ON IRRATIONAL LATTICE ANGLES.

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Contents

Introduction

The aim of this paper is to generalize the notions of ordinary and expanded lattice angles and their sums studied in the work [6] by author to the case of angles with lattice vertices but not necessary lattice rays. We find normal forms and extend the definition

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of lattice sums for a certain special case of such angles. The sum of angles described in the paper seems to be a natural notion of ordinary continued fractions "addition".

The study of lattice angles is an imprescriptible part of modern lattice geometry. Invariants of lattice angles are used in the study of lattice convex polygons and polytopes. Such polygons and polytopes play the principal role in Klein's theory of multidimensional continued fractions (see, for example, the works of F. Klein [11], V. I. Arnold [1], E. Korkina [13], M. Kontsevich and Yu. Suhov [12], G. Lachaud [14], and the author [7]). Lattice polygons and polytopes of the lattice geometry are in the limelight of complex projective toric varieties (see for more information the works of V. I. Danilov [2], G. Ewald [3], T. Oda [15], and W. Fulton [4]).

The studies of lattice angles and measures related to them were started by A. G. Khovanskii, A. Pukhlikov in [9] and [10] in 1992. They introduced and investigated special additive polynomial measure for the expanded notion of polytopes. The relations between sum-formulas of lattice trigonometric functions and lattice angles in Khovanskii-Pukhlikov sense are unknown to the author.

In the work [6] it was studied in details the trigonometry of rational angles and their relation to the triangles. Some properties of rational trigonometric functions follows from the statements of the work [16].

This paper is organized as follows. In the first section we remind the definition and main properties of ordinary continued fractions, and give definitions of ordinary lattice angles. The aim of Section 2 is to introduce trigonometric functions of ordinary lattice angles. Further in Section 4 we denote and study expanded irrational angles. These angles are necessary for the definition of sum of lattice angles. We study equivalence classes (with respect to the group of affine lattice preserving transformations) of expanded lattice angles and show a normal form for such classes. Finally in Section 4 we give definitions of sums of lattice angles. We conclude the papers in Section 5 with related questions and problems.

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1. Basic definitions

1.1. **Ordinary continued fractions.** For any finite sequence $(a_0, a_1, \ldots, a_n)$ where the elements $a_1, \ldots, a_n$ are positive integers and a_0 is an arbitrary integer we associate the following rational q:

$$q = a_0 + \cfrac{1}{a_1 + \cfrac{1}{\ddots \quad \vdots \atop a_{n-1} + \frac{1}{a_n}}}.$$

This representation of the rational q is called an *ordinary continued fraction* for q and denoted by $[a_0, a_1, \ldots, a_n]$. An ordinary continued fraction $[a_0, a_1, \ldots, a_n]$ is said to be *odd* if the number of the elements of the sequence (i.e. $n+1$) is odd, and *even* if the number is even.

Theorem 1.1. *For any rational there exist exactly one odd ordinary continued fraction and exactly one even ordinary continued fraction.* □

We continue with the standard definition of infinite ordinary continued fraction.

Theorem 1.2. *Consider a sequence* $(a_0, a_1, \ldots, a_n, \ldots)$ *of positive integers. There exists the following limit:* $r = \lim\limits_{k\to\infty}\big([a_0, a_1, \ldots, a_k]\big)$. □

This representation of r is called an (*infinite*) *ordinary continued fraction* for r and denoted by $[a_0, a_1, \ldots, a_n, \ldots]$.

Theorem 1.3. *For any irrational there exists and unique an infinite ordinary continued fraction. Any rational does not have infinite ordinary continued fractions.* □

For the proofs of these theorems we refer to the book [5] by A. Ya. Hinchin.

1.2. **Lattice ordinary angles.** A linear (affine) lattice preserving transformation is said to be *lattice.*

Let A, B, and C do not lie in the same straight line. Suppose also that B is lattice. We denote the angle with the vertex at B and the rays BA and BC by $\angle ABC$. If both open rays BA and BC contain lattice points, then we say that the angle $\angle ABC$ is *ordinary rational* angle. If the open ray BA (the open ray BC) contains lattice points, and the remaining open ray of the angle does not contain lattice points, then we say that the angle $\angle ABC$ is *ordinary R-irrational* (*L-irrational*) angle. If the union of open rays BA and BC does not contain lattice points, then we say that the angle $\angle ABC$ is ordinary lattice *LR-irrational* angle.

Definition 1.4. Two ordinary lattice angles $\angle AOB$ and $\angle A'O'B'$ are said to be *$\mathcal{L}$-congruent* if there exist a lattice-affine transformation which takes the vertex O to the vertex O' and the rays OA and OB to the rays $O'A'$ and $O'B'$ respectively. We denote this as follows: $\angle AOB \cong \angle A'O'B'$.

2. Some properties of ordinary lattice angles

2.1. **A few $\mathcal{L}$-congruence invariants.** We start this section with definitions of some important invariants of the group of lattice-affine transformations.

For a lattice segment AB (i.e. a segment with lattice endpoints) we define its *lattice lengths* to be equal to the number of lattice inner points plus one and denote it by $\mathrm{l}\ell(AB)$.

An *lattice area* of the parallelogram $ABCD$ with lattice points A, B, C, D is an index of sublattice generated by the vectors $\overline{AB}$ and $\overline{AC}$ in the whole lattice. We denote the area by $\mathrm{lS}(ABCD)$.

Consider an arbitrary rational angle $\angle ABC$. Let $D = C+\overline{BA}$. The *lattice sine* of $\angle ABC$ is a positive integer

$$\frac{\mathrm{lS}(ABCD)}{\mathrm{l}\ell(BA)\,\mathrm{l}\ell(BC)},$$

we denote it by $\operatorname{lsin}\angle ABC$.

Suppose some points A, B, and V are not in the same straight line. The *integer distance* from the lattice segment AB to the lattice point V is an index of sublattice generated by lattice vectors contained in AB and a vector $\overline{AV}$ in the whole lattice.

For the 3-tuples of lattice points A, B, and C we define the function sgn as follows:

$$\operatorname{sgn}(ABC) = \begin{cases} +1, & \text{if the couple of vectors } \overline{BA} \text{ and } \overline{BC} \text{ defines the positive orientation.} \\ 0, & \text{if the points } A, B, \text{ and } C \text{ are contained in the same straight line.} \\ -1, & \text{if the couple of vectors } \overline{BA} \text{ and } \overline{BC} \text{ defines the negative orientation.} \end{cases}$$

2.2. **LLS-sequences for ordinary angles.** Consider an ordinary angle $\angle AOB$. Let also the vectors $\overline{OA}$ and $\overline{OB}$ be linearly independent.

Denote the closed convex solid cone for the ordinary irrational angle $\angle AOB$ by $C(AOB)$. The boundary of the convex hull of all lattice points of the cone $C(AOB)$ except the origin is homeomorphic to the straight line. The closure in the plane of the intersection of this boundary with the open cone AOB is called the *sail* for the cone $C(AOB)$. A lattice point of the sail is said to be a *vertex* of the sail if there is no segment of the sail containing this point in the interior. The sail of the cone $C(AOB)$ is a broken line with a finite or infinite number of vertices and without self-intersections. We orient the sail in the direction from $\overline{OA}$ to $\overline{OB}$. (For the definition of the sail and its higher dimensional generalization, see, for instance, the works [1], [13], and [7].)

In the case of ordinary R-irrational and rational angle we denote the vertices of the sail by V_i, for $i \geq 0$, according to the orientation of the sail (such that V_0 is contained in the ray OA). In the case of ordinary L-irrational angle we denote the vertices of the sail by V_{-i}, for $i \geq 0$, according to the orientation of the sail (such that V_0 is contained in the ray OB). In the case of ordinary LR-irrational angle we denote the vertices of the sail by V_{-i}, for $i \in \mathbb{Z}$, according to the orientation of the sail (such that V_0 is an arbitrary vertex of the sail).

