NECESSARY FLEXIBILITY CONDITIONS OF SEMIDISCRETE SURFACES

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Abstract. In this paper we study necessary conditions of flexibility for semidiscrete surfaces. For 2-ribbon semidiscrete surfaces we prove their one-parametric finite flexibility. In particular we write down a system of differential equations describing flexions in the case of existence. Further we find infinitesimal criterions of 3-ribbon flexibility. Finally, we discuss the relation between general semidiscrete surface flexibility and 3-ribbon flexibility.

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Introduction

A mapping $f : \mathbb{R} \times \mathbb{Z} \to \mathbb{R}^3$, where the dependence on the continuous parameter is smooth, is called a *semidiscrete surface.* Let us connect $f(t, z)$ with $f(t, z{+}1)$ by segments for all possible pares (t, z). The resulting piecewise smooth surface is a *piecewise ruled surface.* In this paper we study infinitesimal and higher order flexibility conditions for such semidiscrete surfaces. By *flexions* of a semidiscrete surface f we understand deformations

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that isometrically deform corresponding ruled surfaces and in addition that preserve all line segments connecting $f(t,z)$ with $f(t,z{+}1)$.

Many questions on discrete polyhedral surfaces have their origins in classical theory of smooth surfaces. Flexibility is not an exception from this rule. The general theory of flexibility of surfaces and polyhedra is discussed in the overview [11] by I. Kh. Sabitov.

In 1890 [1] L. Bianchi introduced a necessary and sufficient condition for the existence of isometric deformations of a surface preserving some conjugate system (i.e., two independent smooth fields of directions tangent to the surface), see also in [5]. Such surfaces can be understood as certain limits of semidiscrete surfaces.

On the other hand, semidiscrete surfaces are themselves the limits of certain polygonal surfaces (or *meshes*). For the discrete case of flexible meshes much is now known. We refer the reader to [2], [9], [7], and [6] for some recent results in this area. For general relations to the classical case see a recent book [3] by A. I. Bobenko and Yu. B. Suris. It is interesting to notice that the flexibility conditions in the smooth case and the discrete case are of a different nature. Currently there is no clear description of relations between them in terms of limits.

The place of the study of semidiscrete surfaces is between the classical and the discrete cases. Main concepts of semidiscrete theory are described by J. Wallner in [12], and [13]. Some problems related to isothermic semidiscrete surfaces are studied by C. Müller in [8].

We investigate necessary conditions for existence of isometric deformations of semidiscrete surfaces. To avoid pathological behavior related to noncompactness of semidiscrete surfaces we restrict ourselves to compact subsets of the following type. An *n-ribbon surface* is a mapping

$$f : [a,b] \times \{0,\ldots,n\} \to \mathbb{R}^3, \qquad (i,t) \mapsto f_i(t).$$

We also use the notion

$$\Delta f_i(t) = f_{i+1}(t) - f_i(t).$$

While working with a rather abstract semidiscrete or n-ribbon surface f we keep in mind the two-dimensional piecewise-ruled surface associated to it (see Fig. 1).

In present paper we prove that any 2-ribbon surface (as a ruled surface) is flexible and has one degree of freedom in the generic case (Theorem 1.15). This is quite surprising since generic 1-ribbon surfaces have infinitely many degrees of freedom, see, for instance, in [10], Theorem 5.3.10. We also find a system of differential equations for the deformation of 2-ribbon surfaces (System A and Corollary 1.8). In contrast to that, a generic n-ribbon surface is rigid for $n \geq 3$. For the case $n = 3$ we prove the following statement (see Theorem 2.7 and Corollary 2.9).

Infinitesimal flexibility condition.
A 3-ribbon surface is infinitesimally flexible if and only if the following condition holds:

$$\dot{\Lambda} = (H_2 - H_1)\Lambda,$$

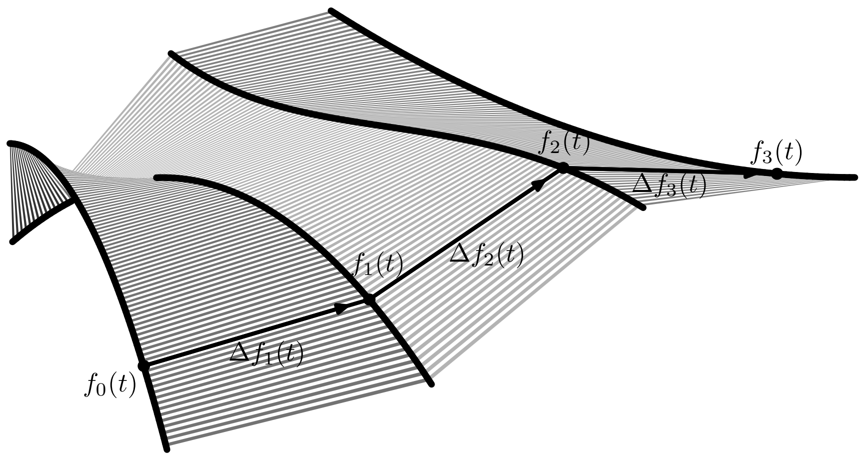

FIGURE 1. A 3-ribbon surface.

where

$$\Lambda = \frac{\big(\dot{f}_1, \ddot{f}_1, \Delta f_0\big)}{\big(\dot{f}_2, \ddot{f}_2, \Delta f_2\big)} \frac{\big(\dot{f}_2, \Delta f_1, \Delta f_2\big)^2}{\big(\dot{f}_1, \Delta f_0, \Delta f_1\big)^2},$$

and

$$H_i(t) = \frac{(\dot{f}_i, \Delta\dot{f}_{i-1}, \Delta f_i) + (\dot{f}_i, \Delta f_{i-1}, \Delta\dot{f}_i)}{(\dot{f}_i, \Delta f_{i-1}, \Delta f_i)}, \quad i = 1, 2.$$

Remark. Throughout this paper we denote the derivative with respect to variable t by the dot symbol.

Having this condition, we also show how to construct inductively the variational isometric conditions of higher orders. Finally, we show that an n-ribbon surface is infinitesimally or finitely flexible if and only if all its 3-ribbon subsurfaces are infinitesimally or finitely flexible (see Theorems 2.13 and 2.14). We say a few words in the case of developable semidiscrete surfaces whose flexions have additional surprising properties.

Organization of the paper. In Section 1 we discuss flexibility of 2-ribbon surfaces. We study infinitesimal flexibility questions for 2-ribbon surfaces in Subsections 1.2 and 1.3. In Subsection 1.2 we give a system of differential equations for infinitesimal flexions, prove the existence of nonzero solutions, and show that all the solutions are proportional to each other. In Subsection 1.3 we define the variational operator of infinitesimal flexion which is studied further in the context of finite flexibility for 2-ribbon surfaces. In Subsection 1.4 we prove that a 2-ribbon surface is finitely flexible and has one degree of freedom if in general position. In Section 2 we work with n-ribbon surfaces. Subsection 2.2 gives infinitesimal flexibility conditions for 3-ribbon surfaces. Subsection 2.3 studies higher order variational conditions for 3-ribbon surfaces. Finally, Subsection 2.4 shows the relations between flexibility of n-ribbon surfaces and infinitesimal and flexibility of 3-ribbon subsurfaces contained in it (in both infinitesimal and finite cases). We conclude the paper with flexibility of developable semidiscrete surfaces in Section 3. In this case flexions have additional geometric properties.

Necessary notions and definitions. Within this paper we traditionally consider t as a smooth argument of a semidiscrete surface f. The time parameter for deformations is λ.

A *perturbation* of a semidiscrete (n-ribbon) surface is a smooth curve $\gamma(\lambda)$ in the space of all sufficiently smooth semidiscrete surfaces. We assume that the curve is parameterized by $\lambda \in [0, \varepsilon]$ for some positive ε such that $\gamma(0) = f$.

Denote by $\mathcal{D}_\gamma f$ the *infinitesimal perturbation* of a semidiscrete (n-ribbon) surface f along the curve γ, i. e. the tangent vector $\frac{\partial \gamma}{\partial \lambda}\big|_{\lambda=0}$.

We say that a perturbation is a *flexion* if it does not change the inner geometry of the surface obtained by joining all the pairs $f_i(t)$ and $f_{i+1}(t)$ by straight segments. In the case of semidiscrete (n-ribbon) surfaces a surface is flexible if the the following quantities are preserved by the perturbation:

$$|\dot{f}_i|, \qquad |\Delta f_i|, \qquad \langle \dot{f}_i, \Delta f_{i-1} \rangle, \qquad \langle \dot{f}_i, \Delta f_i \rangle, \quad \text{and} \quad \langle \dot{f}_i, \dot{f}_{i+1} \rangle$$

(for all possible i and t in the case of an n-ribbon surface).

We say that an infinitesimal perturbation is an *infinitesimal flexion* if it does not change the inner geometry of the surface infinitesimally. In other words, the first derivatives of the quantities listed above are all equal to zero.

1. Finite flexibility of 2-ribbon surfaces

In this section we describe flexions of 2-ribbon surfaces. Such surfaces are defined by three curves f_0, f_1, and f_2. Our main goal here is to prove under some natural genericity assumptions that any 2-ribbon surface is flexible and has one degree of freedom. Our first point is to describe the system of differential equations (System A) that determines infinitesimal flexions corresponding to finite flexions and find solutions to this system (see Subsections 1.1 and 1.2). Further via solutions of System A we define the variational operator of infinitesimal flexion $\mathcal{V}$ (in Subsection 1.3). Finally, to show finite flexibility of 2-ribbon surfaces we study Lipschitz properties for $\mathcal{V}$ (in Subsection 1.4).

1.1. **Basic relations for infinitesimal flexions.** In this small subsection we collect some useful relations.

Proposition 1.1. *For any infinitesimal flexion of a 2-ribbon surface f the following properties hold:*

$$\langle \dot f_1, \mathcal{D}_\gamma \dot f_1\rangle = 0; \tag{1}$$
$$\langle \dot f_1 - \Delta \dot f_0, \mathcal{D}_\gamma \dot f_1 - \mathcal{D}_\gamma \Delta \dot f_0\rangle = 0; \tag{2}$$
$$\langle \dot f_1 + \Delta \dot f_1, \mathcal{D}_\gamma \dot f_1 + \mathcal{D}_\gamma \Delta \dot f_1\rangle = 0; \tag{3}$$
$$\langle \Delta f_0, \mathcal{D}_\gamma \Delta \dot f_0\rangle + \langle \Delta \dot f_0, \mathcal{D}_\gamma \Delta f_0\rangle = 0; \tag{4}$$
$$\langle \Delta f_1, \mathcal{D}_\gamma \Delta \dot f_1\rangle + \langle \Delta \dot f_1, \mathcal{D}_\gamma \Delta f_1\rangle = 0; \tag{5}$$
$$\langle \dot f_1, \mathcal{D}_\gamma \Delta \dot f_0\rangle + \langle \mathcal{D}_\gamma \dot f_1, \Delta \dot f_0\rangle = 0; \tag{6}$$
$$\langle \dot f_1, \mathcal{D}_\gamma \Delta \dot f_1\rangle + \langle \mathcal{D}_\gamma \dot f_1, \Delta \dot f_1\rangle = 0; \tag{7}$$
$$\langle \mathcal{D}_\gamma \ddot f_1, \Delta f_0\rangle + \langle \ddot f_1, \mathcal{D}_\gamma \Delta f_0\rangle = 0; \tag{8}$$
$$\langle \mathcal{D}_\gamma \ddot f_1, \Delta f_1\rangle + \langle \ddot f_1, \mathcal{D}_\gamma \Delta f_1\rangle = 0. \tag{9}$$

Remark 1.2. For a semidiscrete or n-ribbon surface f and a C^2-curve γ the operations $\mathcal{D}_\gamma$, Δ, and $\frac{\partial}{\partial t}$ commute, so we do not pay attention to the order of these operations in compositions.

Proof. The first three equations follow from the fact that infinitesimal flexions preserve the norm of tangent vectors to the curves f_1, f_0, and f_2.

The invariance of the lengths of Δf_0 and Δf_1 implies the fourth and the fifth equations.

Equations (6) and (7) follows from invariance of angles between the vectors $\dot f_1$ and $\Delta \dot f_0$ and the vectors $\dot f_1$ and $\Delta \dot f_0$.

