MULTIDIMENSIONAL GAUSS REDUCTION THEORY FOR CONJUGACY CLASSES OF $SL(n, \mathbb{Z})$

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Abstract. In this paper we describe the set of conjugacy classes in the group $SL(n, \mathbb{Z})$. We expand geometric Gauss Reduction Theory that solves the problem for $SL(2, \mathbb{Z})$ to the multidimensional case. Further we find complete invariant of classes in terms of multidimensional Klein-Voronoi continued fractions, where ς-reduce Hessenberg matrices play the role of reduced matrices.

Contents

Introduction

Two matrices M_1 and M_2 in $SL(n, \mathbb{Z})$ are *integer conjugate* if there exists a matrix X in $GL(n, \mathbb{Z})$ such that

$$M_2 = XM_1X^{-1}.$$

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In this paper we study the following problem.

Problem. Describe the set of integer conjugacy classes in $SL(n, \mathbb{Z})$.

One of the mostly common strategies to solve this kind of problems is to find complete invariants to distinguish the classes, and further if possible to write normal form of conjugacy classes. For instance, in the similar problem for $SL(n, F)$ for an algebraically closed field F one has Jordan Normal Forms as a complete description of conjugacy classes. Jordan blocks form a complete invariant in this case.

If the field is not algebraically closed, the description is much more complicated via Jordan-Chevalley decomposition. In the study of $SL(n, \mathbb{Z})$ we are faced with a group instead of a field. For the general case it is only known the solution of the similarity problem on verification whether two matrices are conjugate or not (see in [1] and [18]). A complete description of the set of integer conjugacy classes in $SL(2, \mathbb{Z})$ is given by Gauss Reduction Theory (see for instance in [30] and [40]). It turns out that it is natural to consider several normal forms for an integer conjugacy class but not necessarily only one.

Currently the main approach to the study of the above problem is as follows: one should try to split $GL(n, \mathbb{Q})$-conjugacy classes into $GL(n, \mathbb{Z})$ conjugacy classes. Then the problem is reduced to certain problems related to orders of algebraic fields defined by the roots of characteristic polynomial of the corresponding matrices (like computing their class numbers, etc.). In this paper we introduce an alternative geometric approach based on generalization of Gauss Reduction Theorem. We will study questions related to three-dimensional case in more details in our forthcoming paper.

Description of the paper. In current paper we present the following main four results.

I. Matrix description of integer conjugacy classes. We consider Hessenberg matrices as a multidimensional analog of reduced matrices in Gauss Reduction Theory. Hessenberg matrices are matrices that vanish below the superdiagonal (for more information see in [50]). These matrices were essentially used in the QR-algorithm for eigenvalue problem, but they were never considered before in the frames of similarity theory. We introduce a natural notion of Hessenberg complexity for Hessenberg matrices, which is a nonnegative integer function, and show that *each integer conjugacy class of irreducible matrices has only finite number of Hessenberg matrices with minimal complexity.* This result is a combination of Theorem 1.8 and Theorem 1.9. We study all related questions in Section 1.

II. Geometric complete invariant of integer conjugacy classes. In Section 2 we introduce the complete invariant of integer conjugacy classes of $GL(n, \mathbb{Z})$ matrices. Recently in [27] we showed a geometric explanation of Gauss Reduction Theory in terms of geometric continued fractions (see also briefly in Subsection 2.1). In Sections 2 we extend this approach to the multidimensional case. We propose a geometric description of integer conjugacy classes in terms of multidimensional continued fractions in the sense of Klein-Voronoi: *the periods of such Klein-Voronoi continued fractions are complete invariants of* *integer conjugacy classes* (Theorem 2.23). In addition we study the group structure of the set of periods (Theorem 2.25).