Definition 2.1. Suppose that the vectors $\overline{OA}$ and $\overline{OB}$ of an ordinary angle $\angle AOB$ are linearly independent. Let V_i be the vertices of the corresponding sail. The sequence of lattice lengths and sines

$$(\mathrm{l}\ell(V_0V_1), \mathrm{lsin}\,\angle V_0V_1V_2, \mathrm{l}\ell(V_1V_2), \mathrm{lsin}\,\angle V_1V_2V_3, \ldots, \mathrm{lsin}\,\angle V_{n-2}V_{n-1}V_n, \mathrm{l}\ell(V_{n-1}V_n)), \text{ or}$$
$$(\mathrm{l}\ell(V_0V_1), \mathrm{lsin}\,\angle V_0V_1V_2, \mathrm{l}\ell(V_1V_2), \mathrm{lsin}\,\angle V_1V_2V_3, \ldots), \text{ or}$$
$$(\ldots, \mathrm{lsin}\,\angle V_{-3}V_{-2}V_{-1}, \mathrm{l}\ell(V_{-2}V_{-1}), \mathrm{lsin}\,\angle V_{-2}V_{-1}V_0, \mathrm{l}\ell(V_{-1}V_0)), \text{ or}$$
$$(\ldots, \mathrm{lsin}\,\angle V_{-2}V_{-1}V_0, \mathrm{l}\ell(V_{-1}V_0), \mathrm{lsin}\,\angle V_{-1}V_0V_1, \mathrm{l}\ell(V_0V_1), \ldots)$$

is called the LLS-sequence for the ordinary angle $\angle AOB$, if this angle is rational, R-irrational, L-irrational, or LR-irrational respectively.

On Figure 1 we show an example of an LR-irrational ordinary angle $\angle AOB$. The convex hull of all lattice points inside is colored with gray, its boundary is the sail of the angle. The lattice lengths of the segments are in black and the lattice sines of the angles are in white respectively. The LLS-sequence of the angle is $(\ldots, 1, 1, 2, 3, 1, 1, 2, \ldots)$.

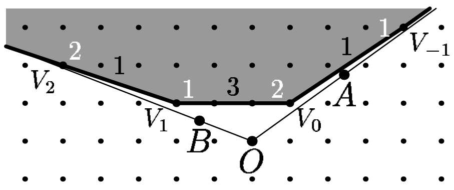


FIGURE 1. LR-irrational angle $\angle AOB$, its sail and LLS-sequence.

Proposition 2.2. a). *The elements of the LLS-sequence for any ordinary rational/irrational angle are positive integers.*
b). *The LLS-sequences of $\mathcal{L}$-congruent ordinary rational/irrational angles coincide.* □

2.3. **Lattice tangents for ordinary rational and R-irrational angles.** Let us give the definitions of lattice tangents for ordinary rational angles.

Definition 2.3. Let the vectors $\overline{OA}$ and $\overline{OB}$ of an ordinary rational angle $\angle AOB$ be linearly independent. Suppose that V_i are the vertices of the corresponding sail. Let

$$(\mathrm{l}\ell(V_0V_1), \mathrm{lsin}\,\angle V_0V_1V_2, \ldots, \mathrm{lsin}\,\angle V_{n-2}V_{n-1}V_n, \mathrm{l}\ell(V_{n-1}V_n))$$

be the LLS-sequence for the angle $\angle AOB$. The *lattice tangent* of the ordinary angle $\angle AOB$ is the following rational:

$$[\mathrm{l}\ell(V_0V_1), \mathrm{lsin}\,\angle V_0V_1V_2, \ldots, \mathrm{lsin}\,\angle V_{n-2}V_{n-1}V_n, \mathrm{l}\ell(V_{n-1}V_n)].$$

We denote it by $\mathrm{ltan}\,\angle AOB$.

Definition 2.4. Let the vectors $\overline{OA}$ and $\overline{OB}$ of an ordinary R-irrational angle $\angle AOB$ be linearly independent. Suppose that V_i are the vertices of the corresponding sail. Let

$$(\mathrm{l}\ell(V_0V_1), \mathrm{lsin}\,\angle V_0V_1V_2, \ldots, \mathrm{lsin}\,\angle V_{n-2}V_{n-1}V_n, \mathrm{l}\ell(V_{n-1}V_n), \ldots)$$

be the LLS-sequence for the angle $\angle AOB$. The *lattice tangent* of the ordinary R-irrational angle $\angle AOB$ is the following irrational:

$$[\mathrm{l}\ell(V_0V_1), \mathrm{lsin}\,\angle V_0V_1V_2, \ldots, \mathrm{lsin}\,\angle V_{n-2}V_{n-1}V_n, \mathrm{l}\ell(V_{n-1}V_n), \ldots].$$

We denote it by $\mathrm{ltan}\,\angle AOB$.

Let A, B, O, V_{-1}, V_1 be as on Figure 1, then

$$\mathrm{ltan}\,\angle V_{-1}OV_1 = [1,2,3] = \frac{10}{7}, \qquad \mathrm{ltan}\,\angle V_1OV_{-1} = [3,2,1] = \frac{10}{3},$$
$$\mathrm{ltan}\,\angle V_{-1}OB = [1,2,3,1,1,2,\ldots], \quad \mathrm{ltan}\,\angle V_1OA = [3,2,1,1,\ldots].$$

Proposition 2.5. a). *For any ordinary rational/R-irrational angle $\angle AOB$ with linearly independent vectors $\overline{OA}$ and $\overline{OB}$ the rational/irrational* $\mathrm{ltan}\,\angle AOB$ *is greater or equivalent to* 1.
b). *The values of the function* ltan *at two $\mathcal{L}$-congruent ordinary angles coincide.* □

2.4. **Lattice arctangent for ordinary rational and R-irrational angles.** Consider the system of coordinates OXY on the space $\mathbb{R}^2$ with the coordinates (x, y) and the origin O. We work with the integer lattice of OXY.

For any reals p_1 and p_2 we denote by α_{p_1,p_2} the angle with the vertex at the origin and two edges $\{(x, p_i x) | x > 0\}$, where $i = 1, 2$.

Definition 2.6. For any real $s \geq 1$, the ordinary angle $\angle AOB$ with the vertex O at the origin, $A = (1, 0)$, and $B = (1, s)$, is called the *lattice arctangent* of s and denoted by $\operatorname{larctan} s$.

The following theorem shows that ltan and larctan are actually inverse to each other.

Theorem 2.7. a). *For any real s, such that $s \geq 1$,*

$$\operatorname{ltan}(\operatorname{larctan} s) = s.$$

b). *For any ordinary rational or R-irrational angle α the following holds:*

$$\operatorname{larctan}(\operatorname{ltan} \alpha) \cong \alpha.$$

Proof. The both statements of the theorem for rational angles were proven in the paper [8].

Let us prove the first statement of Theorem 2.6 for the irrational case. Let $s > 1$ be some irrational real. Suppose that the sail of the angle $\operatorname{larctan} s$ is an infinite broken line $A_0A_1\ldots$ and the corresponding ordinary continued fraction is $[a_0, a_1, a_2, \ldots]$. Let also the coordinates of A_i be (x_i, y_i).

