_i, w_i \rangle - \langle a_i, v_i \rangle \langle a_{i-1}, w_i \rangle}{(a_{i-1}, v_i, w_i)} \prod_{j=1}^i \frac{(a_{j-1}, v_j, w_j)}{(a_j, v_j, w_j)} \right) = 0. \end{aligned}$$



3. INTERNAL FLEXIBILITY CONDITIONS FOR  $(3 \times 3)$ -MESHES

In this section we describe a technique to verify the flexibility of a  $(3 \times 3)$ -mesh with planar faces in terms of inner geometry of a mesh, i.e., in terms of the fixed angles between the edges of the same faces.

Consider a  $(3 \times 3)$ -mesh with a planar quadrangle  $A_1A_2A_3A_4$  in the middle. Denote the angles between the faces adjacent to  $A_iA_{i+1}$  by  $\omega_i$  ( $i = 1, \dots, n$ ).

**Lemma 3.1.** *The following relation holds for any vertex  $A_i$*

$$\begin{aligned} & \cos \alpha_i \cos \beta_i \cos \varphi_i + \sin \alpha_i \sin \beta_i \cos \varphi_i \cos \omega_{i-1} + \sin \alpha_i \cos \beta_i \sin \varphi_i \cos \omega_i - \\ & \cos \alpha_i \sin \beta_i \sin \varphi_i \cos \omega_{i-1} \cos \omega_i + \sin \beta_i \sin \varphi_i \sin \omega_{i-1} \sin \omega_i = \cos \gamma_i. \end{aligned}$$

Denote this equation by  $(R_i)$ .

*Proof.* Note that the values of the angles do not depend on the lengths of vectors  $a_i, a_{i-1}, v_i, w_i$ . So we can choose them equal 1. Let us make calculations in the coordinate system where  $a_i = (1, 0, 0)$ , and  $a_{i-1} = (\cos \alpha, \sin \alpha, 0)$ . Then we have

$$\begin{aligned} v_i &= (\cos \varphi_i, \sin \varphi_i \cos \omega_i, \sin \varphi_i \sin \omega_i), \\ w_i &= \cos \beta_i (\cos \alpha_i, \sin \alpha_i, 0) + \sin \beta_i \left( \cos \omega_{i-1} (\sin \alpha_i, -\cos \alpha_i, 0) + \sin \omega_{i-1} (0, 0, 1) \right) \end{aligned}$$

The formula of the lemma coincides with  $\langle v_i, w_i \rangle = \cos \gamma_i$  and therefore holds.  $\square$

We get the desired equations in 4 steps.

*Step 1.* Denote  $\cos \omega_i$  by  $t_i$ . The first step is to get rid of  $\sin \omega_i \sin \omega_{i-1}$  in formula  $(R_i)$ . This can be done by putting the corresponding term to the right and the rest to the left, than squaring both sides, and finally substituting  $\sin^2 \omega_i \sin^2 \omega_{i-1} = (1 - t_i^2)(1 - t_{i-1}^2)$ . Denote the resulting equation by  $(\overline{R}_i)$ . It leads to

$$C_{1,i} + C_{2,i}t_{i-1} + C_{3,i}t_i + C_{4,i}t_{i-1}^2 + C_{5,i}t_{i-1}t_i + C_{6,i}t_i^2 + C_{7,i}t_{i-1}^2t_i + C_{8,i}t_{i-1}t_i^2 + C_{9,i}t_{i-1}^2t_i^2 = 0,$$