Finally, the last two equations hold since the angles between Δf_0 and $\dot f_1$ and Δf_1 and $\dot f_1$ are preserved by infinitesimal flexions and therefore

$$\frac{\partial}{\partial t}\mathcal{D}_\gamma \langle \dot f_1, \Delta f_0\rangle = 0 \quad \text{and} \quad \frac{\partial}{\partial t}\mathcal{D}_\gamma \langle \dot f_1, \Delta f_1\rangle = 0$$

(in addition we use Equations (6) and (7) respectively). □

1.2. **Infinitesimal flexibility of 2-ribbon surfaces.** In this subsection we write down a system of differential equations (System A) which describe infinitesimal flexions of a 2-ribbon surface in general position. We show the existence of infinitesimal flexions and prove that they are proportional to each other (Theorem 1.9). Let

$$\begin{array}{lll} g_1 = \langle \mathcal{D}_\gamma \dot f_1, \dot f_1\rangle, & g_2 = \langle \mathcal{D}_\gamma \dot f_1, \Delta f_0\rangle, & g_3 = \langle \mathcal{D}_\gamma \dot f_1, \Delta f_1\rangle, \\ g_4 = \langle \mathcal{D}_\gamma \Delta f_0, \dot f_1\rangle, & g_5 = \langle \mathcal{D}_\gamma \Delta f_0, \Delta f_0\rangle, & g_6 = \langle \mathcal{D}_\gamma \Delta f_0, \Delta f_1\rangle, \\ g_7 = \langle \mathcal{D}_\gamma \Delta f_1, \dot f_1\rangle, & g_8 = \langle \mathcal{D}_\gamma \Delta f_1, \Delta f_0\rangle, & g_9 = \langle \mathcal{D}_\gamma \Delta f_1, \Delta f_1\rangle. \end{array} \tag{10}$$

Denote by *System A* the following system of differential equations

$$
\left\{\begin{array}{lcl}
\dot g_1 & = & 0, \\
\dot g_2 & = & \left(\frac{(\dot f_1,\Delta\dot f_0,\Delta f_1)}{(\dot f_1,\Delta f_0,\Delta f_1)}+\frac{(\ddot f_1,\Delta f_0,\Delta f_1)}{(\dot f_1,\Delta f_0,\Delta f_1)}\right)g_2+\frac{(\dot f_1,\Delta f_0,\Delta\dot f_0)}{(\dot f_1,\Delta f_0,\Delta f_1)}g_3-\frac{(\dot f_1,\Delta f_0,\ddot f_1)}{(\dot f_1,\Delta f_0,\Delta f_1)}g_6, \\
\dot g_3 & = & \frac{(\dot f_1,\Delta\dot f_1,\Delta f_1)}{(\dot f_1,\Delta f_0,\Delta f_1)}g_2+\left(\frac{(\dot f_1,\Delta f_0,\Delta\dot f_1)}{(\dot f_1,\Delta f_0,\Delta f_1)}+\frac{(\ddot f_1,\Delta f_0,\Delta f_1)}{(\dot f_1,\Delta f_0,\Delta f_1)}\right)g_3-\frac{(\dot f_1,\ddot f_1,\Delta f_1)}{(\dot f_1,\Delta f_0,\Delta f_1)}g_8, \\
\dot g_4 & = & -\left(\frac{(\dot f_1,\Delta\dot f_0,\Delta f_1)}{(\dot f_1,\Delta f_0,\Delta f_1)}+\frac{(\ddot f_1,\Delta f_0,\Delta f_1)}{(\dot f_1,\Delta f_0,\Delta f_1)}\right)g_2-\frac{(\dot f_1,\Delta f_0,\Delta\dot f_0)}{(\dot f_1,\Delta f_0,\Delta f_1)}g_3+\frac{(\dot f_1,\Delta f_0,\ddot f_1)}{(\dot f_1,\Delta f_0,\Delta f_1)}g_6, \\
\dot g_5 & = & 0, \\
\dot g_6 & = & -\left(\frac{(\Delta f_1,\Delta f_0,\dot f_1\times\Delta f_0)(\dot f_1,\Delta\dot f_0,\Delta f_1)}{|\dot f_1\times\Delta f_0|^2(\dot f_1,\Delta f_0,\Delta f_1)}-\frac{(\dot f_1,\Delta f_1,\dot f_1\times\Delta f_0)(\Delta\dot f_0,\Delta f_0,\Delta f_1)}{|\dot f_1\times\Delta f_0|^2(\dot f_1,\Delta f_0,\Delta f_1)}+\right. \\
 & & \left.\frac{(\dot f_1,\Delta f_0\times\Delta\dot f_0,\Delta f_1)}{|\dot f_1\times\Delta f_0|^2}+\frac{(\dot f_1\times\Delta\dot f_0,\Delta f_0,\Delta f_1)}{|\dot f_1\times\Delta f_0|^2}+\frac{(\Delta\dot f_1,\Delta f_0,\Delta f_1)}{(\dot f_1,\Delta f_0,\Delta f_1)}\right)g_2- \\
 & & \left(\frac{(\Delta f_1,\Delta f_0,\dot f_1\times\Delta f_0)(\dot f_1,\Delta f_0,\Delta\dot f_0)}{|\dot f_1\times\Delta f_0|^2(\dot f_1,\Delta f_0,\Delta f_1)}+\frac{(\dot f_1,\Delta f_0,\Delta f_0\times\Delta\dot f_0)}{|\dot f_1\times\Delta f_0|^2}\right)g_3- \\
 & & \left(\frac{(\dot f_1,\Delta f_1,\dot f_1\times\Delta f_0)(\dot f_1,\Delta f_0,\Delta\dot f_0)}{|\dot f_1\times\Delta f_0|^2(\dot f_1,\Delta f_0,\Delta f_1)}-\frac{(\dot f_1,\Delta f_0,\dot f_1\times\Delta\dot f_0)}{|\dot f_1\times\Delta f_0|^2}-\frac{(\dot f_1,\Delta f_0,\Delta\dot f_1)}{(\dot f_1,\Delta f_0,\Delta f_1)}\right)g_6, \\
\dot g_7 & = & -\frac{(\dot f_1,\Delta\dot f_1,\Delta f_1)}{(\dot f_1,\Delta f_0,\Delta f_1)}g_2-\left(\frac{(\dot f_1,\Delta f_0,\Delta\dot f_1)}{(\dot f_1,\Delta f_0,\Delta f_1)}+\frac{(\ddot f_1,\Delta f_0,\Delta f_1)}{(\dot f_1,\Delta f_0,\Delta f_1)}\right)g_3+\frac{(\dot f_1,\ddot f_1,\Delta f_1)}{(\dot f_1,\Delta f_0,\Delta f_1)}g_8, \\
\dot g_8 & = & -\left(\frac{(\Delta f_0,\Delta f_1,\dot f_1\times\Delta f_1)(\dot f_1,\Delta\dot f_1,\Delta f_1)}{|\dot f_1\times\Delta f_1|^2(\dot f_1,\Delta f_0,\Delta f_1)}+\frac{(\dot f_1,\Delta f_1,\Delta f_1\times\Delta\dot f_1)}{|\dot f_1\times\Delta f_1|^2}\right)g_2- \\
 & & \left(\frac{(\Delta f_0,\Delta f_1,\dot f_1\times\Delta f_1)(\dot f_1,\Delta f_0,\Delta\dot f_1)}{|\dot f_1\times\Delta f_1|^2(\dot f_1,\Delta f_0,\Delta f_1)}-\frac{(\dot f_1,\Delta f_0,\dot f_1\times\Delta f_1)(\Delta\dot f_1,\Delta f_0,\Delta f_1)}{|\dot f_1\times\Delta f_1|^2(\dot f_1,\Delta f_0,\Delta f_1)}+\right. \\
 & & \left.\frac{(\dot f_1,\Delta f_1\times\Delta\dot f_1,\Delta f_0)}{|\dot f_1\times\Delta f_1|^2}+\frac{(\dot f_1\times\Delta\dot f_1,\Delta f_1,\Delta f_0)}{|\dot f_1\times\Delta f_1|^2}+\frac{(\Delta\dot f_0,\Delta f_0,\Delta f_1)}{(\dot f_1,\Delta f_0,\Delta f_1)}\right)g_3- \\
 & & \left(\frac{(\dot f_1,\Delta f_0,\dot f_1\times\Delta f_1)(\dot f_1,\Delta\dot f_1,\Delta f_1)}{|\dot f_1\times\Delta f_1|^2(\dot f_1,\Delta f_0,\Delta f_1)}-\frac{(\dot f_1,\Delta f_1,\dot f_1\times\Delta\dot f_1)}{|\dot f_1\times\Delta f_1|^2}-\frac{(\dot f_1,\Delta\dot f_0,\Delta f_1)}{(\dot f_1,\Delta f_0,\Delta f_1)}\right)g_8, \\
\dot g_9 & = & 0.
\end{array}\right.
$$

Remark 1.3. In Proposition 2.2 below we show an explicit formula for the function g_6+g_8, it is Φ in our notation of Section 2.

Note also that $\dot g_2+\dot g_4=0$ and $\dot g_3+\dot g_7=0$ in System A.

The remaining part of this subsection is dedicated to the proof of Theorem 1.9 on the structure of the space of infinitesimal flexions. In Proposition 1.4 we show that any infinitesimal flexion satisfies System A. Then in Proposition 1.6 we prove that any solution of System A with certain initial data is an infinitesimal flexion. Finally, in Proposition 1.7 we show the uniqueness of the solution of System A for a given initial data. After that we prove Theorem 1.9.

Let us show that any infinitesimal flexion satisfies System A.

Proposition 1.4. *Let $\dot f_1$, Δf_0, and Δf_1 be linearly independent. Then for any infinitesimal flexion $\mathcal{D}_\gamma$ the functions $g_1,\ldots,g_9$ satisfy system A.*

We start the proof with the following general lemma.

Lemma 1.5. *For any infinitesimal flexion $\mathcal{D}_\gamma$ we have the equalities*

$$g_1=g_5=g_9=0, \qquad g_2+g_4=0, \quad \text{and} \quad g_3+g_7=0.$$

Proof. The functions $|\dot{f}_1|$, $|\Delta f_0|$, and $|\Delta f_1|$ are infinitesimally preserved by infinitesimal flexions, hence g_1, g_5, and g_9 vanish.

The invariance of angles between $\dot{f}_1$ and Δf_0, and $\dot{f}_1$ and Δf_1 yield the equations $g_2 + g_4 = 0$ and $g_3 + g_7 = 0$, respectively. □

Proof of Proposition 1.4. From Lemma 1.5 the functions g_1, g_5, and g_9 are equivalent to zero, thus $\dot{g}_1$, $\dot{g}_5$, and $\dot{g}_9$ are equivalent to zero as well.

Let us prove the expression for $\dot{g}_2$ and $\dot{g}_3$. Note that

$$\dot{g}_2 = \langle \mathcal{D}_\gamma \ddot{f}_1, \Delta f_0\rangle + \langle \mathcal{D}_\gamma \dot{f}_1, \Delta \dot{f}_0\rangle.$$

Thus Equations (6) and (8) imply

$$\dot{g}_2 = \langle \mathcal{D}_\gamma \dot{f}_1, \Delta \dot{f}_0\rangle - \langle \ddot{f}_1, \mathcal{D}_\gamma \Delta f_0\rangle.$$

To obtain the expression for $\dot{g}_2$ rewrite $\Delta \dot{f}_0$ and $\ddot{f}_1$ in the basis consisting of vectors $\dot{f}_1$, Δf_0, and Δf_1. The same strategy works for the functions $\dot{g}_3$.

Now we study expressions for $\dot{g}_4$ and $\dot{g}_7$. From Lemma 1.5 we know that $g_4 = -g_2$ and $g_7 = -g_3$ and hence $\dot{g}_4 = -\dot{g}_2$ and $\dot{g}_7 = -\dot{g}_3$. Therefore, the equations for $\dot{g}_4$ and $\dot{g}_7$ are satisfied.

In order to get an expression for $\dot{g}_6$, we first note that $\mathcal{D}_\gamma(\dot{f}_1, \Delta f_0, \Delta \dot{f}_0) = 0$, since the function $(\dot{f}_1, \Delta f_0, \Delta \dot{f}_0)$ is an invariant of an infinitesimal flexion. So we get

$$(\mathcal{D}_\gamma \dot{f}_1, \Delta f_0, \Delta \dot{f}_0) + (\dot{f}_1, \mathcal{D}_\gamma \Delta f_0, \Delta \dot{f}_0) + (\dot{f}_1, \Delta f_0, \mathcal{D}_\gamma \Delta \dot{f}_0) = 0.$$

Rewrite

$$\begin{aligned}(\dot{f}_1, \Delta f_0, \mathcal{D}_\gamma \Delta \dot{f}_0) &= -(\mathcal{D}_\gamma \dot{f}_1, \Delta f_0, \Delta \dot{f}_0) - (\dot{f}_1, \mathcal{D}_\gamma \Delta f_0, \Delta \dot{f}_0)\\ &= -\langle \mathcal{D}_\gamma \dot{f}_1, \Delta f_0 \times \Delta \dot{f}_0\rangle + \langle \mathcal{D}_\gamma \Delta f_0, \dot{f}_1 \times \Delta \dot{f}_0\rangle\\ &= -\tfrac{(\Delta f_0 \times \Delta \dot{f}_0, \Delta f_0, \Delta f_1)}{(\dot{f}_1, \Delta f_0, \Delta f_1)} g_1 - \tfrac{(\dot{f}_1, \Delta f_0 \times \Delta \dot{f}_0, \Delta f_1)}{(\dot{f}_1, \Delta f_0, \Delta f_1)} g_2 - \tfrac{(\dot{f}_1, \Delta f_0, \Delta f_0 \times \Delta \dot{f}_0)}{(\dot{f}_1, \Delta f_0, \Delta f_1)} g_3 +\\ &\quad \tfrac{(\dot{f}_1 \times \Delta \dot{f}_0, \Delta f_0, \Delta f_1)}{(\dot{f}_1, \Delta f_0, \Delta f_1)} g_4 + \tfrac{(\dot{f}_1, \dot{f}_1 \times \Delta \dot{f}_0, \Delta f_1)}{(\dot{f}_1, \Delta f_0, \Delta f_1)} g_5 + \tfrac{(\dot{f}_1, \Delta f_0, \dot{f}_1 \times \Delta \dot{f}_0)}{(\dot{f}_1, \Delta f_0, \Delta f_1)} g_6.\end{aligned}$$

Second, we have

$$\langle \mathcal{D}_\gamma \Delta \dot{f}_0, \Delta f_0\rangle = -\langle \mathcal{D}_\gamma \Delta f_0, \Delta \dot{f}_0\rangle = -\tfrac{(\Delta \dot{f}_0, \Delta f_0, \Delta f_1)}{(\dot{f}_1, \Delta f_0, \Delta f_1)} g_4 - \tfrac{(\dot{f}_1, \Delta \dot{f}_0, \Delta f_1)}{(\dot{f}_1, \Delta f_0, \Delta f_1)} g_5 - \tfrac{(\dot{f}_1, \Delta f_0, \Delta \dot{f}_0)}{(\dot{f}_1, \Delta f_0, \Delta f_1)} g_6.$$

Third, we get

$$\langle \mathcal{D}_\gamma \Delta \dot{f}_0, \dot{f}_1\rangle = -\langle \mathcal{D}_\gamma \dot{f}_1, \Delta \dot{f}_0\rangle = -\tfrac{(\dot{f}_1, \Delta \dot{f}_0, \Delta f_1)}{(\dot{f}_1, \Delta f_0, \Delta f_1)} g_2 - \tfrac{(\dot{f}_1, \Delta f_0, \Delta \dot{f}_0)}{(\dot{f}_1, \Delta f_0, \Delta f_1)} g_3.$$

Fourth,

$$\langle \mathcal{D}_\gamma \Delta \dot f_0, \Delta f_1\rangle = \frac{(\Delta f_1,\Delta f_0,\dot f_1\times\Delta f_0)}{(\dot f_1,\Delta f_0,\dot f_1\times\Delta f_0)}\langle \mathcal{D}_\gamma \Delta \dot f_0, \dot f_1\rangle + \frac{(\dot f_1,\Delta f_1,\dot f_1\times\Delta f_0)}{(\dot f_1,\Delta f_0,\dot f_1\times\Delta f_0)}\langle \mathcal{D}_\gamma \Delta \dot f_0, \Delta f_0\rangle + \frac{(\dot f_1,\Delta f_0,\Delta f_1)}{(\dot f_1,\Delta f_0,\dot f_1\times\Delta f_0)}(\dot f_1,\Delta f_0,\mathcal{D}_\gamma \Delta \dot f_0).$$