We consider the ordinary angles α_i, corresponding to the broken lines $A_0 \ldots A_i$, for $i > 0$. Then,

$$\lim_{i\to\infty} (y_i/x_i) = s/1.$$

By the statement of the theorem for rational angles for any positive integer i the ordinary angle α_i coincides with $\operatorname{larctan}([a_0, a_1, \ldots, a_{2i-2}])$, and hence the coordinates (x_i, y_i) of A_i satisfy

$$y_i/x_i = [a_0, a_1, \ldots, a_{2i-2}].$$

Therefore,

$$\lim_{i\to\infty} ([a_0, a_1, \ldots, a_{2i-2}]) = s.$$

So, we obtain the first statement of the theorem:

$$\operatorname{ltan}(\operatorname{larctan} s) = s.$$

Now we prove the second statement. Consider an ordinary lattice R-irrational angle α. Suppose that the sail of the angle α is the infinite broken line $A_0A_1\ldots$

For any positive integer i we consider the ordinary angle α_i, corresponding to the broken lines $A_0 \ldots A_i$.

For an ordinary angle β denote by $C(\beta)$ the cone, corresponding to β. Note that $C(\beta')$ and $C(\beta'')$ are $\mathcal{L}$-congruent iff $\beta \cong \beta'$.

By the statement of the theorem for rational angles we have:

$$\operatorname{larctan}(\operatorname{ltan} \alpha_i) \cong \alpha_i.$$

Since for any positive integer n the following is true

$$\bigcup_{i=1}^{n} C(\alpha_i) \cong \bigcup_{i=1}^{n} C(\mathrm{larctan}(\mathrm{ltan}\,\alpha_i))$$

we obtain

$$C(\alpha) \cong \bigcup_{i=1}^{\infty} C(\alpha_i) \cong \bigcup_{i=1}^{\infty} C(\mathrm{larctan}(\mathrm{ltan}\,\alpha_i)) \cong C(\mathrm{larctan}(\mathrm{ltan}\,\alpha)).$$

Therefore,

$$\mathrm{larctan}(\mathrm{ltan}\,\alpha) \cong \alpha.$$

This concludes the proof of Theorem 2.7. □

Now we give the following description of ordinary rational and R-irrational angles.

Theorem 2.8. (Description of ordinary rational and R-irrational angles.)
a). *For any finite/infinite sequence of positive integers* $(a_0, a_1, a_2, \ldots)$ *there exists some ordinary rational/R-irrational angle* α *such that* $\mathrm{ltan}\,\alpha = [a_0, a_1, a_2, \ldots]$.
b). *Two ordinary lattice rational/R-irrational angles are* $\mathcal{L}$*-congruent iff they have equivalent lattice tangents.*

Proof. Theorem 2.7a implies the first statement of the theorem.

Let us prove the second statement. Suppose that the ordinary rational/R-irrational angles α and β are $\mathcal{L}$-congruent, then their sails are also $\mathcal{L}$-congruent. Thus their LLS-sequences coincide. Therefore, $\mathrm{ltan}\,\alpha = \mathrm{ltan}\,\beta$.

Suppose now that the lattice tangents for two ordinary rational/R-irrational angles α and β are equivalent. Now we apply Theorem 2.7b and obtain

$$\alpha \cong \mathrm{larctan}(\mathrm{ltan}\,\alpha) = \mathrm{larctan}(\mathrm{ltan}\,\beta) \cong \beta.$$

Therefore, the angles α and β are $\mathcal{L}$-congruent. □

Corollary 2.9. (Description of ordinary L-irrational and LR-irrational angles.)
a). *For any sequence of positive integers* $(\ldots, a_{-2}, a_{-1}, a_0)$ *(or* $(\ldots a_{-1}, a_0, a_1, \ldots)$*) there exists an ordinary L-irrational (LR-irrational) angle with the LLS-sequence equivalent to the given one.*
b). *Two ordinary L-irrational (LR-irrational) angles are* $\mathcal{L}$*-congruent iff they have the same LLS-sequences.*

Proof. The statement on L-irrational angles follows immediately from Theorem 2.8.

Let us construct a LR-angle with a given LLS-sequence $(\ldots a_{-1}, a_0, a_1, \ldots)$. First we construct

$$\alpha_1 = \mathrm{larctan}([a_0, a_1, a_2, \ldots]).$$

Denote the points $(1, 0)$ and $(1, a_0)$ by A_0 and A_1 and construct the angle α_2 that is $\mathcal{L}$-congruent to the angle

$$\mathrm{larctan}([a_0, a_{-1}, a_{-2}, \ldots]),$$

and that has the first two vertices A_1 and A_0 respectively. Now the angle obtained by the rays of α_1 and α_2 that do not contain lattice points is the LR-angle with the given LLS-sequence.

Suppose now we have two LR-angles β_1 and β_2 with the same LLS-sequences. Consider a lattice transformation taking the vertex of β_2 to the vertex of β_1, and one of the segments of β_2 to the segment B_0B_1 of β_1 with the appropriate order in LLS-sequence. Denote this angle by β_2'. Consider the R-angles $\overline{\beta_1}$ and $\overline{\beta_2'}$ corresponding to the sequences of vertices of β_1 and β_2' starting from V_0 in the direction to V_1. These two angles are $\mathcal{L}$-congruent by Theorem 2.8, therefore $\overline{\beta_1}$ and $\overline{\beta_2'}$ coincide. So the angles β_1 and β_2' have a common ray. By the same reason the second ray of β_1 coincides with the second ray of β_2'. Therefore α_1 coincides with β_2' and $\mathcal{L}$-congruent to β_2. □

Remark on zero ordinary angles. Further we use zero ordinary angles and their trigonometric functions. Let A, B, and C be three lattice points of the same lattice straight line. Suppose that B is distinct to A and C and the rays BA and BC coincide. We say that the ordinary lattice angle with the vertex at B and the rays BA and BC is *zero*. Suppose $\angle ABC$ is zero, put by definition

$$\mathrm{lsin}(\angle ABC) = 0, \quad \mathrm{lcos}(\angle ABC) = 1, \quad \mathrm{ltan}(\angle ABC) = 0.$$

Denote by $\mathrm{larctan}(0)$ the angle $\angle AOA$ where $A = (1, 0)$, and O is the origin.

3. Lattice expanded angles

3.1. **Signed LLS-sequences.** In this subsection we work in the oriented two-dimensional real vector space with a fixed lattice. As previously, we fix coordinates OXY on this space.

A finite (infinite to the right, to the left, or both sides) union of ordered lattice segments $\ldots, A_{i-1}A_i, A_iA_{i+1}, A_{i+1}A_{i+2}, \ldots$ is said to be a *lattice oriented finite* (*R-infinite, L-infinite, or LR-infinite*) *broken line*, if any segment of the broken line is not of zero length, and any two consecutive segments are not contained in the same straight line. We denote this broken line by $\ldots A_{i-1}A_iA_{i+1}A_{i+2}\ldots$ We also say that the lattice oriented broken line $\ldots A_{i+2}A_{i+1}A_iA_{i-1}\ldots$ is *inverse* to the broken line $\ldots A_{i-1}A_iA_{i+1}A_{i+2}\ldots$

Definition 3.1. Consider a lattice infinite oriented broken line and a point not in this line. The broken line is said to be *on the unit distance* from the point if all edges of the broken line are on the unit lattice distance from the given point.

Now, let us associate to any lattice oriented broken line on the unit distance from some point the following sequence of non-zero elements.

Definition 3.2. Let $\ldots A_{i-1}A_iA_{i+1}A_{i+2}\ldots$ be a lattice oriented broken line on the unit distance from some lattice point V. Let

$$a_{2i-3} = \mathrm{sgn}(A_{i-2}VA_{i-1})\,\mathrm{sgn}(A_{i-1}VA_i)\,\mathrm{sgn}(A_{i-2}A_{i-1}A_i)\,\mathrm{lsin}\,\angle A_{i-2}A_{i-1}A_i,$$
$$a_{2i-2} = \mathrm{sgn}(A_{i-1}VA_i)\,\mathrm{l}\ell(A_{i-1}A_i)$$

for all possible indexes i. The sequence $(\ldots a_{2i-3}, a_{2i-2}, a_{2i-1}\ldots)$ is called a *signed LLS-sequence* for the lattice oriented finite/infinite broken line on the unit distance from V (or for simplicity just LSLS-sequence).