III. Techniques to construct reduced matrices. In Section 1 we introduce a techniques to construct ς-reduced matrices integer conjugate to a given one. It is based on the following result (Theorem 3.6): *any ς-reduced matrix is obtained from an integer vertex of Klein-Voronoi continued fraction by applying to it the algorithm of Subsection 1.2.*

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1. Hessenberg matrices and conjugacy classes

In this section we study questions of reduction to ς-reduced matrices and investigate families of perfect Hessenberg matrices in general. We start with necessary definitions and notation in Subsection 1.1. In Subsection 1.3 we prove that any integer conjugacy class with irreducible characteristic polynomial has at least one ς-reduced matrix. Further in Subsection 1.4 we show that Hessenberg matrices are defined by their Hessenberg types and characteristic polynomials and deduce the finiteness of ς-reduced matrices in each integer conjugacy class. Finally in Subsection 1.5 we study the structure of the set of perfect Hessenberg matrices. We consider this set as a "book" that contains "pages" enumerated by Hessenberg type. The matrices of the same page are distinguished by characteristic polynomial, only matrices from different pages can be integer conjugate.

1.1. **Notions and definition.**

In this subsection we briefly introduce matrices that generalize the reduced matrices in Gauss Reduction Theory for $SL(2,\mathbb{Z})$.

1.1.1. *Perfect Hessenberg matrices.* A matrix M of the form

$$\begin{pmatrix} a_{1,1} & a_{1,2} & \cdots & a_{1,n-2} & a_{1,n-1} & a_{1,n} \\ a_{2,1} & a_{2,2} & \cdots & a_{2,n-2} & a_{2,n-1} & a_{2,n} \\ 0 & a_{3,2} & \cdots & a_{3,n-2} & a_{3,n-1} & a_{3,n} \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & a_{n-1,n-2} & a_{n-1,n-1} & a_{n-1,n} \\ 0 & 0 & \cdots & 0 & a_{n,n-1} & a_{n,n} \end{pmatrix}$$

is called an (*upper*) *Hessenberg* matrix. Such matrices were first studied by K. Hessenberg in [20] and further used in QR-algorithms (see [16], [51], and [45]). We say that the matrix M is of *Hessenberg type*

$$\langle a_{1,1}, a_{2,1} | a_{1,2}, a_{2,2}, a_{3,2} | \cdots | a_{1,n-1}, a_{2,n-1}, \ldots, a_{n,n-1} \rangle.$$

Definition 1.1. A Hessenberg matrix in $SL(n,\mathbb{Z})$ is said to be *perfect* if for any pair of integers (i,j) satisfying $1 \le i < j+1 \le n$ the following inequalities hold: $0 \le a_{i,j} < a_{j+1,j}$.

In other words all elements of all the columns except the last column of a perfect Hessenberg matrix are nonnegative integers, the maximal elements in these columns are the lowest nonzero ones (i.e., $a_{j+1,j}$, $j = 1, \ldots, n-1$).

1.1.2. *ς-reduced Hessenberg matrices.* We mostly study $SL(n,\mathbb{Z})$-matrices with irreducible characteristic polynomials over rational numbers. Any such matrix is integer conjugate to a perfect Hessenberg matrix with positive Hessenberg complexity (see Theorem 1.8 below). Actually, there are infinitely many perfect Hessenberg matrices integer conjugate to a given one. So we give an additional notion of complexity to reduce the number of such matrices.

Definition 1.2. The integer number

$$\prod_{j=1}^{n-1} |a_{j+1,j}|^{n-j}$$

is called the *Hessenberg complexity* of the matrix M and denoted by $\varsigma(M)$.

Hessenberg complexity has the following geometric meaning. It is equivalent to the volume of the parallelepiped spanned by e_1, $M(e_1)$, $M^2(e_1),\ldots,M^{n-1}(e_1)$, where e_1 is the first basis vector i.e. $(1,0,\ldots,0)$, we discuss this in more details in Subsection 3.1.

An integer Hessenberg matrix has the unit Hessenberg complexity if and only if $a_{2,1} = \cdots = a_{n,n-1} = 1$, such matrices are called *Frobenius* matrices. The elements of the last column of a Frobenius matrix are the coefficients of the characteristic polynomial multiplied alternatively by ± 1.