After a substitution of the four above expressions and simplifications we have

$$\begin{aligned}\langle \mathcal{D}_\gamma \Delta \dot f_0, \Delta f_1\rangle = &-\Bigg(\frac{(\Delta f_1,\Delta f_0,\dot f_1\times\Delta f_0)(\dot f_1,\Delta \dot f_0,\Delta f_1)}{|\dot f_1\times\Delta f_0|^2(\dot f_1,\Delta f_0,\Delta f_1)} - \frac{(\dot f_1,\Delta f_1,\dot f_1\times\Delta f_0)(\Delta \dot f_0,\Delta f_0,\Delta f_1)}{|\dot f_1\times\Delta f_0|^2(\dot f_1,\Delta f_0,\Delta f_1)} + \\ &\frac{(\dot f_1,\Delta f_0\times\Delta \dot f_0,\Delta f_1)}{|\dot f_1\times\Delta f_0|^2} + \frac{(\dot f_1\times\Delta \dot f_0,\Delta f_0,\Delta f_1)}{|\dot f_1\times\Delta f_0|^2}\Bigg) g_2 - \\ &\Bigg(\frac{(\Delta f_1,\Delta f_0,\dot f_1\times\Delta f_0)(\dot f_1,\Delta f_0,\Delta \dot f_0)}{|\dot f_1\times\Delta f_0|^2(\dot f_1,\Delta f_0,\Delta f_1)} + \frac{(\dot f_1,\Delta f_0,\Delta f_0\times\Delta \dot f_0)}{|\dot f_1\times\Delta f_0|^2}\Bigg) g_3 - \\ &\Bigg(\frac{(\dot f_1,\Delta f_1,\dot f_1\times\Delta f_0)(\dot f_1,\Delta f_0,\Delta \dot f_0)}{|\dot f_1\times\Delta f_0|^2(\dot f_1,\Delta f_0,\Delta f_1)} - \frac{(\dot f_1,\Delta f_0,\dot f_1\times\Delta \dot f_0)}{|\dot f_1\times\Delta f_0|^2}\Bigg) g_6.\end{aligned}$$

Further, we get

$$\langle \mathcal{D}_\gamma \Delta f_0, \Delta \dot f_1\rangle = \frac{(\Delta \dot f_1,\Delta f_0,\Delta f_1)}{(\dot f_1,\Delta f_0,\Delta f_1)} g_4 + \frac{(\dot f_1,\Delta \dot f_1,\Delta f_1)}{(\dot f_1,\Delta f_0,\Delta f_1)} g_5 + \frac{(\dot f_1,\Delta f_0,\Delta \dot f_1)}{(\dot f_1,\Delta f_0,\Delta f_1)} g_6.$$

From the last two identities, by substituting $g_5 = 0$ and $g_4 = -g_2$ (see Lemma 1.5), we obtain the expression for $\dot g_6$.

The expression for $\dot g_8$ is calculated in a similar way. This concludes the proof. □

Further we prove that any solution of System A with certain initial data is an infinitesimal flexion.

Proposition 1.6. *Let f be a 2-ribbon surface, $f_i : [a,b] \to \mathbb{R}^3$ for $i = 0,1,2$. Assume that the function $(\dot f_1, \Delta f_0, \Delta f_1)$ has no zeros on $[a,b]$. Then any infinitesimal perturbation $\mathcal{D}_\gamma$ of f satisfying System A and the boundary conditions*

$$\mathcal{D}_\gamma \dot f_1(a) = 0, \quad \mathcal{D}_\gamma \Delta f_1(a) = 0, \quad \text{and} \quad \mathcal{D}_\gamma \Delta f_0(a) = \alpha \dot f_1(a)\times\Delta f_0(a).$$

is an infinitesimal flexion.

Proof. By the definition of an infinitesimal flexion it is enough to check that the following 11 functions are preserved by the infinitesimal perturbation:

$$|\dot f_i|, \qquad |\Delta f_i|, \qquad \langle \dot f_i, \Delta f_{i-1}\rangle, \qquad \langle \dot f_i, \Delta f_i\rangle, \quad \text{and} \quad \langle \dot f_i, \dot f_{i+1}\rangle$$

(for all possible admissible i).

Invariance of $|\dot f_1|$, $|\Delta f_0|$, $|\Delta f_1|$, $\langle \dot f_1, \Delta f_0\rangle$, and $\langle \dot f_1, \Delta f_1\rangle$.

From System A we have

$$\dot g_1 = 0, \quad \dot g_5 = 0, \quad \dot g_9 = 0, \quad \dot g_4 + \dot g_2 = 0, \quad \dot g_7 + \dot g_3 = 0,$$

and hence all five functions under consideration are constants. So it is enough to show that they vanish at some point: we show this at point a.

$$\begin{aligned}
&\mathcal{D}_\gamma\langle \dot f_1(a), \dot f_1(a)\rangle = 2\langle \mathcal{D}_\gamma \dot f_1(a), \dot f_1(a)\rangle = 2\langle 0, \dot f_1(a)\rangle = 0;\\
&\mathcal{D}_\gamma\langle \Delta f_0(a), \Delta f_0(a)\rangle = 2\langle \mathcal{D}_\gamma \Delta f_0(a), \Delta f_0(a)\rangle = 2\langle \alpha \dot f_1(a)\times\Delta f_0(a), \Delta f_0(a)\rangle = 0;\\
&\mathcal{D}_\gamma\langle \Delta f_1(a), \Delta f_1(a)\rangle = 2\langle \mathcal{D}_\gamma \Delta f_1(a), \Delta f_1(a)\rangle = 2\langle 0, \Delta f_1(a)\rangle = 0;\\
&\mathcal{D}_\gamma\langle \dot f_1(a), \Delta f_0(a)\rangle = \langle \mathcal{D}_\gamma \dot f_1(a), \Delta f_0(a)\rangle + \langle \dot f_1(a), \mathcal{D}_\gamma \Delta f_0(a)\rangle = \langle 0, \Delta f_0(a)\rangle +\\
&\qquad\qquad\qquad\qquad\qquad \langle \dot f_1(a), \alpha \dot f_1(a)\times\Delta f_0(a)\rangle = 0;\\
&\mathcal{D}_\gamma\langle \dot f_1(a), \Delta f_1(a)\rangle = \langle \mathcal{D}_\gamma \dot f_1(a), \Delta f_1(a)\rangle + \langle \dot f_1(a), \mathcal{D}_\gamma \Delta f_1(a)\rangle = \langle 0, \Delta f_0(a)\rangle +\\
&\qquad\qquad\qquad\qquad\qquad \langle \dot f_1(a), 0\rangle = 0.
\end{aligned}$$

Invariance of $\langle \dot f_0, \Delta f_0\rangle$ *and* $\langle \dot f_2, \Delta f_1\rangle$. Note that

$$\langle \dot f_0, \Delta f_0\rangle = -\frac{1}{2}\frac{\partial}{\partial t}\langle \Delta f_0, \Delta f_0\rangle + \langle \dot f_1, \Delta f_0\rangle.$$

Hence $\mathcal{D}_\gamma\langle \dot f_0, \Delta f_0\rangle = 0$. Similar reasoning shows that $\mathcal{D}_\gamma\langle \dot f_2, \Delta f_1\rangle = 0$.

Invariance of $\langle \dot f_0, \dot f_1\rangle$ *and* $\langle \dot f_1, \dot f_2\rangle$. Let us prove that $\mathcal{D}_\gamma\langle \dot f_0, \dot f_1\rangle = 0$. First, note that

$$\langle \mathcal{D}_\gamma \dot f_0, \dot f_1\rangle = \langle \mathcal{D}_\gamma \dot f_1, \dot f_1\rangle - \langle \mathcal{D}_\gamma \Delta \dot f_0, \dot f_1\rangle = -\langle \mathcal{D}_\gamma \Delta \dot f_0, \dot f_1\rangle = \langle \mathcal{D}_\gamma \Delta f_0, \ddot f_1\rangle - \frac{\partial}{\partial t}\langle \mathcal{D}_\gamma \Delta f_0, \dot f_1\rangle.$$

Recall that $\frac{\partial}{\partial t}\langle \mathcal{D}_\gamma \Delta f_0, \dot f_1\rangle = \dot g_4 = -\dot g_2$. Let us substitute the expression for $\dot g_2$ from System A and rewrite $\ddot f_1$ in the basis of vectors $\dot f_1$, Δf_0, and Δf_1. One obtains

$$\begin{aligned}
\langle \mathcal{D}_\gamma \Delta f_0, \ddot f_1\rangle + \dot g_2 &= \frac{(\dot f_1, \Delta \dot f_0, \Delta f_1)}{(\dot f_1, \Delta f_0, \Delta f_1)}\langle \mathcal{D}_\gamma \dot f_1, \Delta f_0\rangle + \frac{(\dot f_1, \Delta f_0, \Delta \dot f_0)}{(\dot f_1, \Delta f_0, \Delta f_1)}\langle \mathcal{D}_\gamma \dot f_1, \Delta f_1\rangle =\\
&= \langle \mathcal{D}_\gamma \dot f_1, \Delta \dot f_0\rangle = -\langle \mathcal{D}_\gamma \dot f_1, \dot f_0\rangle.
\end{aligned}$$

Hence

$$\mathcal{D}_\gamma\langle \dot f_0, \dot f_1\rangle = \langle \mathcal{D}_\gamma \dot f_0, \dot f_1\rangle + \langle \mathcal{D}_\gamma \dot f_1, \dot f_0\rangle = 0.$$

It follows that $\langle \dot f_0, \dot f_1\rangle$ is invariant under the infinitesimal perturbation. The proof of the invariance of $\langle \dot f_1, \dot f_2\rangle$ is analogous.

Invariance of $\langle \dot f_0, \dot f_0\rangle$ *and* $\langle \dot f_2, \dot f_2\rangle$. Let us prove that $\mathcal{D}_\gamma\langle \dot f_0, \dot f_0\rangle = 0$.

$$\mathcal{D}_\gamma\langle \dot f_0, \dot f_0\rangle = 2\langle \mathcal{D}_\gamma \dot f_0, \dot f_0\rangle = 2\langle \mathcal{D}_\gamma \Delta \dot f_0, \Delta \dot f_0\rangle + 2\mathcal{D}_\gamma\langle \dot f_1, \dot f_0\rangle - 2\langle \mathcal{D}_\gamma \dot f_1, \dot f_1\rangle.$$

We have already shown that $\mathcal{D}_\gamma\langle \dot f_1, \dot f_0\rangle = 0$ and $\langle \mathcal{D}_\gamma \dot f_1, \dot f_1\rangle = 0$. Hence

$$\mathcal{D}_\gamma\langle \dot f_0, \dot f_0\rangle = 2\langle \mathcal{D}_\gamma \Delta \dot f_0, \Delta \dot f_0\rangle.$$

We rewrite the last $\Delta \dot f_0$ in the last expression in the basis $\dot f_1, \Delta f_0, \dot f_1\times\Delta f_0$ and get

$$\langle \mathcal{D}_\gamma \Delta\dot{f}_0, \Delta\dot{f}_0\rangle = \tfrac{(\Delta\dot{f}_0,\Delta f_0,\dot{f}_1\times\Delta f_0)}{(\dot{f}_1,\Delta f_0,\dot{f}_1\times\Delta f_0)}\langle \mathcal{D}_\gamma \Delta\dot{f}_0, \dot{f}_1\rangle + \tfrac{(\dot{f}_1,\Delta\dot{f}_0,\dot{f}_1\times\Delta f_0)}{(\dot{f}_1,\Delta f_0,\dot{f}_1\times\Delta f_0)}\langle \mathcal{D}_\gamma \Delta\dot{f}_0, \Delta f_0\rangle + \tfrac{(\dot{f}_1,\Delta f_0,\Delta\dot{f}_0)}{(\dot{f}_1,\Delta f_0,\dot{f}_1\times\Delta f_0)}(\mathcal{D}_\gamma \Delta\dot{f}_0, \dot{f}_1, \Delta f_0).$$

Let us rewrite $\langle \mathcal{D}_\gamma \Delta\dot{f}_0, \dot{f}_1\rangle$, $\langle \mathcal{D}_\gamma \Delta\dot{f}_0, \Delta f_0\rangle$, and $(\mathcal{D}_\gamma \Delta\dot{f}_0, \dot{f}_1, \Delta f_0)$ in terms of $g_1, \ldots, g_9$. First, we have:

$$\langle \mathcal{D}_\gamma \Delta\dot{f}_0, \dot{f}_1\rangle = \langle \mathcal{D}_\gamma \dot{f}_0, \dot{f}_1\rangle = -\langle \mathcal{D}_\gamma \dot{f}_1, \dot{f}_0\rangle = -\langle \mathcal{D}_\gamma \dot{f}_1, \Delta\dot{f}_0\rangle.$$

The second equality holds since we have shown that $\mathcal{D}_\gamma\langle \dot{f}_0, \dot{f}_1\rangle = 0$. If we rewrite $\Delta\dot{f}_0$ in the basis $\dot{f}_1, \Delta f_0, \Delta f_1$, we get the following:

$$\langle \mathcal{D}_\gamma \dot{f}_0, \dot{f}_1\rangle = -\tfrac{(\dot{f}_1,\Delta\dot{f}_0,\Delta f_1)}{(\dot{f}_1,\Delta f_0,\Delta f_1)}g_2 - \tfrac{(\dot{f}_1,\Delta f_0,\Delta\dot{f}_0)}{(\dot{f}_1,\Delta f_0,\Delta f_1)}g_3.$$

Second, we have

$$\langle \mathcal{D}_\gamma \Delta\dot{f}_0, \Delta f_0\rangle = -\langle \mathcal{D}_\gamma \Delta f_0, \Delta\dot{f}_0\rangle = \tfrac{(\Delta\dot{f}_0,\Delta f_0,\Delta f_1)}{(\dot{f}_1,\Delta f_0,\Delta f_1)}g_2 - \tfrac{(\dot{f}_1,\Delta f_0,\Delta\dot{f}_0)}{(\dot{f}_1,\Delta f_0,\Delta f_1)}g_6.$$