where

$$\begin{aligned} C_{1,i} &= (\cos \alpha_i \cos \beta_i \cos \varphi_i - \cos \gamma_i)^2 - \sin^2 \beta_i \sin^2 \varphi_i; \\ C_{2,i} &= 2(\cos \alpha_i \cos \beta_i \cos \varphi_i - \cos \gamma_i) \sin \alpha_i \sin \beta_i \cos \varphi_i; \\ C_{3,i} &= 2(\cos \alpha_i \cos \beta_i \cos \varphi_i - \cos \gamma_i) \sin \alpha_i \cos \beta_i \sin \varphi_i; \\ C_{4,i} &= \sin^2 \alpha_i \sin^2 \beta_i \cos^2 \varphi_i + \sin^2 \beta_i \sin^2 \varphi_i; \\ C_{5,i} &= 2(\cos \beta_i \cos \varphi_i - 2 \cos^2 \alpha_i \cos \beta_i \cos \varphi_i + \cos \alpha_i \cos \gamma_i) \sin \beta_i \sin \varphi_i; \\ C_{6,i} &= \sin^2 \alpha_i \cos^2 \beta_i \sin^2 \varphi_i + \sin^2 \beta_i \sin^2 \varphi_i; \\ C_{7,i} &= -2 \sin \alpha_i \cos \alpha_i \sin^2 \beta_i \sin \varphi_i \cos \varphi_i; \\ C_{8,i} &= -2 \sin \alpha_i \cos \alpha_i \sin \beta_i \cos \beta_i \sin^2 \varphi_i; \\ C_{9,i} &= -\sin^2 \alpha_i \sin^2 \beta_i \sin^2 \varphi_i. \end{aligned}$$

*Step 2.* Now we recursively evaluate  $t_i^2$  as a linear function in  $t_i$  whose coefficients are rational functions of  $t_{i-1}$ . We will need only the formulae for  $t_2^2$  and  $t_4^2$ .

*Step 3.* We use  $(\overline{R}_3)$  and the formula for  $t_2^2$  given in Step 2 to evaluate  $t_2$  as a function of  $t_1$  and  $t_3$ , and analogously we employ  $(\overline{R}_4)$  with the formula for  $t_4^2$  given in Step 2 to

evaluate  $t_4$  as a function of  $t_1$  and  $t_3$ . We get

$$t_2 = -\frac{(C_{9,2}t_1^2 + C_{8,2}t_1 + C_{6,2})(C_{6,3}t_3^2 + C_{3,3}t_3 + C_{1,3}) - (C_{9,3}t_3^2 + C_{7,3}t_3 + C_{4,3})(C_{4,2}t_1^2 + C_{2,2}t_1 + C_{1,2})}{(C_{9,2}t_1^2 + C_{8,2}t_1 + C_{6,2})(C_{8,3}t_3^2 + C_{5,3}t_3 + C_{2,3}) - (C_{9,3}t_3^2 + C_{7,3}t_3 + C_{4,3})(C_{7,2}t_1^2 + C_{5,2}t_1 + C_{3,2})},$$

$$t_4 = -\frac{(C_{9,1}t_1^2 + C_{7,1}t_1 + C_{4,1})(C_{4,4}t_3^2 + C_{2,4}t_3 + C_{1,4}) - (C_{9,4}t_3^2 + C_{8,4}t_3 + C_{6,4})(C_{6,1}t_1^2 + C_{3,1}t_1 + C_{1,1})}{(C_{9,1}t_1^2 + C_{7,1}t_1 + C_{4,1})(C_{7,4}t_3^2 + C_{5,4}t_3 + C_{3,4}) - (C_{9,4}t_3^2 + C_{8,4}t_3 + C_{6,4})(C_{8,1}t_1^2 + C_{5,1}t_1 + C_{2,1})}.$$

*Step 4.* Now for  $t_1$  and  $t_3$  we have the system

$$\begin{cases} R_1(t_1, t_4(t_1, t_3)) &= 0 \\ R_2(t_1, t_2(t_1, t_3)) &= 0 \end{cases}.$$

First, note that the functions  $R_1$  and  $R_2$  are rational in variables  $t_1$  and  $t_3$ . So we can restrict ourselves to consider the numerators. Denote these numerators by  $f_1(t_1, t_3)$ , and  $f_2(t_1, t_3)$ . The functions  $f_1$  and  $f_2$  are polynomials of degree 10, and we need to check if these two polynomials define a non-discrete set. This happens only if  $f_1$  and  $f_2$  have a common nonconstant factor. Take a resultant of  $f_1$  and  $f_2$  considered as polynomials in  $t_3$ . This resultant denote it by  $p(t_1)$  is a polynomial in  $t_1$ .

**Theorem 3.2.** *A mesh in  $\Theta_n$  is flexible if and only if all the coefficients of the polynomial  $p(t_1)$  equal zero.*

*Proof.* We give an outline of the proof.