Example 1.3. The following matrix

$$\begin{pmatrix} 1 & 2 & 2 \\ 2 & 3 & 4 \\ 0 & 5 & -1 \end{pmatrix}$$

is a perfect Hessenberg matrix of type $\langle 1,2|2,3,5\rangle$. The Hessenberg complexity of this matrix is $2^2\cdot 5 = 20$.

Definition 1.4. We say that a perfect Hessenberg matrix M is *ς-reduced* if its Hessenberg complexity is the least possible. Otherwise we say that the matrix is *ς-nonreduced.*

In Theorem 1.9 we show that the number of ς-reduced matrices is finite in any integer conjugacy class. Still sometimes there are several ς-reduced perfect Hessenberg matrices integer conjugate to each other, see Example 3.7. This happens also for matrices in $SL(2,\mathbb{Z})$.

1.2. **Perfect Hessenberg matrices conjugate to a given one.** In this subsection we show the algorithm to construct a perfect Hessenberg matrices for a given $SL(n,\mathbb{Z})$-matrix. It is based on the following proposition.

Proposition 1.5. *Let M be an $SL(n,\mathbb{Z})$-matrix with irreducible over $\mathbb{Q}$ characteristic polynomial. For any integer primitive vector v (i.e., with relatively prime coordinates) there exists a unique matrix C such that*

— $C(e_1) = v$;

— the matrix CMC^{-1} is perfect Hessenberg (we denote this matrix by $(M|v)$).

Remark 1.6. This means that for any lattice preserving linear operator any primitive integer vector can be extended to the basis of integer lattice in a way such that the matrix of the operator is perfect Hessenberg in this basis.

Proof. First, we construct the corresponding perfect Hessenberg matrix. Let M be a matrix in $SL(n,\mathbb{Z})$ with irreducible characteristic polynomial, and A be a linear operator with matrix M in some fixed integer basis. Take any primitive integer vector v and consider a set of vector spaces

$$V_i = \operatorname{Span}\big(v, A(v), A^2(v), \ldots, A^{i-1}(v)\big),$$

for $i = 1, \ldots, n$ (here we denote the span of vectors $v_1, \ldots, v_m$ by $\operatorname{Span}(v_1, \ldots, v_m)$). Since the characteristic polynomial of A is irreducible, the dimension of V_i equals i and the set of all spaces V_i forms a complete flag in $\mathbb{R}^n$. Since for any integer j the vector $A^j(v)$ is integer, the spaces V_i contain an integer sublattice of rank i.

Let us inductively construct an integer basis $\{e_i\}$ of a vector space $\mathbb{R}^n$ such that:
— for $i = 1, \ldots, n$, the vectors $e_1, \ldots e_i$ form a basis of the integer sublattice $\mathbb{Z}^n \cap V_i$;
— the matrix of the operator A is perfect Hessenberg in this basis.

Base of induction. We put $e_1 = v$. It is clear that the vector e_1 generate the one-dimensional integer sublattice of V_1.

Step of induction. Suppose we have constructed e_i for all $i \le k$, such that $e_1, \ldots, e_k$ is a basis of integer sublattice contained in V_i. Let us find e_{k+1}.

By construction, $e_1, \ldots, e_k$ generate the integer sublattice $\mathbb{Z}^n \cap V_k$. Hence there exists an integer vector g_{k+1} such that $e_1, \ldots, e_k, g_{k+1}$ generate the integer sublattice $\mathbb{Z}^n \cap V_{k+1}$. Actually g_{k+1} is one of the integer primitive vectors of V_{k+1} with the smallest possible nonzero distance to the space V_k (all such vectors are contained in two hyperplanes parallel to V_k).