Third, with

$$\dot{g}_6 - \langle \mathcal{D}_\gamma \Delta f_0, \Delta\dot{f}_1\rangle = \langle \mathcal{D}_\gamma \Delta\dot{f}_0, \Delta f_1\rangle = \tfrac{(\Delta f_1,\Delta f_0,\dot{f}_1\times\Delta f_0)}{(\dot{f}_1,\Delta f_0,\dot{f}_1\times\Delta f_0)}\langle \mathcal{D}_\gamma \Delta\dot{f}_0, \dot{f}_1\rangle + \tfrac{(\dot{f}_1,\Delta f_1,\dot{f}_1\times\Delta f_0)}{(\dot{f}_1,\Delta f_0,\dot{f}_1\times\Delta f_0)}\langle \mathcal{D}_\gamma \Delta\dot{f}_0, \Delta f_0\rangle + \tfrac{(\dot{f}_1,\Delta f_0,\Delta f_1)}{(\dot{f}_1,\Delta f_0,\dot{f}_1\times\Delta f_0)}(\mathcal{D}_\gamma \Delta\dot{f}_0, \dot{f}_1, \Delta f_0).$$

and the expression for $\dot{g}_6$ from System A we get:

$$(\mathcal{D}_\gamma \Delta\dot{f}_0, \dot{f}_1, \Delta f_0) = -\left(\tfrac{(\dot{f}_1\times\Delta\dot{f}_0,\Delta f_0,\Delta f_1)}{(\dot{f}_1,\Delta f_0,\Delta f_1)} + \tfrac{(\dot{f}_1,\Delta f_0\times\Delta\dot{f}_0,\Delta f_1)}{(\dot{f}_1,\Delta f_0,\Delta f_1)}\right)g_2 - \tfrac{(\dot{f}_1,\Delta f_0,\Delta f_0\times\Delta\dot{f}_0)}{(\dot{f}_1,\Delta f_0,\Delta f_1)}g_3 + \tfrac{(\dot{f}_1,\Delta f_0,\dot{f}_1\times\Delta\dot{f}_0)}{(\dot{f}_1,\Delta f_0,\Delta f_1)}g_6.$$

Finally, we combine these three expressions and arrive at

$$\begin{aligned}\langle \mathcal{D}_\gamma \Delta\dot{f}_0, \Delta\dot{f}_0\rangle = \Big(&-\tfrac{(\Delta\dot{f}_0,\Delta f_0,\dot{f}_1\times\Delta f_0)(\dot{f}_1,\Delta\dot{f}_0,\Delta f_1)}{(\dot{f}_1,\Delta f_0,\dot{f}_1\times\Delta f_0)(\dot{f}_1,\Delta f_0,\Delta f_1)} + \tfrac{(\dot{f}_1,\Delta\dot{f}_0,\dot{f}_1\times\Delta f_0)(\Delta\dot{f}_0,\Delta f_0,\Delta f_1)}{(\dot{f}_1,\Delta f_0,\dot{f}_1\times\Delta f_0)(\dot{f}_1,\Delta f_0,\Delta f_1)} - \\ &\tfrac{(\dot{f}_1,\Delta f_0,\Delta\dot{f}_0)(\dot{f}_1\times\Delta\dot{f}_0,\Delta f_0,\Delta f_1)}{(\dot{f}_1,\Delta f_0,\dot{f}_1\times\Delta f_0)(\dot{f}_1,\Delta f_0,\Delta f_1)} - \tfrac{(\dot{f}_1,\Delta f_0,\Delta\dot{f}_0)(\dot{f}_1,\Delta f_0\times\Delta\dot{f}_0,\Delta f_1)}{(\dot{f}_1,\Delta f_0,\dot{f}_1\times\Delta f_0)(\dot{f}_1,\Delta f_0,\Delta f_1)}\Big)g_2 + \\ &\Big(-\tfrac{(\Delta\dot{f}_0,\Delta f_0,\dot{f}_1\times\Delta f_0)(\dot{f}_1,\Delta f_0,\Delta\dot{f}_0)}{(\dot{f}_1,\Delta f_0,\dot{f}_1\times\Delta f_0)(\dot{f}_1,\Delta f_0,\Delta f_1)} - \tfrac{(\dot{f}_1,\Delta f_0,\Delta\dot{f}_0)(\dot{f}_1,\Delta f_0,\Delta f_0\times\Delta\dot{f}_0)}{(\dot{f}_1,\Delta f_0,\dot{f}_1\times\Delta f_0)(\dot{f}_1,\Delta f_0,\Delta f_1)}\Big)g_3 + \\ &\Big(-\tfrac{(\dot{f}_1,\Delta\dot{f}_0,\dot{f}_1\times\Delta f_0)(\dot{f}_1,\Delta f_0,\Delta\dot{f}_0)}{(\dot{f}_1,\Delta f_0,\dot{f}_1\times\Delta f_0)(\dot{f}_1,\Delta f_0,\Delta f_1)} + \tfrac{(\dot{f}_1,\Delta f_0,\Delta\dot{f}_0)(\dot{f}_1,\Delta f_0,\dot{f}_1\times\Delta\dot{f}_0)}{(\dot{f}_1,\Delta f_0,\dot{f}_1\times\Delta f_0)(\dot{f}_1,\Delta f_0,\Delta f_1)}\Big)g_6.\end{aligned}$$

It is clear that the coefficients of g_3 and g_6 vanish identically. Let us study the coefficient of g_2.

Consider the following mixed product $(\Delta\dot{f}_0, \Delta\dot{f}_0, \dot{f}_1\times\Delta f_0)$, it is identical to zero. Let us rewrite $\Delta\dot{f}_0$ in the second position of the mixed product in the basis $\dot{f}_0$, Δf_0, Δf_1. We get the relation

$$\begin{aligned}&\tfrac{(\Delta\dot f_0,\Delta f_0,\Delta f_1)}{(\dot f_1,\Delta f_0,\Delta f_1)}(\Delta\dot f_0,\dot f_1,\dot f_1\times\Delta f_0)+\tfrac{(\dot f_1,\Delta\dot f_0,\Delta f_1)}{(\dot f_1,\Delta f_0,\Delta f_1)}(\Delta\dot f_0,\Delta f_0,\dot f_1\times\Delta f_0)\\ &=-\tfrac{(\dot f_1,\Delta f_0,\Delta\dot f_0)}{(\dot f_1,\Delta f_0,\Delta f_1)}(\Delta\dot f_0,\Delta f_1,\dot f_1\times\Delta f_0).\end{aligned}$$

We apply this identity to the first two summands of the coefficient of g_2 and get the following expression for the coefficient of g_2:

$$\tfrac{(\dot f_1,\Delta f_0,\Delta\dot f_0)(\Delta\dot f_0,\Delta f_1,\dot f_1\times\Delta f_0)}{(\dot f_1,\Delta f_0,\Delta f_1)|\dot f_1\times\Delta f_0|^2}-\tfrac{(\dot f_1,\Delta f_0\times\Delta\dot f_0,\Delta f_1)(\dot f_1,\Delta f_0,\Delta\dot f_0)}{(\dot f_1,\Delta f_0,\Delta f_1)|\dot f_1\times\Delta f_0|^2}-\tfrac{(\dot f_1\times\Delta\dot f_0,\Delta f_0,\Delta f_1)(\dot f_1,\Delta f_0,\Delta\dot f_0)}{(\dot f_1,\Delta f_0,\Delta f_1)|\dot f_1\times\Delta f_0|^2}.$$

We rewrite this as

$$\tfrac{(\dot f_1,\Delta f_0,\Delta\dot f_0)}{(\dot f_1,\Delta f_0,\Delta f_1)|\dot f_1\times\Delta f_0|^2}\Big((\Delta\dot f_0,\Delta f_1,\dot f_1\times\Delta f_0)-(\dot f_1,\Delta f_0\times\Delta\dot f_0,\Delta f_1)-(\dot f_1\times\Delta\dot f_0,\Delta f_0,\Delta f_1)\Big).$$

Let us study the expression in the brackets.

$$\begin{aligned}&(\Delta\dot f_0,\Delta f_1,\dot f_1\times\Delta f_0)-(\dot f_1,\Delta f_0\times\Delta\dot f_0,\Delta f_1)-(\dot f_1\times\Delta\dot f_0,\Delta f_0,\Delta f_1)=\\ &-\big(\Delta\dot f_0\times(\dot f_1\times\Delta f_0)+\dot f_1\times(\Delta f_0\times\Delta\dot f_0)+\Delta f_0\times(\Delta\dot f_0\times\dot f_1),\Delta f_1\big)=(0,\Delta f_1)=0.\end{aligned}$$

The second equality holds by the Jacobi identity. Hence the coefficient of g_2 is zero. Therefore,

$$\mathcal{D}_\gamma\langle\dot f_0,\dot f_0\rangle=2\langle\mathcal{D}_\gamma\Delta\dot f_0,\Delta\dot f_0\rangle=0,$$

and $\langle\dot f_0,\dot f_0\rangle$ is invariant under the infinitesimal perturbation.

The proof of the invariance of $\langle\dot f_2,\dot f_2\rangle$ repeats the proof for $\langle\dot f_0,\dot f_0\rangle$.

So we have checked the invariance of all the 11 functions in the definition of an infinitesimal flexion. Hence the infinitesimal perturbation $\mathcal{D}_\gamma$ is an infinitesimal flexion. □

In the following proposition we prove that System A has a unique solution for any single 2-ribbon surface f (not for a deformation) and initial data for g_i at one point $f(t_0)$. Recall that t is an argument of f.

Proposition 1.7. *Let f be a 2-ribbon surface, $f_i:[a,b]\to\mathbb{R}^3$ for $i=0,1,2$. For any collection of initial data $g_i(t_0)=c_i$ there exists a unique solution of System A. This solution is extended for all $t<T_0$, where*

$$T_0=\min\big\{T>t_0\mid\big(\dot f_1(T),\Delta f_0(T),\Delta f_1(T)\big)=0\big\}.$$

Proof. The system of differential equations for $t_0\le t<T_0$ is a system of homogeneous linear equations with variable coefficients and hence for any collection of initial data it has a unique solution. □

The initial conditions of the last proposition can be reformulated in terms of infinitesimal flexion $\mathcal{D}_\gamma\dot f_1$ at a single point t_0 itself.

Corollary 1.8. *Let f be a 2-ribbon surface, $f_i:[a,b]\to\mathbb{R}^3$ for $i=0,1,2$. For any collection of initial data*

$$\mathcal{D}_\gamma\dot f_1(t_0)=v_1,\quad\mathcal{D}_\gamma\Delta f_0(t_0)=v_2,\quad\textit{and}\quad\mathcal{D}_\gamma\Delta f_1(t_0)=v_3.$$

there exists a unique solution of System A. This solution is extended for all $t < T_0$*, where*

$$T_0 = \min\big\{T > t_0 \mid \big(\dot f_1(T), \Delta f_0(T), \Delta f_1(T)\big) = 0\big\}.$$

Proof. The corollary follows directly from Proposition 1.7 after obtaining the "initial values c_i" from the vectors v_i:

$$\begin{array}{lll} c_1 = \langle v_1, \dot f_1\rangle, & c_2 = \langle v_1, \Delta f_0\rangle, & c_3 = \langle v_1, \Delta f_1\rangle, \\ c_4 = \langle v_2, \dot f_1\rangle, & c_5 = \langle v_2, \Delta f_0\rangle, & c_6 = \langle v_2, \Delta f_1\rangle, \\ c_7 = \langle v_3, \dot f_1\rangle, & c_8 = \langle v_3, \Delta f_0\rangle, & c_9 = \langle v_3, \Delta f_1\rangle. \end{array}$$

□

Now we have all the ingredients to prove the general theorem on the structure of the space of infinitesimal flexions.

Theorem 1.9. *Consider a 2-ribbon surface defined by curves* $f_i : [a, b] \to \mathbb{R}^3$ *for* $i = 0, 1, 2$*, where* f_0 *and* f_2 *are* C^1*-smooth and* f_1 *is* C^2*-smooth. Assume that the function* $(\dot f_1, \Delta f_0, \Delta f_1)$ *has no zeroes on* $[a, b]$*. The space of infinitesimal flexions of such surfaces (up to isometries) is one-dimensional.*

Proof. Uniqueness. Any infinitesimal flexion is isometrically equivalent to an infinitesimal flexion which satisfies

$$\mathcal{D}_\gamma \dot f_1(a) = 0, \quad \mathcal{D}_\gamma \Delta f_1(a) = 0, \quad \text{and} \quad \mathcal{D}_\gamma \Delta f_0(a) = \alpha \dot f_1(a) \times \Delta f_0(a).$$

Consider functions g_i defined by Equations (10). By Proposition 1.4 these functions satisfy System A. Hence by Corollary 1.8, the functions g_i are uniquely defined by f and the initial conditions for infinitesimal flexions. Recall that elements of an arbitrary Euclidean vector $v = (c_1, c_2, c_3)$ are uniquely determined by its scalar products with an arbitrary basis:

$$\begin{pmatrix} c_1 \\ c_2 \\ c_3 \end{pmatrix} = \begin{pmatrix} \langle e_1, e_1\rangle & \langle e_1, e_2\rangle & \langle e_1, e_3\rangle \\ \langle e_2, e_1\rangle & \langle e_2, e_2\rangle & \langle e_2, e_3\rangle \\ \langle e_3, e_1\rangle & \langle e_3, e_2\rangle & \langle e_3, e_3\rangle \end{pmatrix}^{-1} \begin{pmatrix} \langle e_1, v\rangle \\ \langle e_2, v\rangle \\ \langle e_3, v\rangle \end{pmatrix}.$$

Therefore, the infinitesimal flexion is uniquely defined by the functions g_i. Hence the dimension of infinitesimal flexions is at most one (the parameter γ is the unique parameter of this flexion).