On Figure 2 we identify geometrically the signs of elements of the LSLS-sequence for a lattice oriented V-broken line on Figure 2.

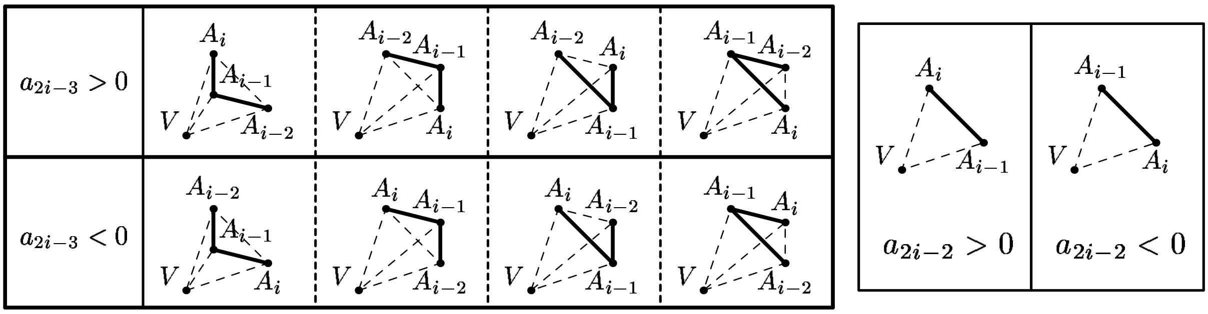


FIGURE 2. All possible different combinatorial cases for angles and segments of an LSLS-sequence.

Proposition 3.3. *An LSLS-sequence for the given lattice oriented broken line and the point is invariant under the group action of orientation preserving lattice-affine transformations.*

Proof. The statement holds, since the functions sgn, $\mathrm{l}\ell$, and lsin are invariant under the group action of orientation preserving lattice-affine transformations. □

3.2. **$\mathcal{L}_+$-congruence of lattice oriented broken lines on the unit distance from the lattice points.** Two lattice oriented broken lines on the unit distance from lattice points V_1 and V_2 are said to be *$\mathcal{L}_+$-congruent* iff there exists an orientation preserving lattice-affine transformation taking V_1 to V_2 and the first broken line to the second.

Let us formulate a necessary and sufficient conditions for two lattice oriented broken lines on the unit distance from two lattice points to be $\mathcal{L}_+$-congruent.

Theorem 3.4. *The LSLS-sequences of two lattice finite or infinite oriented broken lines on the unit distance from lattice points V_1 and V_2 respectively coincide, iff there exists an orientation preserving lattice-affine transformation taking the point V_1 to V_2 and one oriented broken line to the other.*

Proof. The case of finite broken lines was studied [6], we skip the proof here.

The LSLS-sequence for any lattice infinite oriented broken line on the unit distance is invariant under the group action of orientation preserving lattice-affine transformations, since functions sgn, $\mathrm{l}\ell$, lsin are invariant. Therefore, the LSLS-sequences for two $\mathcal{L}_+$-congruent broken lines coincide.

Suppose now that we have two lattice oriented infinite broken lines $\ldots A_{i-1}A_iA_{i+1}\ldots$ and $\ldots B_{i-1}B_iB_{i+1}\ldots$ on the unit distance from the points V_1 and V_2, and with the same LSLS-sequences. Consider the lattice-affine transformation ξ that takes the point V_1 to V_2, A_i to B_i, and A_{i+1} to B_{i+1} for some integer i. Since $\mathrm{sgn}(A_iVA_{i+1}) = \mathrm{sgn}(B_iVB_{i+1})$, the lattice-affine transformation ξ is orientation preserving. By Theorem 3.4 for the finite case the transformation ξ takes any finite oriented broken line $A_sA_{s+1}\ldots A_t$ containing the segment A_iA_{i+1} to the oriented broken line $B_sB_{s+1}\ldots B_t$. Therefore, the transformation ξ takes the lattice oriented infinite broken line $\ldots A_{i-1}A_iA_{i+1}\ldots$ to the oriented broken line $\ldots B_{i-1}B_iB_{i+1}\ldots$ and the lattice point V_1 to the lattice point V_2.

This concludes the proof of Theorem 3.4 for the infinite broken lines. □

3.3. Equivalence classes of almost positive lattice infinite oriented broken lines and corresponding expanded infinite angles.

We start this section with the following general definition.

Definition 3.5. We say that the lattice infinite oriented broken line on the unit distance from some lattice point is *almost positive* if the elements of the corresponding LSLS-sequence are all positive, except for a finite number of elements.

Let l be the lattice (finite or infinite) oriented broken line $\ldots A_{n-1}A_n\ldots A_mA_{m+1}\ldots$ Denote by $l(-\infty, A_n)$ the broken line $\ldots A_{n-1}A_n$. Denote by $l(A_m, +\infty)$ the broken line $A_mA_{m+1}\ldots$ Denote by $l(A_n, A_m)$ the broken line $A_n\ldots A_m$.

Definition 3.6. Two lattice oriented infinite broken lines l_1 and l_2 on unit distance from V are said to be *equivalent* if there exist two vertices W_{11} and W_{12} of the broken line l_1 and two vertices W_{21} and W_{22} of the broken line l_2 such that the following three conditions are satisfied:
i) the broken line $l_1(W_{12}, +\infty)$ coincides (edge by edge) with the broken line $l_2(W_{22}, +\infty)$;
ii) the broken line $l_1(-\infty, W_{11})$ coincides with the broken line $l_2(-\infty, W_{21})$;
iii) the closed broken line generated by $l_1(W_{11}, W_{12})$ and the inverse of $l_2(W_{21}, W_{22})$ is homotopy equivalent to the point on $\mathbb{R}^2 \setminus \{V\}$.

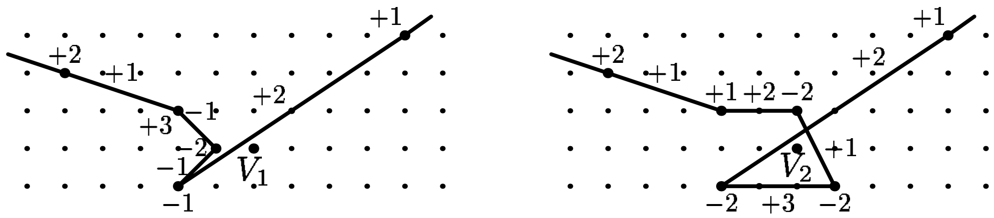
Now we give the definition of expanded angles.

Definition 3.7. An equivalence class of lattice finite (R/L/LR-infinite) oriented broken lines on unit distance from V containing the broken line l is called the *expanded finite (R/L/LR-infinite) angle for the equivalence class of* l at the *vertex* V and denoted by $\angle(V; l)$ (or, for short, *expanded R/L/LR-infinite angle*).

Remark 3.8. Since all the sails for ordinary angles are lattice oriented broken lines, the set of all ordinary irrational angles is naturally embedded into the set of expanded irrational angles. An ordinary angle with a sail S corresponds to the expanded angle with the equivalence class of the broken line S.

Definition 3.9. Two expanded angles Φ_1 and Φ_2 are said to be $\mathcal{L}_+$*-congruent* iff there exists an orientation preserving lattice-affine transformation sending the class of lattice oriented broken lines corresponding to Φ_1 to the class of lattice oriented broken lines corresponding to Φ_2. We denote it by $\Phi_1 \hat{\cong} \Phi_2$.