Since $A(e_k)$ is contained in V_{k+1}, it is decomposable in the basis $e_1, \ldots, e_k, g_{k+1}$ with integer coefficients:

$$A(e_k) = \sum_{i=1}^{k} q_{i,k} e_i + a_{k+1,k} g_{k+1}.$$

For $i = 1, \ldots, k$ we define $b_{i,k}$, and $a_{i,k}$ as integer quotients and reminders:

$$q_{i,k} = b_{i,k} \cdot |a_{k+1,k}| + a_{i,k},$$

where $0 \le a_{i,k} < a_{k+1,k}$. Rewrite

$$A(e_k) = |a_{k+1,k}| \left(\operatorname{sign}(a_{k+1,k}) g_{k+1} + \sum_{i=1}^{k} b_{i,k} e_i \right) + \sum_{i=1}^{k} a_{i,k} e_i.$$

satisfying $0 \le a_{i,k} < a_{k+1,k}$ for $i = 1, \ldots, k$. Finally we put

$$e_{k+1} = \operatorname{sign}(a_{k+1,k}) g_{k+1} + \sum_{i=1}^{k} b_{i,k} e_i.$$

Since the characteristic polynomial of A is irreducible, the integer spaces V_i are not invariant subspaces of A, hence $e_1, \ldots, e_{k+1}$ are linearly independent and generate V_{k+1}.

The matrix $\hat{M}$ of the operator A in the basis $\{e_i\}$ is of Hessenberg type

$$\Big\langle a_{1,1}, |a_{2,1}| \Big| a_{1,2}, a_{2,2}, |a_{3,2}| \Big| \cdots \Big| a_{1,n-1}, \ldots, a_{n-1,n-1}, |a_{n,n-1}| \Big\rangle.$$

By the definition, $\hat{M}$ is a perfect Hessenberg matrix. The matrices M and $\hat{M}$ represent the same operator A in two different integer bases, hence M and $\hat{M}$ are integer conjugate.

Since the characteristic polynomial of A is irreducible, the integer spaces V_i are not invariant subspaces of A. Hence, the integers $a_{i+1,i}$ are nonzero for $i = 1, \ldots, n-1$. Therefore, the Hessenberg complexity of $\hat{M}$ is positive.

Denote by C the transition matrix to the basis $\{e_i\}$. Then we have $C(e_1) = v$ and $\hat{M} = CMC^{-1}$ is perfect Hessenberg.

Finally, we say a few words about uniqueness of $\hat{M}$. The spaces V_i are uniquely defined. The vector e_1 is uniquely defined. On each step there is a unique way to define e_{k+1}. Hence the transition matrix C is uniquely defined. Therefore, such matrix $\hat{M}$ is unique. □

Let us briefly outline the algorithm used in the proof of Proposition 1.5.

Algorithm to construct perfect Hessenberg matrices.

Input Data. We are given by a matrix M of a lattice preserving operator A with irreducible characteristic polynomial and an integer vector v.

Step 1. We put $e_1 = v$.

Inductive Step k. Suppose we have constructed e_i for all $i \leq k$. For g_{k+1} we take one of the integer primitive vectors of V_{k+1} with the smallest possible nonzero distance to the space V_k. Find the coordinates $q_{i,k}$ for $i = 1, \ldots, k$ and $a_{k+1,k}$ from the decomposition

$$A(e_k) = \sum_{i=1}^{k} q_{i,k} e_i + a_{k+1,k} g_{k+1}.$$

For $i = 1, \ldots, k$ find $b_{i,k}$, and $a_{i,k}$ as integer quotients and reminders:

$$q_{i,k} = b_{i,k} \cdot |a_{k+1,k}| + a_{i,k}.$$

Then we have

$$e_{k+1} = \operatorname{sign}(a_{k+1,k}) g_{k+1} + \sum_{i=1}^{k} b_{i,k} e_i.$$

Finally, let C be a transition matrix to the basis $\{e_k\}$.

Output Data. In the output we have the perfect Hessenberg matrix CMC^{-1}.

We use the following corollary in the proof of Theorem 3.6 below.

Corollary 1.7. *Consider an $SL(n, \mathbb{Z})$-operators A with matrix M. Let B be an arbitrary $GL(n, \mathbb{Z})$-operator commuting with A. Then for an arbitrary v we have*

$$(M|v) = (M|B(v)).$$

Proof. Since A commutes with B, each step of the above algorithm is invariant produces the same data for both v and $B(v)$. Hence $(M|v) = (M|B(v))$. □

1.3. **Reduction to ς-reduced Hessenberg matrices.** In this subsection we show the existence of ς-reduced Hessenberg matrices integer conjugate to a given one. We show how to find it explicitly in Subsection 3.3 after we introduce Klein-Voronoi continued fractions.