Existence. By Corollary 1.8 there exists an infinitesimal deformation satisfying system A and the initial values

$$\mathcal{D}_\gamma \dot f_1(a) = 0, \quad \mathcal{D}_\gamma \Delta f_1(a) = 0, \quad \text{and} \quad \mathcal{D}_\gamma \Delta f_0(a) = \dot f_1(a) \times \Delta f_0(a).$$

By Proposition 1.6 this infinitesimal deformation is an infinitesimal flexion. Since the function $(\dot f_1, \Delta f_0, \Delta f_1)$ has no zeroes, $\dot f_1(a) \times \Delta f_0(a)$ is a nonzero vector and hence the infinitesimal deformation is nonvanishing. □

1.3. **Variational operator of infinitesimal flexions.** Let us fix an orthonormal basis (e_1, e_2, e_3) in $\mathbb{R}^3$. Suppose that we know the coordinates of a 2-ribbon surface $f : [a,b] \times \{0,1,2\} \to \mathbb{R}^3$ in this basis. Denote the coordinate functions for $\dot f_1$, Δf_0, and Δf_1 as follows

$$\dot f_1(t) = \big(h_1(t), h_2(t), h_3(t)\big),\ \Delta f_0(t) = \big(h_4(t), h_5(t), h_6(t)\big),\ \Delta f_1(t) = \big(h_7(t), h_8(t), h_9(t)\big).$$

Denote by Ω_9^1 the Banach space $(C^1[a,b])^9$ with the norm

$$\|(h_1, \ldots, h_9)\| = \max_{1\le i\le 9}(\max(\sup|h_i|, \sup|\dot h_i|)).$$

Note that any 2-ribbon surface f is defined by the curves $\dot f_1$, Δf_0, and Δf_1 up to a translation. So the space Ω_9^1 is actually the space of all 2-ribbon surfaces with one endpoint fixed, say $f_1(a) = (0,0,0)$.

We say that a point $h = (h_1, \ldots, h_9)$ is in *general position* if the determinant

$$\det\begin{pmatrix} h_1 & h_2 & h_3 \\ h_4 & h_5 & h_6 \\ h_7 & h_8 & h_9 \end{pmatrix} \neq 0$$

for any point in the segment $[a,b]$. This condition obviously corresponds to

$$(\dot f_1, \Delta f_0, \Delta f_1) \neq 0.$$

Definition 1.10. Denote by $\mathcal{V} : [0,\Lambda] \times \Omega_9^1 \to \Omega_9^1$ the variational *operator of infinitesimal flexion* in coordinates $(h_1, \ldots, h_9)$:

$$\begin{aligned} \mathcal{V}_{3(l-1)+m}(\lambda, h) = \ & \frac{(e_m, \Delta f_0, \Delta f_1)}{(\dot f_1, \Delta f_0, \Delta f_1)} g_{3(l-1)+1}(h) + \frac{(\dot f_1, e_m, \Delta f_1)}{(\dot f_1, \Delta f_0, \Delta f_1)} g_{3(l-1)+2}(h) + \\ & \frac{(\dot f_1, \Delta f_0, e_m)}{(\dot f_1, \Delta f_0, \Delta f_1)} g_{3(l-1)+3}(h). \end{aligned} \tag{11}$$

for $(1 \le l, m \le 3)$. Here $g_1(h), \ldots, g_9(h)$ is a solution of System A at point f with the initial conditions corresponding to

$$\mathcal{D}_\gamma \dot f_1(a) = 0, \quad \mathcal{D}_\gamma \Delta f_1(a) = 0, \quad \text{and} \quad \mathcal{D}_\gamma \Delta f_0(a) = \dot f_1(a) \times \Delta f_0(a),$$

i. e.,

$$\begin{array}{lll} g_1(a) = 0, & g_2(a) = 0, & g_3(a) = 0, \\ g_4(a) = 0, & g_5(a) = 0, & g_6(a) = (\dot f_1(a), \Delta f_0(a), \Delta f_1(a)\rangle, \\ g_7(a) = 0, & g_8(a) = 0, & g_9(a) = 0. \end{array}$$

Note that the variational operator of infinitesimal flexion $\mathcal{V}$ is autonomous, it does not depend on time parameter λ.

Remark 1.11. Let us show in brief how to find the coordinates of the perturbation $\mathcal{D}_\gamma f$ in the basis e_1, e_2, e_3 satisfying

$$\mathcal{D}_\gamma f_1(a) = 0, \qquad \mathcal{D}_\gamma \dot f_1(a) = 0, \qquad \mathcal{D}_\gamma \Delta f_0(a) = \dot f_1(a) \times \Delta f_0(a), \quad \text{and} \quad \mathcal{D}_\gamma \Delta f_1(a) = 0.$$

First, one should solve System A with the above initial data, then substitute the obtained solution $(g_1, \ldots, g_9)$ to Equations (11). Now we have the coordinates of $\mathcal{D}_\gamma \dot{f}_1$, $\mathcal{D}_\gamma \Delta f_0$, and $\mathcal{D}_\gamma \Delta f_1$. Having the additional condition $\mathcal{D}_\gamma f_1(0) = 0$ one can construct $\mathcal{D}_\gamma f_1$, $\mathcal{D}_\gamma f_0$, and $\mathcal{D}_\gamma f_2$:

$$\mathcal{D}_\gamma f_1(t_0) = \int_a^{t_0} \mathcal{D}_\gamma \dot{f}_1(t) d(t), \quad \mathcal{D}_\gamma f_0 = \mathcal{D}_\gamma f_1 - \mathcal{D}_\gamma \Delta f_0, \quad \mathcal{D}_\gamma f_2 = \mathcal{D}_\gamma f_1 + \mathcal{D}_\gamma \Delta f_1.$$

1.4. **Finite flexibility of 2-ribbon surfaces.** In previous subsection we showed that any 2-ribbon surface in general position is infinitesimally flexible and the space of its infinitesimal flexions is one-dimensional. The aim of this subsection is to show that a 2-ribbon surface in general position is flexible and has one degree of freedom.

We start with the discussion of the initial value problem for the following differential equation on the set of all points Ω_9^1 in general position (here λ is the time parameter):

$$\frac{\partial h}{\partial \lambda} = \mathcal{V}(\lambda, h).$$

To solve the initial value problem we study local Lipschitz properties for $\mathcal{V}$.

Definition 1.12. Consider a Banach space E with a norm $| * |_E$ and let U be a subset of $[0, \Lambda] \times E$. We say that a functional $\mathcal{F} : U \to E$ *locally satisfies a Lipschitz condition* if for any point (λ_0, p) in U there exist a neighborhood V of the point and a constant K such that for any pair of points (λ, p_1) and (λ, p_2) in V the inequality

$$|\mathcal{F}(\lambda, p_1) - \mathcal{F}(\lambda, p_2)|_E \leq K |p_1 - p_2|_E$$

holds.

First we verify a Lipschitz condition for the following operator. Define $\mathcal{G} : [0, \Lambda] \times \Omega_9^1 \to \Omega_9^1$ by

$$\mathcal{G}_i(\lambda, h) = g_i(h), \quad i = 1, \ldots, 9,$$

where $g_i(h)$ are defined by Equations (10).

Lemma 1.13. *The functional $\mathcal{G}$ locally satisfies a Lipschitz condition at any point in general position.*

Proof. Consider a point $h \in U$. The element $(g_1, \ldots, g_9)$ itself satisfies a linear system of differential equations (System A). The coefficients of this system depend only on a point of Ω_9^1. Since the point h is in general position, there exists an integer constant K such that for a sufficiently small neighborhood V_h of h the dependence is K-Lipschitz, i.e., for p and q from V_h all the coefficients satisfy the inequality

$$|c(p) - c(q)| < K \|p - q\|.$$

Hence the solutions for $t \in [a, b]$ satisfy the Lipschitz condition on V_h as well: for some constants $\overline{K}_l$ we have

$$\sup(|g_l(p) - g_l(q)|) < \overline{K}_l \|p - q\|, \quad l = 1, \ldots, 9.$$

From System A we know that the $\dot g_l$ linearly depend on $g_1, \dots, g_9$, therefore, we get the Lipschitz condition for the derivatives: for some constants $\tilde K_l$ we have

$$\sup(|\dot g_l(p) - \dot g_l(q)|) < \tilde K_l \|p-q\|, \quad l = 1, \dots, 9.$$

Thus there exists a real number $\hat K$ such that for all points p and q in V_h,

$$\|\mathcal{G}(\lambda, p) - \mathcal{G}(\lambda, q)\| = \max_{1\le l\le 9} \big(\max\big(\sup|g_l(p) - g_l(q)|, \sup|\dot g_l(p) - \dot g_l(q)| \big)\big) < \hat K \|p-q\|.$$

Therefore, $\mathcal{G}$ satisfies a Lipschitz condition on V_h. □

Lemma 1.13 and Expression (11) directly imply the following statement.

Corollary 1.14. *The functional $\mathcal{V}$ locally satisfies a Lipschitz condition at points in general position.* □

Now we prove the following theorem on finite flexibility of 2-ribbon surfaces in general position.

Theorem 1.15. *Consider a 2-ribbon surface defined by a C^2-curve f_1 and C^1-curves f_0 and f_2 defined on a segment $[a, b]$. Assume that $(\dot f_1, \Delta f_0, \Delta f_1)$ does not have zeros on $[a, b]$. Then the set of all flexions of such surface (up to isometries) is one-dimensional.*

Proof. As we show in Corollary 1.14, the operator $\mathcal{V}$ satisfies a Lipschitz condition in some neighborhood of the point p related to $\dot f_1$, Δf_0, and Δf_1. From the general theory of differential equations on Banach spaces (see for instance the first section of the second chapter of [4]) it follows that this condition implies local existence and uniqueness of a solution of the initial value problem for the following differential equation

$$\frac{\partial h}{\partial \lambda} = \mathcal{V}(\lambda, h)$$

in some neighborhood of h.

Since the 2-ribbon surface (f_0, f_1, f_2) with a fixed endpoint $f_0(a)$ is uniquely defined by $(\dot f_1, \Delta f_0, \Delta f_1) \in \Omega_9^1$, we get the statement of the theorem. □

2. Flexibility of n-ribbon surfaces

In this section we study necessary flexibility conditions of n-ribbon surfaces. We find these conditions for 3-ribbon surfaces, and we show how they are related to the conditions for n-ribbon surfaces.

2.1. **Preliminary statements on infinitesimal flexion of 3-ribbon surfaces.**

In this subsection we prove certain relations that we further use in the proof of the statement on infinitesimal flexibility conditions for 3-ribbon surfaces.

Remark 2.1. As we have shown in Section 1 the notions of finite flexibility and infinitesimal flexibility coincide for the 2-ribbon case. Still in this subsection we say *infinitesimal flexions* of a 2-ribbon surface to indicate that an infinitesimal flexion of an n-ribbon surface coincides with finite flexions of all its 2-ribbon surfaces.

Consider the following function

$$\Phi = \langle \Delta f_0, \Delta f_1 \rangle.$$

This function plays a central role in our further description of the flexibility conditions of 3-ribbon and n-ribbon surfaces (see Theorem 2.7 and Theorem 2.13). Let $\mathcal{D}_\gamma\Phi$ be the infinitesimal flexion of Φ. Via this function we describe monodromy conditions for finite flexibility. Proposition 2.2 and Corollary 2.6 deliver necessary tools to describe continuous and discrete parts of the monodromy condition on Φ.

2.1.1. *Continuous shift.* Here we study the dependence of the infinitesimal flexion $\mathcal{D}_\gamma\Phi$ on the argument t.

Proposition 2.2. (On continuous shift.) *Suppose $\dot{f}_1$, Δf_0, and Δf_1 are linearly independent on the segment $[t_1, t_2]$. Then for an infinitesimal flexion $\mathcal{D}_\gamma\Phi$ the following condition holds:*

$$\mathcal{D}_\gamma\Phi(t_2) = \mathcal{D}_\gamma\Phi(t_1) \cdot \exp\left(\int\limits_{t_1}^{t_2} \frac{(\dot{f}_1, \Delta\dot{f}_0, \Delta f_1) + (\dot{f}_1, \Delta f_0, \Delta\dot{f}_1)}{(\dot{f}_1, \Delta f_0, \Delta f_1)} dt \right).$$

This is a direct consequence of the next lemma.

Lemma 2.3. *Let $\dot{f}_1$, Δf_0, and Δf_1 be linearly independent, then we have*

$$\mathcal{D}_\gamma\dot{\Phi} = \frac{(\dot{f}_1, \Delta\dot{f}_0, \Delta f_1) + (\dot{f}_1, \Delta f_0, \Delta\dot{f}_1)}{(\dot{f}_1, \Delta f_0, \Delta f_1)} \mathcal{D}_\gamma\Phi.$$

Proof. Note that

$$\begin{aligned} \mathcal{D}_\gamma\Phi &= \langle \mathcal{D}_\gamma\Delta f_0, \Delta f_1 \rangle + \langle \Delta f_0, \mathcal{D}_\gamma\Delta f_1 \rangle, \qquad \text{and} \\ \mathcal{D}_\gamma\dot{\Phi} &= \langle \mathcal{D}_\gamma\Delta\dot{f}_0, \Delta f_1 \rangle + \langle \mathcal{D}_\gamma\Delta f_0, \Delta\dot{f}_1 \rangle + \langle \Delta\dot{f}_0, \mathcal{D}_\gamma\Delta f_1 \rangle + \langle \Delta f_0, \mathcal{D}_\gamma\Delta\dot{f}_1 \rangle. \end{aligned}$$

Let us prove the statement of the lemma for an arbitrary point t_0. Without loss of generality we fix $\mathcal{D}_\gamma\dot{f}_1(t_0) = 0$ and $\mathcal{D}_\gamma\Delta f_1(t_0) = 0$ (this is possible since any flexion is isometric to a flexion with such properties and isometries of flexions do not change the functions in the formula of the lemma). Then $\mathcal{D}_\gamma\Delta f_0(t_0)$ is proportional to $\dot{f}_1(t_0)\times\Delta f_0(t_0)$, and hence there exists some real number α with

$$\mathcal{D}_\gamma\Delta f_0(t_0) = \alpha\dot{f}_1(t_0)\times\Delta f_0(t_0).$$

Thus we immediately get

$$\mathcal{D}_\gamma\Phi(t_0) = \big\langle \mathcal{D}_\gamma\Delta f_0(t), \Delta f_1(t) \big\rangle = \alpha\big(\dot{f}_1(t_0), \Delta f_0(t_0), \Delta f_1(t_0)\big).$$

Let us express the summands for $\mathcal{D}_\gamma\dot{\Phi}(t_0)$. We start with $\big\langle \mathcal{D}_\gamma\Delta\dot{f}_0(t_0), \Delta f_1(t_0) \big\rangle$. First we note that