On Figure 3 we show two LR-infinite broken lines, we also indicate their LSLS-sequences. (We suppose that outside of the pictures the broken lines are the same.) These broken lines define two non-equivalent expanded LR-infinite angles. The broken line of Figure 3a and the sail of Figure 1a defines equivalent expanded angles.



a:$(\ldots 1, 2, -1, -1, -2, -1, 3, 1, 2 \ldots)$. b):$(\ldots 1, 2, -2, 3, -2, 1, -2, 2, 1, 1, 2 \ldots)$.

FIGURE 3. Examples of expanded angles for two particular LSLS-sequences.

3.4. **Revolution number for expanded rational, L- and R-irrational angles.** First we define the revolution number for the case of finite broken lines.

Let $r = \{V + \lambda\overline{v} | \lambda \geq 0\}$ be the oriented ray for an arbitrary vector $\overline{v}$ with the vertex at V, and AB be an oriented (from A to B) segment not contained in the ray r. Suppose also, that the vertex V of the ray r is not contained in the segment AB. We denote by $\#(r, V, AB)$ the following number:

$$\#(r, V, AB) = \begin{cases} 0, & \text{if the segment } AB \text{ does not intersect the ray } r \\ \frac{1}{2}\operatorname{sgn}\Big(A(A+\overline{v})B\Big), & \text{if the segment } AB \text{ intersects the ray } r \text{ at } A \text{ or} \\ & \quad \text{at } B \\ \operatorname{sgn}\Big(A(A+\overline{v})B\Big), & \text{if the segment } AB \text{ intersects the ray } r \text{ at the} \\ & \quad \text{interior point of } AB \end{cases},$$

and call it the *intersection number* of the ray r and the segment AB.

Definition 3.10. Let $A_0A_1 \ldots A_n$ be some lattice oriented broken line, and let r be an oriented ray $\{V + \lambda\overline{v} | \lambda \geq 0\}$. Suppose that the ray r does not contain the edges of the broken line, and the broken line does not contain the point V. We call the number

$$\sum_{i=1}^{n} \#(r, V, A_{i-1}A_i)$$

the *intersection number* of the ray r and the lattice oriented broken line $A_0A_1 \ldots A_n$, and denote it by $\#(r, V, A_0A_1 \ldots A_n)$.

Definition 3.11. Consider an arbitrary expanded angle $\angle(V, A_0A_1 \ldots A_n)$. Denote the rays $\{V + \lambda\overline{VA_0} | \lambda \geq 0\}$ and $\{V - \lambda\overline{VA_0} | \lambda \geq 0\}$ by r_+ and r_- respectively. The number

$$\frac{1}{2}\big(\#(r_+, V, A_0A_1 \ldots A_n) + \#(r_-, V, A_0A_1 \ldots A_n)\big)$$

is called the *lattice revolution number* for the expanded angle $\angle(V, A_0A_1 \ldots A_n)$, and denoted by $\#(\angle(V, A_0A_1 \ldots A_n))$.

The revolution number of any expanded angle is well-defined and is invariant under the group action of the orientation preserving lattice-affine transformations (see [6] for more details).

Let now us extend the revolution number to the case of almost positive infinite oriented broken lines.

Definition 3.12. Let $\ldots A_{i-1}A_iA_{i+1}\ldots$ be some lattice R-, L- or LR-infinite almost positive oriented broken line, and $r=\{V+\lambda\overline{v}|\lambda\ge 0\}$ be the oriented ray for an arbitrary vector $\overline{v}$ with the vertex at V. Suppose that all straight lines containing the edges of the broken line do not the vertex V. We call the number

$$\begin{array}{ll}\lim\limits_{n\to+\infty}\#(r,V,A_0A_1\ldots A_n) & \text{if the broken line is R-infinite,}\\ \lim\limits_{n\to+\infty}\#(r,V,A_{-n}\ldots A_{-1}A_0) & \text{if the broken line is L-infinite,}\\ \lim\limits_{n\to+\infty}\#(r,V,A_{-n}A_{-n+1}\ldots A_n) & \text{if the broken line is LR-infinite}\end{array}$$

the *intersection number* of the ray r and the lattice almost positive infinite oriented broken line broken line $\ldots A_{i-1}A_iA_{i+1}\ldots$ and denote it by $\#(r,V,\ldots A_{i-1}A_iA_{i+1}\ldots)$.

Proposition 3.13. *The intersection number of a ray r and an almost positive lattice infinite oriented broken line is well-defined.*

Proof. Consider an almost positive lattice infinite oriented broken line l. Let us show that the broken line l intersects the ray r only finitely many times.

By Definition 3.5 there exist vertices W_1 and W_2 of this broken line such that the LSLS-sequence for the lattice oriented broken line $l(-\infty,W_1)$ contains only positive elements, and the LSLS-sequence for the oriented broken line $l(W_2,+\infty)$ also contains only positive elements.

The positivity of the LLS-sequences implies that the lattice oriented broken lines $l(-\infty,W_1)$, and $l(W_2,+\infty)$ are the sails for some angles with the vertex V. Thus, these two broken lines intersect the ray r at most once each. Therefore, the broken line l intersects the ray r at most once at the part $l(-\infty,W_1)$, only a finite number times at the part $l(W_1,W_2)$, and at most once at the part $l(W_2,+\infty)$.

So, the lattice infinite oriented broken line l intersects the ray r only finitely many times, and, therefore, the corresponding intersection number is well-defined. □

Now we give a definition of the lattice revolution number for expanded R-irrational and L-irrational angles.

Definition 3.14. a). Consider an arbitrary R-infinite (or L-infinite) expanded angle $\angle(V,l)$, where V is some lattice point, and l is a lattice infinite oriented almost-positive broken line. Let A_0 be the first (the last) vertex of l. Denote the rays $\{V+\lambda\overline{VA_0}|\lambda\ge 0\}$ and $\{V-\lambda\overline{VA_0}|\lambda\ge 0\}$ by r_+ and r_- respectively. The following number

$$\frac{1}{2}\big(\#(r_+,V,l)+\#(r_-,V,l)\big)$$

is called the *lattice revolution number* for the expanded irrational angle $\angle(V,l)$, and denoted by $\#(\angle(V,l))$.

The revolution numbers for the angles defined by the broken lines of Figure 3a and 3b are respectively: $1/4$ and $5/4$.

Proposition 3.15. *The revolution number of an R-irrational (or L-irrational) expanded angle is well-defined.*

Proof. Consider an arbitrary expanded R-irrational angle $\angle(V, A_0A_1\ldots)$. Let

$$r_+ = \{V + \lambda\overline{VA_0}|\lambda \geq 0\} \quad \text{and} \quad r_- = \{V - \lambda\overline{VA_0}|\lambda \geq 0\}.$$

Suppose that

$$\angle V, A_0A_1A_2\ldots = \angle V', A'_0A'_1A'_2\ldots$$

This implies that $V = V'$, $A_0 = A'_0$, $A_{n+k} = A'_{m+k}$ for some integers n and m and any non-negative integer k, and the broken lines $A_0A_1\ldots A_nA'_{m-1}\ldots A'_1A'_0$ is homotopy equivalent to the point on $\mathbb{R}^2 \setminus \{V\}$. Thus,

$$\begin{aligned}&\#(\angle V, A_0A_1\ldots) - \#(\angle V', A'_0A'_1\ldots) = \\ &\tfrac{1}{2}\big(\#(r_+, A_0A_1\ldots A_nA'_{m-1}\ldots A'_1A'_0) + \#(r_-, A_0A_1\ldots A_nA'_{m-1}\ldots A'_1A'_0)\big) \\ &= 0{+}0 = 0.\end{aligned}$$

And hence

$$\#(\angle V, A_0A_1A_2\ldots) = \#(\angle V', A'_0A'_1A'_2\ldots).$$

Therefore, the revolution number of any expanded R-irrational angle is well-defined.