It is clear that if a mesh is flexible then the set of common solutions of  $R_i = 0$  for  $i = 1, 2, 3, 4$  is nondiscrete and therefore one of the equations can be eliminated. By Proposition 1.1 in the case of meshes in  $\Theta_n$  all the  $t_i$ 's are nonconstant while the mesh is moving, so the resultant  $p(t_1)$  is identically zero.

Suppose now that all the coefficients of the polynomial  $p(t_1)$  equal zero, then there are at most three independent equations among  $R_i = 0$  for  $i = 1, 2, 3, 4$ . The common set of solutions of the equations  $R_i = 0$  is almost always at least one-dimensional.

The dimension of this set is zero only if the corresponding mesh is a limit of a sequence of flexible meshes with possible motion curves shrinking to this mesh. Such mesh is infinitesimally flexible with more than one degrees of freedom. By Proposition 1.1 such meshes are not in  $\Theta_n$ .  $\square$

**Corollary 3.3.** *Flexibility of meshes in  $\Theta_n$  is determined by the shapes of faces (but not of an embedding in the space as a mesh).*  $\square$

To simplify symbolic calculations we recommend to compute the resultants of  $f_1(n, t_3)$  and  $f_2(n, t_3)$  for  $-50 \leq n \leq 50$ . If all 101 ( $=10^2 + 1$ ) resultants evaluate to zero, than the polynomial  $p(t_1)$ , which has at most 100 zeroes, has all zero coefficients.

Finally, we give an algorithm that calculates the functions  $f_1(x, y)$  and  $f_2(x, y)$  (as  $f[1]$  and  $f[2]$  in the output below).

```
TwoEquations:=proc(alpha,beta,gama,phi,N) local i,C,R,t,f,tn,td;
for i from 1 to 4 do
  C[1,i]:=cos(alpha[i])*cos(beta[i])*cos(phi[i])-cos(gama[i]))^2-
  sin(beta[i])^2*sin(phi[i])^2;
```

```

...
# Here we write all the formulae for C[2,i],...,C[9,i] from Step 1
end do:

tn[2] := -((C[6,2]+C[8,2]*t[1]+C[9,2]*t[1]^2)*(C[1,3]+C[6,3]*t[3]^2+C[3,3]*t[3]) -
(C[7,3]*t[3]+C[4,3]+C[9,3]*t[3]^2)*(C[1,2]+C[2,2]*t[1]+C[4,2]*t[1]^2)):
td[2] := ((C[6,2]+C[8,2]*t[1]+C[9,2]*t[1]^2)*(C[8,3]*t[3]^2+C[5,3]*t[3]+C[2,3]) -
(C[7,3]*t[3]+C[4,3]+C[9,3]*t[3]^2)*(C[3,2]+C[5,2]*t[1]+C[7,2]*t[1]^2)):

tn[4] := -((C[1,4]+C[2,4]*t[3]+C[4,4]*t[3]^2)*(C[4,1]+C[7,1]*t[1]+C[9,1]*t[1]^2) -
(C[6,4]+C[8,4]*t[3]+C[9,4]*t[3]^2)*(C[1,1]+C[3,1]*t[1]+C[6,1]*t[1]^2)):
td[4] := ((C[3,4]+C[5,4]*t[3]+C[7,4]*t[3]^2)*(C[4,1]+C[7,1]*t[1]+C[9,1]*t[1]^2) -
(C[6,4]+C[8,4]*t[3]+C[9,4]*t[3]^2)*(C[2,1]+C[5,1]*t[1]+C[8,1]*t[1]^2)):

RBar[1] := (C[1,1]+C[6,1]*t[1]^2+C[3,1]*t[1])*td[4]^2+
(C[8,1]*t[1]^2+C[5,1]*t[1]+C[2,1])*td[4]*tn[4]+
(C[7,1]*t[1]+C[4,1]+C[9,1]*t[1]^2)*tn[4]^2:
RBar[2] := (C[1,2]+C[2,2]*t[1]+C[4,2]*t[1]^2)*td[2]^2+
(C[3,2]+C[5,2]*t[1]+C[7,2]*t[1]^2)*td[2]*tn[2]+
(C[6,2]+C[8,2]*t[1]+C[9,2]*t[1]^2)*tn[2]^2:

f[1] := (x,y)->(subs({t[1]=x,t[3]=y},simplify(RBar[1]))):
f[2] := (x,y)->(subs({t[1]=x,t[3]=y},simplify(RBar[2]))):

return(f):
end proc;