Theorem 1.8. *For any matrix M in $SL(n,\mathbb{Z})$ with irreducible characteristic polynomial there exists a ς-reduced Hessenberg matrix $\tilde{M}$ with positive Hessenberg complexity such that M is integer conjugate to $\tilde{M}$.*

Proof. By Proposition 1.2 there exists at least one perfect Hessenberg matrix integer conjugate to M. Since the set of values of Hessenberg complexity is discrete and bounded from below, there exists a perfect Hessenberg matrix $\tilde{M}$ integer conjugate to M and with minimal possible Hessenberg complexity. By definition $\tilde{M}$ is a ς-reduced Hessenberg matrix. □

1.4. **Finiteness of ς-reduced Hessenberg matrices.** In this subsection we prove the following theorem.

Theorem 1.9. *For any $SL(n,\mathbb{Z})$-matrix M with irreducible characteristic polynomial there exists finitely many ς-reduced Hessenberg matrices integer conjugate to M.*

In the proof of this theorem we use the following general proposition.

Proposition 1.10. *Any Hessenberg matrix with positive Hessenberg complexity is uniquely defined by its Hessenberg type and the characteristic polynomial.* □

Proof. Consider a Hessenberg matrix $M = (a_{i,j})$ of a given Hessenberg type with positive Hessenberg complexity. From the Hessenberg type of M we know all its columns except for the last one. Let us show that the last column in is uniquely defined by the coefficients of its characteristic polynomial. Let the characteristic polynomial of M be

$$x^n + c_{n-1}x^{n-1} + \cdots + c_1 x + c_0.$$

Direct calculations show that for any k the coefficient c_k is a polynomial in $a_{i,j}$ variables that does not depend on $a_{1,n}, \ldots, a_{k,n}$. The unique monomial for c_k containing $a_{k+1,n}$ is

$$\left(\prod_{j=k+1}^{n-1} a_{j+1,j}\right) a_{k+1,n}.$$

Since the Hessenberg complexity of M is nonzero, the product in the brackets is nonzero. Hence $a_{k+1,n}$ is a function of c_k and the elements $a_{i,j}$ contained in the first $n-1$ columns. This concludes the proof of the proposition. □

The following example shows that simply Hessenberg complexity together with characteristic polynomial do not distinguish all the integer conjugacy classes.

Example 1.11. The following two matrices

$$\begin{pmatrix} 0 & 1 & 3 \\ 1 & 0 & 0 \\ 0 & 3 & 8 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} 0 & 2 & 5 \\ 1 & 1 & 2 \\ 0 & 3 & 7 \end{pmatrix}$$

are not integer conjugate but have the same Hessenberg complexity equal to 3 and the same characteristic polynomials.

Proof of Theorem 1.9. The existence of ς-reduced Hessenberg matrices integer conjugate to M follows from Theorem 1.8. By definition they all have the same Hessenberg complexity (say, c). The number of Hessenberg types whose Hessenberg complexity equals c is finite. It is clear that the integer conjugate matrices have the same characteristic polynomial, hence by Proposition 1.10 there exists at most one Hessenberg matrix of a given Hessenberg type integer conjugate to M. Therefore, there is only a finite number of ς-reduced Hessenberg matrices integer conjugate to M. □

1.5. **Families of Hessenberg matrices with given Hessenberg type.** Denote by $H(\Omega)$ the set of all Hessenberg matrices in $SL(n,\mathbb{Z})$ of Hessenberg type Ω.