The proof for L-irrational angles repeats the proof for R-irrational angles and is omitted here. □

Proposition 3.16. *The revolution number of expanded R/L-irrational angles is invariant under the group action of the orientation preserving lattice-affine transformations.* □

Let us finally give the definition of trigonometric functions for the expanded angles and describe some relations between ordinary and expanded angles.

Definition 3.17. Consider an arbitrary expanded angle Φ with the normal form $k\pi{+}\varphi$ for some ordinary (possible zero) angle φ and for an integer k.
a). The ordinary angle φ is said to be *associated* with the expanded angle Φ.
b). The numbers $\mathrm{ltan}(\varphi)$, $\mathrm{lsin}(\varphi)$, and $\mathrm{lcos}(\varphi)$ are called the *lattice tangent*, the *lattice sine*, and the *lattice cosine* of the expanded angle Φ.

3.5. **Normal forms of expanded rational angles.** In this section we list the results of [6] in rational case.

We use the following notation. By the sequence

$$\big((a_0, \ldots, a_n) \times k\text{-times}, b_0, \ldots, b_m\big),$$

where $k \geq 0$, we denote the following sequence:

$$(\underbrace{a_0, \ldots, a_n, a_0, \ldots, a_n, \quad \ldots \quad, a_0, \ldots, a_n}_{k\text{-times}}, b_0, \ldots, b_m).$$

Definition 3.18. I). Suppose O be the origin, A_0 be the point $(1, 0)$. We say that the expanded angle $\angle(O, A_0)$ is *of the type* **I** and denote it by $0\pi + \mathrm{larctan}(0)$ (or 0, for short). The empty sequence is said to be *characteristic* for the angle $0\pi + \mathrm{larctan}(0)$.

Consider a lattice oriented broken line $A_0A_1\ldots A_s$ on the unit distance from the origin

O. Let also A_0 be the point $(1,0)$, and the point A_1 be on the straight line $x=1$. If the LSLS-sequence of the expanded angle $\Phi_0=\angle(O,A_0A_1\ldots A_s)$ coincides with the following sequence (we call it *characteristic sequence* for the corresponding angle):
II$_k$) $\big((1,-2,1,-2)\times(k-1)\text{-times},1,-2,1\big)$, where $k\ge 1$, then we denote the angle Φ_0 by $k\pi+\operatorname{larctan}(0)$ (or $k\pi$, for short) and say that Φ_0 is *of the type* $\mathbf{II}_k$;
III$_k$) $\big((-1,2,-1,2)\times(k-1)\text{-times},-1,2,-1\big)$, where $k\ge 1$, then we denote the angle Φ_0 by $-k\pi+\operatorname{larctan}(0)$ (or $-k\pi$, for short) and say that Φ_0 is *of the type* $\mathbf{III}_k$;
IV$_k$) $\big((1,-2,1,-2)\times k\text{-times},a_0,\ldots,a_{2n}\big)$, where $k\ge 0$, $n\ge 0$, $a_i>0$, for $i=0,\ldots,2n$, then we denote the angle Φ_0 by $k\pi+\operatorname{larctan}([a_0,a_1,\ldots,a_{2n}])$ and say that Φ_0 is *of the type* $\mathbf{IV}_k$;
V$_k$) $\big((-1,2,-1,2)\times k\text{-times},a_0,\ldots,a_{2n}\big)$, where $k>0$, $n\ge 0$, $a_i>0$, for $i=0,\ldots,2n$, then we denote the angle Φ_0 by $-k\pi+\operatorname{larctan}([a_0,a_1,\ldots,a_{2n}])$ and say that Φ_0 is *of the type* $\mathbf{V}_k$.

Theorem 3.19. *For any expanded rational angle Φ there exist and unique a type among the types* **I**-**V** *and a unique rational expanded angle Φ_0 of that type such that Φ_0 is $\mathcal{L}_+$-congruent to Φ.*
The expanded angle Φ_0 is said to be *the normal form* for the expanded angle Φ. □

Further we use the following lemma of [6].

Lemma 3.20. *Let m, $k\ge 1$, and $a_i>0$ for $i=0,\ldots,2n$ be some integers.*
a). *Suppose the LSLS-sequences for the expanded angles Φ_1 and Φ_2 are respectively*

$$\big((1,-2,1,-2)\times(k{-}1)\text{-}times,1,-2,1,-2,a_0,\ldots,a_{2n}\big)\quad and$$
$$\big((1,-2,1,-2)\times(k{-}1)\text{-}times,1,-2,1,m,a_0,\ldots,a_{2n}\big),$$

then Φ_1 is $\mathcal{L}_+$-congruent to Φ_2.
b). *Suppose the LSLS-sequences for the expanded angles Φ_1 and Φ_2 are respectively*

$$\big((-1,2,-1,2)\times(k{-}1)\text{-}times,-1,2,-1,m,a_0,\ldots,a_{2n}\big)\quad and$$
$$\big((-1,2,-1,2)\times(k{-}1)\text{-}times,-1,2,-1,2,a_0,\ldots,a_{2n}\big),$$

then Φ_1 is $\mathcal{L}_+$-congruent to Φ_2. □

3.6. **Normal forms of expanded R- and L-irrational angles.** In this section we formulate and prove a theorem on normal forms of expanded lattice R-irrational and L-irrational angles.

For the theorems of this section we introduce the following notation. By the sequence

$$\big((a_0,\ldots,a_n)\times k\text{-times},b_0,b_1\ldots\big),$$

where $k\ge 0$, we denote the sequence:

$$(\underbrace{a_0,\ldots,a_n,a_0,\ldots,a_n,\quad\ldots\quad,a_0,\ldots,a_n}_{k\text{-times}},b_0,b_1,\ldots).$$

By the sequence

$$\big(\ldots,b_{-2},b_{-1},b_0,(a_0,\ldots,a_n)\times k\text{-times}\big),$$

where $k \geq 0$, we denote the sequence:

$$(\ldots, b_{-2}, b_{-1}, b_0, \underbrace{a_0, \ldots, a_n, a_0, \ldots, a_n, \quad \ldots \quad, a_0, \ldots, a_n}_{k\text{-times}}).$$

We start with the case of expanded R-irrational angles.

Definition 3.21. Consider a lattice R-infinite oriented broken line $A_0A_1\ldots$ on the unit distance from the origin O. Let also A_0 be the point $(1, 0)$, and the point A_1 be on the line $x = 1$. If the LSLS-sequence of the expanded R-irrational angle $\Phi_0 = \angle(O, A_0A_1\ldots)$ coincides with the following sequence (we call it *characteristic sequence* for the corresponding angle):
$\mathbf{IV}_k$) $\big((1, -2, 1, -2) \times k\text{-times}, a_0, a_1, \ldots\big)$, where $k \geq 0$, $a_i > 0$, for $i \geq 0$, then we denote the angle Φ_0 by $k\pi+\operatorname{larctan}([a_0, a_1, \ldots])$ and say that Φ_0 is *of the type* $\mathbf{IV}_k$;
$\mathbf{V}_k$) $\big((-1, 2, -1, 2) \times k\text{-times}, a_0, a_1, \ldots\big)$, where $k > 0$, $a_i > 0$, for $i \geq 0$, then we denote the angle Φ_0 by $-k\pi + \operatorname{larctan}([a_0, a_1, \ldots])$ and say that Φ_0 is *of the type* $\mathbf{V}_k$.

Theorem 3.22. *For any expanded R-irrational angle Φ there exist and unique a type among the types* **IV**-**V** *and a unique expanded R-irrational angle Φ of that type such that Φ_0 is $\mathcal{L}_+$-congruent to Φ_0.*
The expanded R-irrational angle Φ_0 is said to be *the normal form* for the expanded R-irrational angle Φ.