(i) $$\Delta f_1 = \frac{(\Delta f_1, \Delta f_0, \dot{f}_1\times\Delta\dot{f}_0)}{(\dot{f}_1, \Delta f_0, \dot{f}_1\times\Delta f_0)}\dot{f}_1 + \frac{(\dot{f}_1, \Delta f_1, \dot{f}_1\times\Delta\dot{f}_0)}{(\dot{f}_1, \Delta f_0, \dot{f}_1\times\Delta f_0)}\Delta f_0 + \frac{(\dot{f}_1, \Delta f_0, \Delta f_1)}{(\dot{f}_1, \Delta f_0, \dot{f}_1\times\Delta f_0)} \dot{f}_1\times\Delta\dot{f}_0.$$

Equation (6) implies

(ii) $$\big\langle \mathcal{D}_\gamma\Delta\dot{f}_0(t_0), \dot{f}_1(t_0) \big\rangle = -\big\langle \mathcal{D}_\gamma\dot{f}_1(t_0), \Delta\dot{f}_0(t_0) \big\rangle = -\big\langle 0, \Delta\dot{f}_0(t_0) \big\rangle = 0.$$

From Equation (4) we have

(iii) $\quad \big\langle \mathcal{D}_\gamma \Delta \dot{f}_0(t_0), \Delta f_0(t_0)\big\rangle = -\big\langle \mathcal{D}_\gamma \Delta f_0(t_0), \Delta \dot{f}_0(t_0)\big\rangle = -\alpha\big(\dot{f}_1(t_0), \Delta f_0(t_0), \Delta \dot{f}_0(t_0)\big).$

The function $(\Delta \dot{f}_0, \dot{f}_1, \Delta f_0)$ is invariant of an infinitesimal flexion, therefore:

$$(\mathcal{D}_\gamma \Delta \dot{f}_0, \dot{f}_1, \Delta f_0) + (\Delta \dot{f}_0, \mathcal{D}_\gamma \dot{f}_1, \Delta f_0) + (\Delta \dot{f}_0, \dot{f}_1, \mathcal{D}_\gamma \Delta f_0) = 0,$$

and hence

(iv) $$\begin{aligned} \big\langle \mathcal{D}_\gamma \Delta \dot{f}_0(t_0), \dot{f}_1(t_0) \times \Delta f_0(t_0)\big\rangle &= -\big(\Delta \dot{f}_0(t_0), \dot{f}_1(t_0), \mathcal{D}_\gamma \Delta f_0(t_0)\big) \\ &= -\alpha\big(\Delta \dot{f}_0(t_0), \dot{f}_1(t_0), \dot{f}_1(t_0) \times \Delta f_0(t_0)\big). \end{aligned}$$

Now we decompose $\Delta \dot{f}_0(t_0)$ in the last formula in the basis of vectors $\dot{f}_1(t_0)$, $\Delta f_0(t_0)$, and $\Delta f_1(t_0)$:

$$\begin{aligned} \big(\Delta \dot{f}_0(t_0), \dot{f}_1(t_0), \dot{f}_1(t_0) \times \Delta f_0(t_0)\big) = &\frac{\big(\dot{f}_1(t_0), \Delta \dot{f}_0(t_0), \Delta f_1(t_0)\big)}{\big(\dot{f}_1(t_0), \Delta f_0(t_0), \Delta f_1(t_0)\big)} \big(\Delta f_0(t_0), \dot{f}_1(t_0), \dot{f}_1(t_0) \times \Delta f_0(t_0)\big) + \\ &\frac{\big(\dot{f}_1(t_0), \Delta f_0(t_0), \Delta \dot{f}_0(t_0)\big)}{\big(\dot{f}_1(t_0), \Delta f_0(t_0), \Delta f_1(t_0)\big)} \big(\Delta f_1(t_0), \dot{f}_1(t_0), \dot{f}_1(t_0) \times \Delta f_0(t_0)\big). \end{aligned}$$

Therefore, after substitution (i) of Δf_2 we apply (ii), (iii), (iv), and the last expression and get

$$\begin{aligned} \big\langle \mathcal{D}_\gamma \Delta \dot{f}_0(t_0), \Delta f_1(t_0)\big\rangle = &-\alpha \frac{\big(\dot{f}_1(t_0), \Delta f_1(t_0), \dot{f}_1(t_0) \times \Delta f_0(t_0)\big)}{\big(\dot{f}_1(t_0), \Delta f_0(t_0), \dot{f}_1(t_0) \times \Delta f_0(t_0)\big)} \big(\dot{f}_1(t_0), \Delta f_0(t_0), \Delta \dot{f}_0(t_0)\big) - \\ &\alpha \frac{\big(\dot{f}_1(t_0), \Delta \dot{f}_0(t_0), \Delta f_1(t_0)\big)}{\big(\dot{f}_1(t_0), \Delta f_0(t_0), \dot{f}_1(t_0) \times \Delta f_0(t_0)\big)} \big(\Delta f_0(t_0), \dot{f}_1(t_0), \dot{f}_1(t_0) \times \Delta f_0(t_0)\big) - \\ &\alpha \frac{\big(\dot{f}_1(t_0), \Delta f_0(t_0), \Delta \dot{f}_0(t_0)\big)}{\big(\dot{f}_1(t_0), \Delta f_0(t_0), \dot{f}_1(t_0) \times \Delta f_0(t_0)\big)} \big(\Delta f_1(t_0), \dot{f}_1(t_0), \dot{f}_1(t_0) \times \Delta f_0(t_0)\big) \\ = &- \alpha\big(\dot{f}_1(t_0), \Delta f_1(t_0), \Delta \dot{f}_0(t_0)\big). \end{aligned}$$

Similar calculations for the summand $\big\langle \Delta f_0(t_0), \mathcal{D}_\gamma \Delta \dot{f}_1(t_0)\big\rangle$ (applying Equations (3), (5), and (7) and the conditions $\mathcal{D}_\gamma \dot{f}_1(t_0) = 0$ and $\mathcal{D}_\gamma \Delta f_1(t_0) = 0$) show that

$$\big\langle \Delta f_0(t_0), \mathcal{D}_\gamma \Delta \dot{f}_1(t_0)\big\rangle = 0.$$

Further we have

$$\begin{aligned} &\big\langle \mathcal{D}_\gamma \Delta f_0(t_0), \Delta \dot{f}_1(t_0)\big\rangle = \alpha\big(\dot{f}_1(t_0), \Delta f_0(t_0), \Delta \dot{f}_1(t_0)\big), \\ &\big\langle \Delta \dot{f}_0(t_0), \mathcal{D}_\gamma \Delta f_1(t_0)\big\rangle = 0. \end{aligned}$$

Therefore,

$$\mathcal{D}_\gamma \dot{\Phi}(t_0) = \alpha\Big(\big(\dot{f}_1(t_0), \Delta \dot{f}_0(t_0), \Delta f_1(t_0)\big) + \big(\dot{f}_1(t_0), \Delta f_0(t_0), \Delta \dot{f}_1(t_0)\big)\Big),$$

and consequently

$$\mathcal{D}_\gamma \dot{\Phi}(t_0) = \frac{\big(\dot{f}_1(t_0), \Delta \dot{f}_0(t_0), \Delta f_1(t_0)\big) + \big(\dot{f}_1(t_0), \Delta f_0(t_0), \Delta \dot{f}_1(t_0)\big)}{\big(\dot{f}_1(t_0), \Delta f_0(t_0), \Delta f_1(t_0)\big)} \mathcal{D}_\gamma \Phi(t_0).$$

Thus Lemma 2.3 holds for all possible values of t_0. □

2.1.2. *Discrete shift.* Any 3-ribbon surface contain 2-ribbon surfaces as a subsurfaces. Each of them has an infinitesimal flexion $\mathcal{D}_\gamma\Phi_i$ ($i=1,2$). Here we show the relation between $\mathcal{D}_\gamma\Phi_1$ and $\mathcal{D}_\gamma\Phi_2$ for the same values of argument t.

First, in Proposition 2.4 we show a relation for $\mathcal{D}_\gamma\langle \ddot{f}_1, \ddot{f}_1\rangle$ and $\mathcal{D}_\gamma\langle \ddot{f}_2, \ddot{f}_2\rangle$. Second, in Proposition 2.5 we give a link between $\mathcal{D}_\gamma\langle \ddot{f}_1, \ddot{f}_1\rangle$ and $\mathcal{D}_\gamma\Phi_i$. This will result in the formula of Corollary 2.6 on the relation between $\mathcal{D}_\gamma\Phi_1$ and $\mathcal{D}_\gamma\Phi_2$.

We start with a formula expressing $\mathcal{D}_\gamma\langle \ddot{f}_2, \ddot{f}_2\rangle$ via $\mathcal{D}_\gamma\langle \ddot{f}_1, \ddot{f}_1\rangle$.

Proposition 2.4. *We have the following equation:*

$$\mathcal{D}_\gamma\langle \ddot{f}_2, \ddot{f}_2\rangle = \frac{(\dot{f}_2, \ddot{f}_2, \Delta f_1)}{(\dot{f}_1, \ddot{f}_1, \Delta f_1)} \mathcal{D}_\gamma\langle \ddot{f}_1, \ddot{f}_1\rangle.$$

Proof. We do calculations at a point t_0 again assuming that $\mathcal{D}_\gamma \dot{f}_1(t_0)=0$ and $\mathcal{D}_\gamma\Delta f_1(t_0)=0$ (by choosing an appropriate isometric representative of the deformation). Let us show that $\mathcal{D}_\gamma \dot{f}_2(t_0)=0$. First, note that

$$\mathcal{D}_\gamma \dot{f}_2(t_0) = \mathcal{D}_\gamma \dot{f}_1(t_0) + \mathcal{D}_\gamma \Delta \dot{f}_1(t_0) = \mathcal{D}_\gamma \Delta \dot{f}_1(t_0).$$

Secondly we show that the inner products of $\mathcal{D}_\gamma\Delta\dot{f}_1(t_0)$ and the vectors $\dot{f}_1(t_0)$, $\Delta f_1(t_0)$, and $\dot{f}_1(t_0)\times\Delta f_1(t_0)$ are all zero (this would imply that $\mathcal{D}_\gamma\Delta\dot{f}_1(t_0)=0$).

From Equation (7) we have

$$\big\langle \mathcal{D}_\gamma\Delta\dot{f}_1(t_0), \dot{f}_1(t_0)\big\rangle = -\big\langle \mathcal{D}_\gamma\dot{f}_1(t_0), \Delta\dot{f}_1(t_0)\big\rangle = -\big\langle 0, \Delta\dot{f}_1(t_0)\big\rangle = 0.$$

Further, from Equations (5), we get

$$\big\langle \mathcal{D}_\gamma\Delta\dot{f}_1(t_0), \Delta f_1(t_0)\big\rangle = -\big\langle \mathcal{D}_\gamma\Delta f_1(t_0), \Delta\dot{f}_1(t_0)\big\rangle = 0.$$

Finally, from the equation $\mathcal{D}_\gamma(\dot{f}_1, \Delta f_1, \Delta\dot{f}_1)=0$ we obtain

$$\begin{aligned}&\big\langle \mathcal{D}_\gamma\Delta\dot{f}_1(t_0), \dot{f}_1(t_0)\times\Delta f_1(t_0)\big\rangle = \\ &-\big(\Delta\dot{f}_1(t_0), \mathcal{D}_\gamma\dot{f}_1(t_0), \Delta f_1(t_0)\big) - \big(\Delta\dot{f}_1(t_0), \dot{f}_1(t_0), \mathcal{D}_\gamma\Delta f_1(t_0)\big) = 0.\end{aligned}$$

Therefore, $\mathcal{D}_\gamma\Delta\dot{f}_1(t_0)=0$, and hence $\mathcal{D}_\gamma\dot{f}_2(t_0)=0$.

From Equation (1) and Equation (9) we get

$$\begin{aligned}&\big\langle \mathcal{D}_\gamma\ddot{f}_1(t_0), \dot{f}_1(t_0)\big\rangle = \tfrac{\partial}{\partial t}\big\langle \mathcal{D}_\gamma\dot{f}_1(t_0), \dot{f}_1(t_0)\big\rangle - \big\langle \ddot{f}_1(t_0), \mathcal{D}_\gamma\dot{f}_1(t_0)\big\rangle = 0 - \big\langle \ddot{f}_1(t_0), 0\big\rangle = 0;\\ &\big\langle \mathcal{D}_\gamma\ddot{f}_1(t_0), \Delta f_1(t_0)\big\rangle = -\big\langle \ddot{f}_1(t_0), \mathcal{D}_\gamma\Delta f_1(t_0)\big\rangle = -\big\langle \ddot{f}_1(t_0), 0\big\rangle = 0.\end{aligned}$$

Therefore, for some real number β_1 we have

$$\mathcal{D}_\gamma\ddot{f}_1(t_0) = \beta_1\dot{f}_1(t_0)\times\Delta f_1(t_0).$$

By a similar reasoning (since we have shown that $\mathcal{D}_\gamma \dot{f}_2(t_0)=0$) we get

$$\mathcal{D}_\gamma \ddot{f}_2(t_0) = \beta_2 \dot{f}_2(t_0)\times\Delta f_1(t_0).$$

Since $\frac{\partial}{\partial t}\big(\mathcal{D}_\gamma(\dot{f}_1, \Delta f_1, \dot{f}_2)\big)=0$, at point t_0 we have

$$\big(\mathcal{D}_\gamma \ddot{f}_1(t_0), \Delta f_1(t_0), \dot{f}_2(t_0)\big) + \big(\dot{f}_1(t_0), \Delta f_1(t_0), \mathcal{D}_\gamma \ddot{f}_2(t_0)\big) = 0.$$

Hence,

$$\beta_1\big(\dot{f}_1(t_0)\times\Delta f_1(t_0), \Delta f_1(t_0), \dot{f}_2(t_0)\big) + \beta_2\big(\dot{f}_1(t_0), \Delta f_1(t_0), \dot{f}_2(t_0)\times\Delta f_1(t_0)\big) = 0,$$

and, therefore $\beta_1=\beta_2$. This implies

$$\mathcal{D}_\gamma\big\langle \ddot{f}_1(t_0), \ddot{f}_1(t_0)\big\rangle = 2\big\langle \mathcal{D}_\gamma \ddot{f}_1(t_0), \ddot{f}(t_0)\big\rangle = 2\beta_1\big(\dot{f}_1(t_0), \Delta f_1(t_0), \ddot{f}_1(t_0)\big)$$

and

$$\mathcal{D}_\gamma\big\langle \ddot{f}_2(t_0), \ddot{f}_2(t_0)\big\rangle = 2\beta_1\big(\dot{f}_2(t_0), \Delta f_1(t_0), \ddot{f}_2(t_0)\big).$$

The last two formulas imply the statement of Proposition 2.4. □

Now let us relate $\mathcal{D}_\gamma\langle \ddot{f}_1, \ddot{f}_1\rangle$ and $\mathcal{D}_\gamma\Phi$.