For an arbitrary Hessenberg type

$$\Omega = \langle a_{1,1}, a_{1,2} | a_{2,1}, a_{2,2}, a_{2,3} | \cdots | a_{n-1,1}, \ldots, a_{n-1,2}, a_{n-1,n} \rangle$$

and $k = 1, \ldots, n{-}1$ we denote by $v_k(\Omega)$ the vector $(a_{k,1}, \ldots, a_{k,k+1}, 0, \ldots, 0)$, and by $M_k(\Omega)$ — the matrix with zero first $n{-}1$ columns and the last one equals to $v_k(\Omega)$.

Denote by $\sigma(\Omega)$ the $(n{-}1)$-dimensional simplex with vertices O, $O{+}v_1, \ldots, O{+}v_{n-1}$ where O is the origin.

Definition 1.12. The *integer volume* of a simplex σ with integer vertices is the index of the sublattice generated by the edges of σ in the lattice of all integer vectors in the plane spanned by σ.

Theorem 1.13. *Let Ω be a Hessenberg type.*

i). The set $H(\Omega)$ is not empty if and only if the integer volume of $\sigma(\Omega)$ equals one.

ii). Suppose that $M_0 \in H(\Omega)$, then $H(\Omega)$ is an integer affine $(n{-}1)$-dimensional sublattice in the lattice of all integer $(n \times n)$-matrices. More precisely,

$$H(\Omega) = \left\{ M_0 + \sum_{i=1}^{n-1} c_i M_i(\Omega) \middle| c_1, \ldots, c_{n-1} \in \mathbb{Z} \right\}.$$

The proof of Theorem 1.13 is based on Lemma 1.15 which we show after the following definition.

Definition 1.14. Consider an integer vector v and a k-dimensional plane π containing the integer sublattice of rank k such that v is not in π. The *integer distance* from v to π is the index of the sublattice generated by the integer vectors of the set $\pi \cup \{u\}$ in the whole integer lattice of the $(k{+}1)$-dimensional plane spanning v and π.

Lemma 1.15. *Consider a Hessenberg matrix M of type Ω, let its last column be an integer vector vector v. The matrix M is in $SL(n,\mathbb{Z})$ if and only if the following conditions hold:*
— the integer volume of $\sigma(\Omega)$ equals one;
— the integer distance from the vector v to the integer hyperplane containing $\sigma(\Omega)$ equals one.

Proof. Let A be an operator with Hessenberg matrix M in the basis $\{e_i\}$ of integer lattice.

Suppose that M is in $SL(n,\mathbb{Z})$, then the operator A preserves all integer volumes and integer distances. Since the integer volume of the coordinate $(n-1)$-dimensional simplex S_e^{n-1} with vertices

$$O, O+e_1, \ldots, O+e_{n-1}$$

equals one, the integer volume of the image $\sigma(\Omega) = A(S_e^{n-1})$ equals one. Notice that

$$A(\mathrm{Span}(S_e^{n-1})) = \mathrm{Span}(\sigma(\Omega)) \qquad \text{and} \qquad A(e_n) = v.$$

Since the integer distance from the point $O+e_n$ to the plane spanned by the vectors $e_1, \ldots, e_{n-1}$ equals one, the integer distance from the point $O+v$ to the integer hyperplane containing $\sigma(\Omega)$ also equals one.

Suppose now that both conditions of the lemma hold. Then the operator A takes the integer lattice (generated by $e_1, \ldots, e_n$) to itself bijectively. Therefore, M is in $SL(n,\mathbb{Z})$. □

Proof of Theorem 1.13. (i) Suppose the integer volume of $\sigma(\Omega)$ equals one. Then we choose v to be at unite integer distance to the plane $\mathrm{Span}(\sigma(\Omega))$. Then by Lemma 1.15 we get the matrix. Conversely if $H(\Omega)$ contains an $SL(n,\mathbb{Z})$-matrix, then by Lemma 1.15 the integer volume of $\sigma(\Omega)$ equals 1.

Statement (ii) is straightforward, since the determinant of the matrix is additive with respect to the operation of addition of vectors in the last column. □

We conclude this subsection with a particular example.