Proof. First, we prove that any two distinct expanded R-irrational angles listed in Definition 3.21 are not $\mathcal{L}_+$-congruent. Let us note that the revolution numbers of expanded angles distinguish the types of the angles. The revolution number for the expanded angles of the type $\mathbf{IV}_k$ is $1/4+1/2k$ where $k \geq 0$. The revolution number for the expanded angles of the type $\mathbf{V}_k$ is $1/4-1/2k$ where $k > 0$.

We now prove that the normal forms of the same type $\mathbf{IV}_k$ (or $\mathbf{V}_k$) are not $\mathcal{L}_+$-congruent for any integer k. Consider the expanded R-infinite angle $\Phi = k\pi+\operatorname{larctan}([a_0, a_1, \ldots])$. Suppose that a lattice oriented broken line $A_0A_1A_2\ldots$ on the unit distance from O defines the angle Φ. Let also that the LSLS-sequence for this broken line be characteristic.

If k is even, then the ordinary R-irrational angle with the sail $A_{2k}A_{2k+1}\ldots$ is $\mathcal{L}_+$-congruent to $\operatorname{larctan}([a_0, a_1, \ldots])$. This angle is a proper lattice-affine invariant for the expanded R-irrational angle Φ (since $A_{2k} = A_0$). This invariant distinguish the expanded R-irrational angles of type $\mathbf{IV}_k$ (or $\mathbf{V}_k$) for even k.

If k is odd, then denote $B_i = V+\overline{A_iV}$. The ordinary R-irrational angle with the sail $B_{2k}B_{2k+1}\ldots$ is $\mathcal{L}_+$-congruent to the angle $\operatorname{larctan}([a_0, a_1, \ldots])$. This angle is a proper lattice-affine invariant of the expanded R-irrational angle Φ (since $B_{2k} = V+\overline{A_0V}$). This invariant distinguish the expanded R-irrational angles $\mathbf{IV}_k$ (or $\mathbf{V}_k$) for odd k.

Therefore, the expanded angles listed in Definition 3.21 are not $\mathcal{L}_+$-congruent.

Secondly, we prove that an arbitrary expanded R-irrational angle is $\mathcal{L}_+$-congruent to some of the expanded angles listed in Definition 3.21.

Consider an arbitrary expanded R-irrational angle $\Phi = \angle(V, A_0A_1\ldots)$. Suppose that $\#(\Phi) = 1/4+k/2$ for some non-negative integer k. By Proposition 3.13 there exist an integer positive number n_0 such that the lattice oriented broken line $A_{n_0}A_{n_0+1}\ldots$ does not intersect the rays $r_+ = \{V+\lambda\overline{VA_0}|\lambda \geq 0\}$ and $r_- = \{V-\lambda\overline{VA_0}|\lambda \geq 0\}$, and the LSLS-sequence $(a_{2n_0-2}, a_{2n_0-1}\ldots)$ for the oriented broken line $A_{n_0}A_{n_0+1}\ldots$ does not contain non-positive elements.

By Theorem 3.19 there exist integers k and m, and a lattice oriented broken line

$$A_0B_1B_2\ldots B_{2k}B_{2k+1}\ldots B_{2k+m}A_{n_0}$$

with LLS-sequence of the form

$$\big((1,-2,1,-2)\times k\text{-times}, b_0, b_1, \ldots, b_{2m-2}\big),$$

where all b_i are positives.

Consider now the lattice oriented infinite broken line $A_0B_1B_2\ldots B_{2k+m-1}A_{n_0}A_{n_0+1}\cdots$ The LLS-sequence for this broken line is as follows

$$\big((1,-2,1,-2)\times k\text{-times}, b_0, b_1, \ldots, b_{2m-2}, v, a_{2n_0-2}a_{2n_0-1}\ldots\big),$$

where v is (not necessary positive) integer.

Note that the lattice oriented broken line $A_0B_1B_2\ldots B_{2k+m}A_{n_0}$ is a sail for the angle $\angle A_0VA_{n_0}$ and the broken line $A_{n_0}A_{n_0+1}\ldots$ is a sail for some R-irrational angle (we denote it by α). Let H_1 be the convex hull of all lattice points of the angle $\angle A_0VA_{n_0}$ except the origin, and H_2 be the convex hull of all lattice points of the angle α except the origin. Note that H_1 intersects H_2 in the ray with the vertex at A_{n_0}.

The lattice oriented infinite broken line $B_{2k}B_{2k+2}\ldots B_{2k+m}A_{n_0}A_{n_0+1}\ldots$ intersects the ray r_+ in the unique point B_{2k} and does not intersect the ray r_-. Hence there exists a straight line l intersecting both boundaries of H_1 and H_2, such that the open half-plane with the boundary straight line l containing the origin does not intersect the sets H_1 and H_2.

Denote $B_0 = A_0$ and $B_{2k+m+1} = A_{n_0}$. The intersection of the straight line l with H_1 is either a point B_s (for $2k \leq s \leq 2k+m+1$), or a boundary segment B_sB_{s+1} for some integer s satisfying $2k \leq s \leq 2k+m$. The intersection of l with H_2 is either a point A_t for some integer $t \geq n_0$, or a boundary segment $A_{t-1}A_t$ for some integer $t > n_0$.

Since the triangle $\triangle VA_tB_s$ does not contain interior points of H_1 and H_2, the lattice points of $\triangle VA_tB_s$ distinct to B are on the segment A_tB_s. Hence, the segment A_tB_s is on unit lattice distance to the vertex V. Therefore, the lattice infinite oriented broken line

$$A_0B_1B_2\ldots B_sA_tA_{t+1}\ldots$$

is on lattice unit distance.

Since the lattice oriented broken line $B_k\ldots B_sA_tA_{t+1}\ldots$ is convex, it is a sail for some R-irrational angle. (Actually, the case $B_s = A_t = A_{n_0}$ is also possible, then delete one of the copies of A_{n_0} from the sequence.) We denote this broken like by $C_{2k+1}C_{2k+2}\ldots$ The corresponding LSLS-sequence is $(c_{4k}, c_{4k+1}, c_{4k+2}, \ldots)$, where $c_i > 0$ for $i \geq 4k$. Thus the LSLS-sequence for the lattice ordered broken line $A_0B_1B_2\ldots B_{2k}C_{2k+1}C_{2k+2}\ldots$ is

$$\big((1,-2,1,-2)\times(k-1)\text{-times}, 1,-2,1,w,(c_{4k}, c_{4k+1}, c_{4k+2}, \ldots\big),$$

where w is an integer that is not necessary equivalent to -2.

Consider an expanded angle $\angle(V, A_0B_1B_2\ldots B_{2k}C_{2k+1})$. By Lemma 3.20 there exists a lattice oriented broken line $C_0\ldots C_{2k+1}$ with the vertices $C_0 = A_0$ and C_{2k+1} of the same equivalence class, such that $C_{2k} = B_{2k}$, and the LSLS-sequence for it is

$$\big((1,-2,1,-2)\times k\text{-times}, c_{4k}, c_{4k+1}\big).$$

Therefore, the lattice oriented R-infinite broken line $C_0C_1\ldots$ for the angle $\angle(V, A_0A_1\ldots)$ has the LSLS-sequence coinciding with the characteristic sequence for the angle $k\pi + \operatorname{larctan}([c_{4k}, c_{4k+1},\ldots])$. Therefore,

$$\Phi \hat{\cong} k\pi + \operatorname{larctan}([c_{4k}, c_{4k+1},\ldots]).$$

This concludes the proof of the theorem for the case of nonnegative integer k.

The proof for the case of negative k repeats the proof for the nonnegative case and is omitted here. □

Let us give the definition of trigonometric functions for expanded R-irrational angles.