Proposition 2.5. *Suppose $\dot{f}_1$, Δf_0, and Δf_1 are linearly independent. Then the following equation holds:*

$$\mathcal{D}_\gamma\langle \ddot{f}_1, \ddot{f}_1\rangle = 2\frac{(\dot{f}_1, \ddot{f}_1, \Delta f_0)(\dot{f}_1, \ddot{f}_1, \Delta f_1)}{(\dot{f}_1, \Delta f_0, \Delta f_1)^2}\mathcal{D}_\gamma\Phi.$$

Proof. We restrict ourselves to the case of a point. Without loss of generality we assume that $\mathcal{D}_\gamma \dot{f}_1(t_0)=0$ and $\mathcal{D}_\gamma\Delta f_1(t_0)=0$. So as we have seen before, there exists α such that

$$\mathcal{D}_\gamma \Delta f_0(t_0) = \alpha \dot{f}_1(t_0)\times\Delta f_0(t_0)$$

and hence

$$\mathcal{D}_\gamma\Phi(t_0) = \alpha\big(\dot{f}_1(t_0), \Delta f_0(t_0), \Delta f_1(t_0)\big).$$

Let us calculate $\mathcal{D}_\gamma\langle \ddot{f}_1, \ddot{f}_1\rangle = 2\langle \mathcal{D}_\gamma \ddot{f}_1, \ddot{f}_1\rangle$. Decompose

$$\ddot{f}_1 = \frac{(\ddot{f}_1, \Delta f_0, \Delta f_1)}{(\dot{f}_1, \Delta f_0, \Delta f_1)}\dot{f}_1 + \frac{(\dot{f}_1, \ddot{f}_1, \Delta f_1)}{(\dot{f}_1, \Delta f_0, \Delta f_1)}\Delta f_0 + \frac{(\dot{f}_1, \Delta f_0, \ddot{f}_1)}{(\dot{f}_1, \Delta f_0, \Delta f_1)}\Delta f_1.$$

Since

$$\big\langle \mathcal{D}_\gamma \ddot{f}_1(t_0), \dot{f}_1(t_0)\big\rangle = 0, \qquad \text{and} \qquad \big\langle \mathcal{D}_\gamma \ddot{f}_1(t_0), \Delta f_1(t_0)\big\rangle = 0,$$

we get

$$\mathcal{D}_\gamma\langle \ddot{f}_1(t_0), \ddot{f}_1(t_0)\rangle = 2\frac{\big(\dot{f}_1(t_0), \ddot{f}_1(t_0), \Delta f_1(t_0)\big)}{\big(\dot{f}_1(t_0), \Delta f_0(t_0), \Delta f_1(t_0)\big)}\big\langle \mathcal{D}_\gamma \ddot{f}_1(t_0), \Delta f_0(t_0)\big\rangle.$$

By Equation (8) we have

$$\big\langle \mathcal{D}_\gamma \ddot{f}_1, \Delta f_0\big\rangle = -\big\langle \ddot{f}_1, \mathcal{D}_\gamma \Delta f_0\big\rangle.$$

Hence after the substitution of $\mathcal{D}_\gamma \Delta f_0(t_0)$ in the first summand one gets

$$\big\langle \mathcal{D}_\gamma \ddot{f}_1, \Delta f_0 \big\rangle = \alpha\big(\dot{f}_1(t_0), \ddot{f}_1(t_0), \Delta f_0(t_0)\big) = \frac{\big(\dot{f}_1(t_0), \ddot{f}_1(t_0), \Delta f_0(t_0)\big)}{\big(\dot{f}_1(t_0), \Delta f_0(t_0), \Delta f_1(t_0)\big)} \mathcal{D}_\gamma \Phi(t_0).$$

Therefore, we obtain

$$\mathcal{D}_\gamma \langle \ddot{f}_1(t_0), \ddot{f}_1(t_0)\rangle = 2\frac{\big(\dot{f}_1(t_0), \ddot{f}_1(t_0), \Delta f_1(t_0)\big)\big(\dot{f}_1(t_0), \ddot{f}_1(t_0), \Delta f_0(t_0)\big)}{\big(\dot{f}_1(t_0), \Delta f_0(t_0), \Delta f_1(t_0)\big)^2} \mathcal{D}_\gamma \Phi(t_0).$$

Since the statement does not depend on the choice of the basis and invariant under isometries, we get the statement for all the points. □

We introduce the abbreviations

$$\Phi_1 = \langle \Delta f_0, \Delta f_1\rangle \qquad \text{and} \qquad \Phi_2 = \langle \Delta f_1, \Delta f_2\rangle.$$

Let us show a formula of a discrete shift.

Corollary 2.6. (On discrete shift.) *Suppose $\dot{f}_1$, Δf_0, and Δf_1 are linearly independent. Then the following holds:*

$$\mathcal{D}_\gamma \Phi_2(t) = \frac{\big(\dot{f}_1(t), \ddot{f}_1(t), \Delta f_0(t)\big)}{\big(\dot{f}_2(t), \ddot{f}_2(t), \Delta f_2(t)\big)} \frac{\big(\dot{f}_2(t), \Delta f_1(t), \Delta f_2(t)\big)^2}{\big(\dot{f}_1(t), \Delta f_0(t), \Delta f_1(t)\big)^2} \mathcal{D}_\gamma \Phi_1(t).$$

Proof. The statement follows directly from Propositions 2.4 and 2.5. □

2.2. **Infinitesimal flexibility of 3-ribbon surfaces.** In this subsection we write down the infinitesimal flexibility monodromy conditions for 3-ribbon surfaces (via continuous shifts of Proposition 2.2 and discrete shifts of Corollary 2.6). Recall that

$$\Lambda(t) = \frac{\big(\dot{f}_1(t), \ddot{f}_1(t), \Delta f_0(t)\big)}{\big(\dot{f}_2(t), \ddot{f}_2(t), \Delta f_2(t)\big)} \frac{\big(\dot{f}_2(t), \Delta f_1(t), \Delta f_2(t)\big)^2}{\big(\dot{f}_1(t), \Delta f_0(t), \Delta f_1(t)\big)^2},$$

and

$$H_i(t) = \frac{(\dot{f}_i(t), \Delta \dot{f}_{i-1}(t), \Delta f_i(t)) + (\dot{f}_i(t), \Delta f_{i-1}(t), \Delta \dot{f}_i(t))}{(\dot{f}_i(t), \Delta f_{i-1}(t), \Delta f_i(t))}, \quad i = 1, 2.$$

Theorem 2.7. *Consider a 3-ribbon surface f with linearly independent $\dot{f}_1$, Δf_0, and Δf_1 at all admissible points. The surface f is infinitesimally flexible if and only if for any t_1 and t_2 in the interval $[a, b]$ we have*

$$\Lambda(t_2) \cdot \exp\Big(\int_{t_1}^{t_2} H_1(t)dt\Big) = \Lambda(t_1) \cdot \exp\Big(\int_{t_1}^{t_2} H_2(t)dt\Big).$$

Proof. By Corollary 2.6 we get relations between $\mathcal{D}_\gamma\Phi_1(t_i)$ and $\mathcal{D}_\gamma\Phi_2(t_i)$ for $i = 1, 2$. On the other hand, Proposition 2.2 relates $\mathcal{D}_\gamma\Phi_i(t_1)$ and $\mathcal{D}_\gamma\Phi_i(t_2)$ for $i = 1, 2$. These four relations define the monodromy condition for Φ_i that is the condition in the theorem and, therefore, it holds if a surface is infinitesimally flexible.

Suppose now the condition holds. Then the flexion is uniquely defined by the value of $\mathcal{D}_\gamma\Phi_1$ at a point t_0. □

Remark 2.8. Let us simplify the expressions for Λ and H_i performing the following normalization for a fixed parameter λ. Denote

$$\begin{aligned} w_0 &= f_1 - \tfrac{1}{(\dot f_1, \Delta f_0, \Delta f_1)}\Delta f_0;\\ w_1 &= f_1;\\ w_2 &= f_2;\\ w_3 &= f_2 + \tfrac{1}{(\dot f_2, \Delta f_1, \Delta f_2)}\Delta f_2. \end{aligned}$$

Here the semidiscrete surface f is flexible if and only if w is flexible. In addition for the semidiscrete surface w we get

$$\big(\dot w_1(t), \Delta w_0(t), \Delta w_1(t)\big) = 1 \qquad \text{and} \qquad \big(\dot w_2(t), \Delta w_1(t), \Delta w_2(t)\big) = 1$$

for all arguments t. Therefore we get the expressions for Λ and H_i as follows:

$$\Lambda = \frac{\big(\dot w_1, \ddot w_1, \Delta w_0\big)}{\big(\dot w_2, \ddot w_2, \Delta w_2\big)},$$

and

$$H_i = -(\ddot w_i, \Delta w_{i-1}, \Delta w_i), \quad i = 1, 2.$$

Notice that this expression holds momentary, i.e. only for a fixed time parameter λ, so it cannot be use for finite deformations.

2.3. **Higher order variational conditions of flexibility for 3-ribbon surfaces.** In this subsection we say a few words about higher order variational conditions of flexibility for 3-ribbon surfaces. We give an algorithm to rewrite these conditions in terms of the coefficients of the infinitesimal flexion defined by the system of differential equations (System A).

We introduce a further auxiliary function by letting

$$\chi = \dot\Lambda - (H_2 - H_1)\Lambda.$$

Corollary 2.9. *A 3-ribbon surface is infinitesimally flexible if and only if the following condition holds:*

$$\chi = 0.$$

Proof. This condition is obtained from the condition of Theorem 2.7 by differentiating w.r.t t_2 at the point t_1. Therefore, these conditions are equivalent. □

From the infinitesimal flexibility condition of Corollary 2.9 one constructs many other conditions of flexibility. If a 3-ribbon surface has a flexion, depending on a parameter λ, then $\chi(\lambda) = 0$ at all points. This implies the following statement.

Proposition 2.10. *If a 3-ribbon surface is flexible then for any positive integer m we have*

$$\mathcal{D}_\gamma^m \chi = 0,$$

where $\mathcal{D}_\gamma^m \chi = \frac{\partial^m \chi}{\partial \lambda^m}$. □

Let us briefly describe a technique to calculate $\mathcal{D}_\gamma^m(\chi)$.

Step 1. To simplify the expressions we write:

$$f_{i,1} = \dot{f}_i, \qquad f_{i,2} = \Delta f_{i-1}, \qquad f_{i,3} = \Delta f_i.$$

Further we let

$$\begin{array}{ll} f_{i,jk} = \langle f_{i,j}, f_{i,k}\rangle, & f_{i,jkl} = (f_{i,j}, f_{i,k}, f_{i,l}); \\ g_{i,jk} = \big\langle \mathcal{D}_\gamma f_{i,j}, f_{i,k}\big\rangle, & g_{i,jkl} = \big(\mathcal{D}_\gamma f_{i,j}, f_{i,k}, f_{i,l}\big). \end{array}$$

Here we are interested in derivatives at an arbitrary value of a curve argument t but at a fixed parameter of deformation $\lambda = 0$.

Note that $g_{i,jkk} = 0$.

The functions $f_{i,jk}$ and $f_{i,jkl}$ are calculated from the initial data for the 3-ribbon surface f.

Let us find the expressions for $g_{i,jk}$. Without loss of generality we fix

$$\mathcal{D}_\gamma \dot{f}_1(a) = 0, \quad \mathcal{D}_\gamma \Delta f_1(a) = 0, \quad \mathcal{D}_\gamma \dot{f}_2(a) = 0, \quad \mathcal{D}_\gamma \Delta f_0(a) = \dot{f}_1(a) \times \Delta f_0(a),$$
$$\text{and} \quad \mathcal{D}_\gamma \Delta f_0(a) = \alpha \dot{f}_2(a) \times \Delta f_2(a)$$

for a starting point a. We find α from Corollary 2.6 (on discrete shift). First, we have

$$\mathcal{D}_\gamma \Phi_1(a) = \langle \dot{f}_1(a) \times \Delta f_0(a), \Delta f_1(a)\rangle \quad \text{and} \quad \mathcal{D}_\gamma \Phi_2(a) = \alpha \langle \Delta f_1(a), \dot{f}_2(a) \times \Delta f_2(a)\rangle.$$

Therefore, from Corollary 2.6 we have

$$\alpha = \frac{\big(\dot{f}_1(a), \ddot{f}_1(a), \Delta f_0(a)\big)}{\big(\dot{f}_2(a), \ddot{f}_2(a), \Delta f_2(a)\big)} \frac{\big(\dot{f}_2(a), \Delta f_2(a), \Delta f_1(a)\big)}{\big(\dot{f}_1(a), \Delta f_0(a), \Delta f_1(a)\big)}.$$

All the functions $g_{i,jk}$ are found as the corresponding solutions of two systems of differential equations (System A for $i = 1, 2$) according to Corollary 1.8. Since $\chi = 0$, these solutions are compatible.

For the functions $g_{i,jkl}$ we have

$$\begin{aligned} g_{i,jkl} &= \big\langle \mathcal{D}_\gamma f_{i,j}, f_{i,k} \times f_{i,l}\big\rangle \\ &= \frac{(f_{i,k} \times f_{i,l}, f_{i,2}, f_{i,3})}{f_{i,123}} g_{i,j1} + \frac{(f_{i,1}, f_{i,k} \times f_{i,l}, f_{i,3})}{f_{i,123}} g_{i,j2} + \frac{(f_{i,1}, f_{i,2}, f_{i,k} \times f_{i,l})}{f_{i,123}} g_{i,j3}. \end{aligned}$$

To avoid cross products in the above expression we use Lagrange's formula:

$$(a, b, c \times d) = \langle a, b \times (c \times d)\rangle = \langle a, c\langle b, d\rangle - d\langle a, b\rangle\rangle = \langle a, c\rangle\langle b, d\rangle - \langle a, d\rangle\langle a, b\rangle.$$

Step 2. Define

$$f^m_{i,jk}=\left\langle f^{(m)}_{i,j}, f_{i,k}\right\rangle, \qquad f^m_{i,jkl}=\left(f^{(m)}_{i,j}, f_{i,k}, f_{i,l}\right).$$
$$g^m_{i,jk}=\left\langle \mathcal{D}_\gamma f^{(m)}_{i,j}, f_{i,k}\right\rangle, \quad g^m_{i,jkl}=\left(\mathcal{D}_\gamma f^{(m)}_{i,j}, f_{i,k}, f_{i,l}\right).$$

As before, the functions $f^m_{i,jk}$ and $f^m_{i,jkl}$ are calculated from the initial data for the 3-ribbon surface f.