```

Now we can compute resultants of  $f[1](y_0, y)$  and  $f[2](y_0, y)$  for any value  $y_0$  with any precision  $N$ , or symbolically.

```

y0:=1;
N:=50;
f:=TwoEquations(alpha,beta,gama,phi,N):
ff[1]:=evalf[N](f[1](y0,y)):
ff[2]:=evalf[N](f[2](y0,y)):
evalf[N](resultant(ff[1]/coeftayl(ff[1],y=0,4),ff[2]/coeftayl(ff[2],y=0,4),y));

```

Still we do not know which of the listed 101 equations are independent, the first interesting problem in this direction is the following.

#### 4. SOME QUESTIONS FOR FURTHER STUDY

**4.1. Geometry of conditions.** In Section 1 we studied the geometric conditions for infinitesimal rigidity of meshes with planar faces, the following question arise here.

**Problem 1.** What is the geometric infinitesimal flexibility condition in case of meshes with nonplanar faces?

A similar problem remains open for the case of flexibility conditions introduced in Section 2.

**Problem 2.** Find a geometric interpretation of the condition  $\chi^{(n)}(M) = 0$  for  $n \geq 1$ .

We do not know how the condition  $\chi'(M) = 0$  is related to the second-order rigidity condition defined by A. I. Bobenko, et al. in [3].

**4.2. Codimension calculation.** In Section 2 we described the closure of all flexible meshes  $F_{\Theta_n}$  by algebraic equations. We conjecture that it is a complete intersection. It is most probable that all the components of the closure are of the same dimension.

*Remark 4.1.* Let us say a few words about the case of  $(3 \times 3)$ -meshes. Direct calculation of the tangent space in the points of  $F_{\Theta_4}$  based on the results of the second section shows that the codimension of some of the irreducible components of  $F_{\Theta_4}$  is greater than or equal to 6. On the other hand, the Voss surfaces (c.f. [8] and [3]) forms a set of flexible meshes of codimension 8. In general we suspect that all the irreducible components of  $F_{\Theta_4}$  has the same codimension between 6 and 8. Note that for codimension 8 we know the following *families of flexible meshes*:

i) Voss surfaces (defined by equations  $\alpha_i = \gamma_i, \beta_i = \varphi_i$  for  $i = 1, 2, 3, 4$ ).

ii) symmetric meshes where the angles at the vertices  $A_1$  and  $A_2$  coincide with the angles at the vertices  $A_4$  and  $A_3$ :

$$\alpha_1 = \alpha_4, \beta_1 = \varphi_4, \gamma_1 = \gamma_4, \varphi_1 = \beta_4; \quad \alpha_2 = \alpha_3, \beta_2 = \varphi_3, \gamma_2 = \gamma_3, \varphi_2 = \beta_3.$$

iii) symmetric meshes where the angles at the vertices  $A_1$  and  $A_4$  are the same as at the vertices  $A_2$  and  $A_3$ .

iv) meshes obtained from the cases i)–iii) by replacing some  $v_i$  and/or  $w_j$  by  $-v_i$  and/or  $-w_j$ . For instance, if we replace  $v_1$  by  $-v_1$  in the case of Voss meshes, then we get equations  $\alpha_1 + \gamma_1 = \pi, \beta_1 + \varphi_1 = \pi$ , and  $\alpha_i = \gamma_i, \beta_i = \varphi_i$  for  $i = 2, 3, 4$ .

If the codimension of  $F_{\Theta_4}$  is 8, then the natural problem is *to complete the list of flexible meshes in general position*.

All this leads to the following problem.

**Problem 3.** Find the codimension of the closure of  $F_{\Theta_n}$  for  $n = 4, 5, \dots$

The same question is interesting for the case of an algebraic set defined by 101 equations in Section 3, *does this set has the same codimension?*

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