Definition 3.23. Consider an arbitrary expanded R-irrational angle Φ with the normal form $k\pi+\varphi$ for some integer k.
a). The ordinary R-irrational angle φ is said to be *associated* with the expanded R-irrational angle Φ.
b). The number $\operatorname{ltan}(\varphi)$ is called the lattice *tangent* of the expanded R-irrational angle Φ.

We continue now with the case of expanded L-irrational angles.

Definition 3.24. The expanded irrational angle $\angle(V,\ldots A_{i+2}A_{i+1}A_i\ldots)$ is said to be transpose to the expanded irrational angle $\angle(V,\ldots A_iA_{i+1}A_{i+2}\ldots)$ and denoted by $(\angle(V,\ldots A_iA_{i+1}A_{i+2}\ldots))^t$.

Definition 3.25. Consider a lattice L-infinite oriented broken line $\ldots A_{-1}A_0$ on the unit distance from the origin O. Let also A_0 be the point $(1,0)$, and the point A_{-1} be on the straight line $x = 1$. If the LSLS-sequence of the expanded L-irrational angle $\Phi_0 = \angle(O,\ldots A_{-1}A_0)$ coincides with the following sequence (we call it *characteristic sequence* for the corresponding angle):
IV$_k$) $\big(\ldots, a_{-1}, a_0, (-2,1,-2,1)\times k\text{-times}\big)$, where $k \ge 0$, $a_i > 0$, for $i \le 0$, then we denote the angle Φ_0 by $k\pi + \operatorname{larctan}^t([a_0, a_{-1},\ldots])$ and say that Φ_0 is *of the type* **IV**$_k$;
V$_k$) $\big(\ldots, a_{-1}, a_0, (2,-1,2,-1)\times k\text{-times}\big)$, where $k > 0$, $a_i > 0$, for $i \le 0$, then we denote the angle Φ_0 by $-k\pi + \operatorname{larctan}^t([a_0, a_{-1},\ldots])$ and say that Φ_0 is *of the type* **V**$_k$.

Theorem 3.26. *For any expanded L-irrational angle Φ there exist a unique type among the types* **IV**-**V** *and a unique expanded L-irrational angle Φ_0 of that type such that Φ is $\mathcal{L}_+$-congruent to Φ_0.*
The expanded L-irrational angle Φ_0 is said to be *the normal form* for the expanded L-irrational angle Φ.

Proof. After transposing the set of all angles and change of the orientation of the plane the statement of Theorem 3.26 coincide with the statement of Theorem 3.22. □

4. Sums of expanded angles and expanded irrational angles

Now we can give definitions of sums of ordinary angles, and ordinary R-irrational or/and L-irrational angles.

Definition 4.1. Consider expanded angles Φ_i, where $i = 1, \ldots, t$, an expanded R-irrational angle Φ_r, and an expanded L-irrational angle Φ_l. Let the characteristic LSLS-sequences for the normal forms of the angles Φ_i be $(a_{0,i}, a_{1,i}, \ldots, a_{2n_i,i})$; of Φ_r — be $(a_{0,r}, a_{1,r}, \ldots)$, and of Φ_l — be $(\ldots, a_{-1,l}, a_{0,l})$.
Let $M = (m_1, \ldots, m_{t-1})$ be some $(t-1)$-tuple of integers. The normal form of any expanded angle, corresponding to the following LSLS-sequence

$$\begin{aligned}\big(a_{0,1}, a_{1,1}, \ldots, a_{2n_1,1}, m_1, a_{0,2}, a_{1,2}, \ldots, a_{2n_2,2}, m_2, \ldots\\ \ldots, m_{t-1}, a_{0,t}, a_{1,t}, \ldots, a_{2n_t,t}\big)\end{aligned}$$

is called the *M-sum of expanded angles* Φ_i $(i = 1, \ldots, t)$.
Let $M_R = (m_1, \ldots, m_{t-1}, m_r)$ be some t-tuple of integers. The normal form of any expanded angle, corresponding to the following LSLS-sequence

$$\begin{aligned}\big(a_{0,1}, a_{1,1}, \ldots, a_{2n_1,1}, m_1, a_{0,2}, a_{1,2}, \ldots, a_{2n_2,2}, m_2, \ldots\\ \ldots, m_{t-1}, a_{0,t}, a_{1,t}, \ldots, a_{2n_t,t} m_r, a_{0,r}, a_{1,r}, \ldots\big)\end{aligned}$$

is called the *M_R-sum of expanded angles* Φ_i $(i = 1, \ldots, t)$ and Φ_r.
Let $M_L = (m_l, m_1, \ldots, m_{t-1})$ be some t-tuple of integers. The normal form for any expanded angle, corresponding to the following LSLS-sequence

$$\begin{aligned}\big(\ldots, a_{-1,l}, a_{0,l}, m_l, a_{0,1}, a_{1,1}, \ldots, a_{2n_1,1}, m_1, a_{0,2}, a_{1,2}, \ldots, a_{2n_2,2}, m_2, \ldots\\ \ldots, m_{t-1}, a_{0,t}, a_{1,t}, \ldots, a_{2n_t,t}\big)\end{aligned}$$

is called the *M_L-sum of expanded angles Φ_l, and* Φ_i $(i = 1, \ldots, t)$.
Let $M_{LR} = (m_l, m_1, \ldots, m_{t-1}, m_r)$ be some $(t+1)$-tuple of integers. Any expanded LR-irrational angle, corresponding to the following LSLS-sequence

$$\begin{aligned}\big(\ldots, a_{-1,l}, a_{0,l}, m_l, a_{0,1}, a_{1,1}, \ldots, a_{2n_1,1}, m_1, a_{0,2}, a_{1,2}, \ldots, a_{2n_2,2}, m_2, \ldots\\ \ldots, m_{t-1}, a_{0,t}, a_{1,t}, \ldots, a_{2n_t,t} m_r, a_{0,r}, a_{1,r}, \ldots\big)\end{aligned}$$

is called a *M_{LR}-sum of expanded angles* Φ_l, Φ_i $(i = 1, \ldots, t)$ and Φ_r.

We denote the $(a_1, a_2, \ldots, a_{n-1})$-sum of angles $\Phi_1, \ldots, \Phi_n$ by

$$\Phi_1 +_{a_1} \Phi_2 +_{a_2} \ldots +_{a_{n-1}} \Phi_n.$$

Finally we give a few examples of sums.

$$\begin{aligned}\operatorname{larctan} 1 +_n \operatorname{larctan} 1 &\;\hat{\cong}\; \operatorname{larctan} \tfrac{n+2}{n+1} \ (n > 0),\\ \operatorname{larctan} 3 +_{-1} \operatorname{larctan} \tfrac{5}{3} +_{-2} \operatorname{larctan} 1 &\;\hat{\cong}\; 2\pi,\\ \operatorname{larctan} \tfrac{3}{2} +_1 \operatorname{larctan} \tfrac{1+\sqrt{5}}{2} &\;\hat{\cong}\; \operatorname{larctan} \tfrac{1+\sqrt{5}}{2}.\end{aligned}$$

5. Related questions and problems

We conclude the paper with the following questions and problems.

Problem 1. a). Find a natural definition of lattice tangents for L-irrational angles, and LR-irrational angles.
b). Find a natural definition of lattice sines and cosines for irrational angles (see also in [6]).

Problem 2. Does there exist a natural definition of the sums of
a) any expanded LR-irrational angle and any expanded angle;
b) any expanded R-irrational angle and any expanded angle;
c) any expanded angle and any expanded L-irrational angle?

Problem 3. Find an effective algorithm to verify whether two given almost-positive LSLS-sequences define $\mathcal{L}$-congruent expanded irrational angles, or not.

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