First, let us find the expressions for $g^m_{i,jkl}$ by induction on m.

Induction base. For $m=1$ we get the formulae from Step 1.

Induction step. Suppose that we know the expressions for m let us find $g^{m+1}_{i,jkl}$. We have

$$\frac{\partial\left(g^{m-1}_{i,jkl}\right)}{\partial t}=g^m_{i,jkl}+\left(\mathcal{D}_\gamma f^{(m-1)}_{i,j}, \dot{f}_{i,k}, f_{i,l}\right)+\left(\mathcal{D}_\gamma f^{(m-1)}_{i,j}, f_{i,k}, \dot{f}_{i,l}\right).$$

The expression in the left hand part is a function that is known by induction. The last two summands of the hand right part are also expressed inductively after rewriting $\dot{f}_{i,k}$, and $\dot{f}_{i,l}$ in the basis $f_{i,1}$, $f_{i,2}$, $f_{i,3}$.

Secondly, decomposing $\mathcal{D}_\gamma f^m_{i,j}$ in the basis $f_{i,1}$, $f_{i,2}$, $f_{i,3}$ we get

$$g^m_{i,jk}=\frac{g^m_{i,j23}f_{i,1k}+g^m_{i,j31}f_{i,2k}+g^m_{i,j12}f_{i,3k}}{f_{i,123}}.$$

Step 3. Note that

$$\left\langle f^{(m_1)}_{i,j_1}, f^{(m_2)}_{i,j_2}\right\rangle=\frac{f^{m_1}_{i,j_1 23}f^{m_2}_{i,j_2 1}+f^{m_1}_{i,j_1 31}f^{m_2}_{i,j_2 2}+f^{m_1}_{i,j_1 12}f^{m_2}_{i,j_2 3}}{f_{i,123}},$$
$$\left\langle \mathcal{D}_\gamma f^{(m_1)}_{i,j_1}, f^{(m_2)}_{i,j_2}\right\rangle=\frac{g^{m_1}_{i,j_1 23}f^{m_2}_{i,j_2 1}+g^{m_1}_{i,j_1 31}f^{m_2}_{i,j_2 2}+g^{m_1}_{i,j_1 12}f^{m_2}_{i,j_2 3}}{f_{i,123}},$$

and

$$\left(f^{(m_1)}_{i,j_1}, f^{(m_2)}_{i,j_2}, f^{(m_3)}_{i,j_3}\right)=\frac{f^{m_2}_{i,j_1 12}f^{m_3}_{i,j_2 31}}{(f_{i,123})^2}f^{m_1}_{i,j_3 32}+\frac{f^{m_2}_{i,j_1 12}f^{m_3}_{i,j_2 23}}{(f_{i,123})^2}f^{m_1}_{i,j_3 31}+\frac{f^{m_2}_{i,j_1 23}f^{m_3}_{i,j_2 12}}{(f_{i,123})^2}f^{m_1}_{i,j_3 13}+$$
$$\frac{f^{m_2}_{i,j_1 23}f^{m_3}_{i,j_2 31}}{(f_{i,123})^2}f^{m_1}_{i,j_3 12}+\frac{f^{m_2}_{i,j_1 31}f^{m_3}_{i,j_2 12}}{(f_{i,123})^2}f^{m_1}_{i,j_3 23}+\frac{f^{m_2}_{i,j_1 31}f^{m_3}_{i,j_2 23}}{(f_{i,123})^2}f^{m_1}_{i,j_3 21},$$
$$\left(\mathcal{D}_\gamma f^{(m_1)}_{i,j_1}, f^{(m_2)}_{i,j_2}, f^{(m_3)}_{i,j_3}\right)=\frac{f^{m_2}_{i,j_1 12}f^{m_3}_{i,j_2 31}}{(f_{i,123})^2}g^{m_1}_{i,j_3 32}+\frac{f^{m_2}_{i,j_1 12}f^{m_3}_{i,j_2 23}}{(f_{i,123})^2}g^{m_1}_{i,j_3 31}+\frac{f^{m_2}_{i,j_1 23}f^{m_3}_{i,j_2 12}}{(f_{i,123})^2}g^{m_1}_{i,j_3 13}+$$
$$\frac{f^{m_2}_{i,j_1 23}f^{m_3}_{i,j_2 31}}{(f_{i,123})^2}g^{m_1}_{i,j_3 12}+\frac{f^{m_2}_{i,j_1 31}f^{m_3}_{i,j_2 12}}{(f_{i,123})^2}g^{m_1}_{i,j_3 23}+\frac{f^{m_2}_{i,j_1 31}f^{m_3}_{i,j_2 23}}{(f_{i,123})^2}g^{m_1}_{i,j_3 21}.$$

Proposition 2.11. *Suppose that we get a rational polynomial expression T in variables $f^*_{i,**}$, $f^*_{i,***}$, $g^*_{i,**}$, and $g^*_{i,***}$. Then $\mathcal{D}_\gamma T$ is also a rational polynomial expression in variables $f^*_{i,**}$, $f^*_{i,***}$, $g^*_{i,**}$, and $g^*_{i,***}$.*

Proof. Steps 1–3 give all the tools to write the expression for $\mathcal{D}_\gamma T$ explicitly. □

Proposition 2.12. *For any positive integer m the function $\mathcal{D}^m_\gamma \chi$ is a rational polynomial expression in variables $f^*_{i,**}$, $f^*_{i,***}$, $g^*_{i,**}$, and $g^*_{i,***}$.* □

Therefore, we can apply Steps 1–3 and Proposition 2.11 to calculate $\mathcal{D}^m_\gamma \chi$ using induction on m.

We conclude this subsection with a few words on sufficient conditions for flexibility. We start with an open problem.

Problem 1. Find a sufficient condition for flexibility of semidiscrete and n-ribbon surfaces.

For the case of 3-ribbon surfaces we have the following conjecture.

Conjecture 2. Consider a 3-ribbon surface f. Let $\mathcal{D}^m_\gamma \chi = 0$ for all non-negative integers m (where $\mathcal{D}^0_\gamma \chi = \chi$). Then f is locally flexible.

We also conjecture that it is enough to take only a finite number of these conditions. Then the following question is actual: *What is the number of independent conditions of isometric deformation*?

2.4. **An n-ribbon surface and its 3-ribbon subsurfaces.** Let us finally describe a relation between (finite and infinitesimal) flexibility of n-ribbon surfaces and flexibility of all 3-ribbon subsurfaces contained in them.

We start with theorem on infinitesimal flexibility.

Theorem 2.13. *Consider an n-ribbon surface satisfying the genericity condition: $\dot{f}_i$, Δf_{i-1}, and Δf_i are not coplanar at any admissible point t_0 and integer i. Then this surface is infinitesimally flexible if and only if any 3-ribbon surface contained in the surface is infinitesimally flexible.*

Proof. The proof is straightforward. All the conditions for infinitesimal flexion are exactly the conditions for 3-ribbon surfaces of Theorem 2.7. □

For the finite flexibility we have the following.

Theorem 2.14. *Consider an n-ribbon surface satisfying the genericity condition: $\dot{f}_i$, Δf_{i-1}, and Δf_i are not coplanar at any admissible point t_0 and integer i. Then this surface is flexible if and only if any 3-ribbon surface contained in the surface is flexible.*

Remark 2.15. We think of this theorem as of a semidiscrete analogue to the statement of the paper [2] on conjugate nets and all (3×3)-meshes that they contain. In this paper we do not study phenomena related to non-compactness and hence we restrict ourselves to the case of compact n-ribbons surfaces.

Proof. The "only if" part of the statement is straightforward. We prove the converse by induction on the number of ribbons in a surface.

Induction base. By assumption any 3-ribbon subsurface contained in the surface is flexible. It has one degree of freedom, since by Theorem 1.9 any 2-ribbon subsurface of a 3-ribbon surface has at most one degree of freedom, while the genericity condition holds in a certain neighborhood of a starting position.

Induction step. Suppose we know that any k-ribbon subsurface is flexible and has one degree of freedom in some neighborhood (for $k \geq 3$).

Let us prove the statement for any $(k{+}1)$-ribbon subsurface. We consider a $(k{+}1)$-ribbon subsurface as the union of two k-ribbon subsurfaces that intersect in a $(k{-}1)$-ribbon subsurface. By the induction assumption this $(k{-}1)$-ribbon surface has one degree of freedom compatible with the flexions of both k-ribbon subsurfaces. Therefore, the flexion of the $(k{-}1)$-ribbon subsurface is uniquely extended to the both k-ribbon subsurfaces. This implies flexibility of the $(k{+}1)$-ribbon with one degree of freedom. □

3. Flexions of developable semidiscrete surfaces

Suppose that all ribbons of a semidiscrete surface are developable, i.e. the vectors $\dot{f}_i$, Δf_i, and $\dot{f}_{i+1}$ are linearly dependent. We call such semidiscrete surfaces *developable*. In this section we describe an additional property for flexions of developable semidiscrete surfaces.

Proposition 3.1. *Consider a developable 2-ribbon semidiscrete surface f. Let*

$$\Delta f_i(t) = a_i(t)\dot{f}_i(t) + b_i(t)\dot{f}_{i+1}(t)$$

for $i = 0, 1$. Then for the function $H_i(t)$ we have

$$H_i(t) = \frac{1}{b_i(t)} - \frac{1}{a_{i-1}(t)}.$$

Proof. The expression is obtained from the expression in the definition of H_i (on page 20) after the substitutions

$$\Delta f_j(t) = a_j(t)\dot{f}_j(t) + b_j(t)\dot{f}_{j+1}(t) \qquad \text{and} \qquad \Delta \dot{f}_j(t) = \dot{f}_{j+1}(t) - \dot{f}_j(t)$$

for $j = i-1,\, i$. □

This fact gives a surprising corollary concerning the flexion of a 2-ribbon developable surface. The degree of freedom for a flexion of a generic 2-ribbon developable surface is 1, as can easily be seen from the genericity condition for 2-ribbon surfaces. So a flexion is unique up to the choice of a parameter. Denote by $\alpha(t)$ the angle between Δf_0 and Δf_1.

Corollary 3.2. *Consider a flexion of a 2-ribbon developable surface f. Let us choose the parameter γ of the flexion such that $\cos(\alpha(t_0))$ changes linearly in γ. Then for any t the value $\cos(\alpha(t))$ changes linearly in γ.*

Proof. Let $\hat{f}$ be the 2-ribbon surface defined by

$$\Delta\hat{f}_0 = \frac{\Delta f_0}{|\Delta f_0|}, \qquad \hat{f}_1 = f_1, \qquad \text{and} \qquad \Delta\hat{f}_1 = \frac{\Delta f_1}{|\Delta f_1|}.$$

The 2-ribbon surface $\hat{f}$ is in some sense a normalization of a surface f, so $\hat{f}$ is developable, and the flexions for $\hat{f}$ and for f coincide.

For the surface of $\hat{f}$ we get

$$\cos\alpha = \langle \Delta\hat{f}_0, \Delta\hat{f}_1 \rangle = \Phi,$$

since $|\Delta\hat{f}_0| = 1$ and $|\Delta\hat{f}_1| = 1$. Now the statement of the corollary for $\hat{f}$ follows from Proposition 2.2 and the inner geometry expression of Proposition 3.1 for the function under integration (that is actually H_1).

Since $\hat{f}$ is a normalization of f the statement of the corollary holds for f as well. □

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References

[1] L. Bianchi, *Sopra alcone nuove classi di superficie e di sistemi tripli ortogonali*, Ann. Matem. v. 18 (1890), pp. 301–358.
[2] A. I. Bobenko, T. Hoffmann, W. K. Schief, *On the integrability of infinitesimal and finite deformations of polyhedral surfaces*, Discrete Differential Geometry, A. I. Bobenko, P. Schröder, J. M. Sullivan, G. M. Ziegler, (eds.), Series: Oberwolfach Seminars, v. 38 (2008), pp. 67–93.
[3] A. I. Bobenko, Yu. B. Suris, *Discrete differential geometry. Integrable structure*, Graduate Studies in Mathematics, 98, American Mathematical Society, Providence, RI, 2008.
[4] H. Cartan, *Calcul différentiel*, Hermann, Paris, 1967.
[5] L. P. Eisenhart, *A Treatise on the Differential Geometry of Curves and Surfaces*, Dover Publications, New York, 1960.
[6] O. Karpenkov, *On the flexibility of Kokotsakis meshes*, Geom. Dedicata, v. 147 (2010), no. 1, pp. 15–28.
[7] A. Kokotsakis, *Über bewegliche Polyeder*, Math. Ann., v. 107 (1932), pp. 627–647.
[8] Ch. Müller, J. Wallner, *Semi-discrete isothermic surfaces*, Geometry Preprint 2010/02, TU Graz, February 2010. http://www.geometrie.tugraz.at/mueller/conformal.pdf
[9] H. Pottmann, J. Wallner, *Infinitesimally flexible meshes and discrete minimal surfaces*, Monatshefte Math., v. 153 (2008), pp. 347–365.
[10] H. Pottmann, J. Wallner, *Computational line geometry. Mathematics and Visualization*, Springer-Verlag, Berlin, 2001.
[11] I. Kh. Sabitov, *Local theory of bendings of surfaces, Geometry, III*, Encyclopaedia Math. Sci., v. 48, pp. 179–256, Springer, Berlin, 1992.
[12] J. Wallner, *Semidiscrete surface representations*, In A. Bobenko et al., editors, Discrete Differential Geometry, Oberwolfach Reports. 2009. Abstracts from the workshop held January 12–17, 2009.

[13] J. Wallner, *On the semidiscrete differential geometry of A-surfaces and K-surfaces*, Geometry Preprint 2010/01, TU Graz, February 2010. http://www.geometrie.tugraz.at/wallner/asymp.pdf

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