Example 1.16. Let us consider matrices of Hessenberg type $\langle 0,1|1,0,2\rangle$. All matrices of that type form a two-parametric family

$$H(\langle 0,1|1,0,2\rangle) = \left\{ \begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 0 \\ 0 & 2 & 1 \end{pmatrix} + m \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix} + n \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 2 \end{pmatrix} \middle| m, n \in \mathbb{Z} \right\}.$$

We denote

$$H^{(1,0,1)}_{\langle 0,1|1,0,2\rangle}(m,n) = \begin{pmatrix} 0 & 1 & n+1 \\ 1 & 0 & m \\ 0 & 2 & 2n+1 \end{pmatrix}$$

with integer parameters m and n. The discriminant of the matrix $H^{(1,0,1)}_{\langle 0,1|1,0,2\rangle}(m,n)$ equals

$$-44 - 44n^2 - 56mn - 32n^3 + 32m^3 + 16m^2n^2 + 16mn^2 + 16m^2n - 56n - 8m + 52m^2.$$

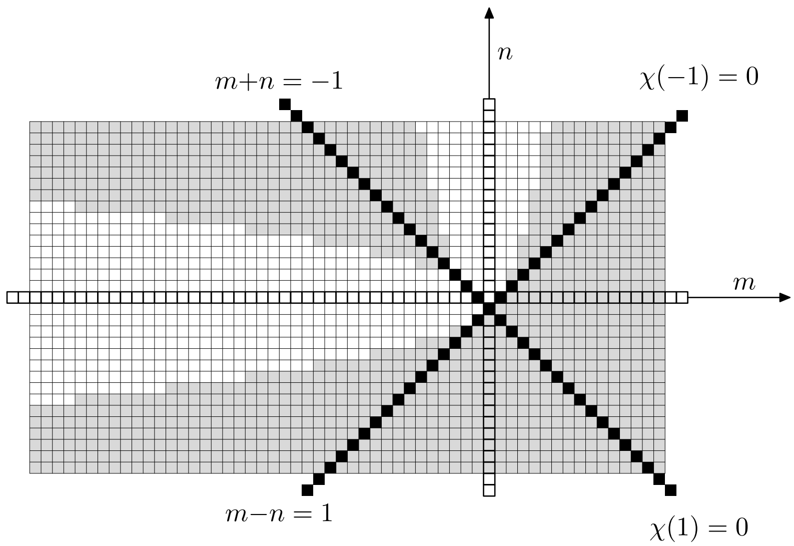


FIGURE 1. The family of matrices of Hessenberg type $\langle 0,1|1,0,2\rangle$.

The set of matrices with negative discriminant for the given family coincides with the union of integer solutions of the following inequalities:

$$2m \le -n^2 - n - 2 \quad \text{and} \quad 2n \le m^2 + m.$$

In Figure 1 the square in the intersection of the m-th column and the n-th row corresponds to the matrix $H^{(1,0,1)}_{\langle 0,1|1,0,2\rangle}(m,n)$. Black squares correspond to the matrices with reducible characteristic polynomials. Light gray squares correspond to the matrices with three real eigenvalues. The rest have a pair of complex conjugate eigenvalues.

2. Complete geometric invariant of conjugacy classes

In this section we introduce a geometric complete invariant of integer conjugacy classes: multidimensional continued fractions in the sense of Klein-Voronoi. We start with the classical two-dimensional case corresponding to Gauss Reduction Theory in Subsection 2.1, where we show a relation between integer conjugacy classes of matrices and certain continued geometric continued fractions. We give general definitions of Klein-Voronoi continued fractions in all dimensions in Subsection 2.2.2. Klein-Voronoi continued fractions for matrices of $SL(n,\mathbb{Z})$ possess additional combinatorial periodicity, we discuss it in Subsection 2.2.3. Finally in Subsection 2.3 we show that Klein-Voronoi continued fractions classify integer conjugacy classes of $SL(n,\mathbb{Z})$-matrices.

2.1. **Geometry of Gauss Reduction Theory.** In this subsection we briefly discuss geometry of two-dimensional case. We skip all the proofs in this subsection, for a more detailed exposition with all proofs we refer to [27] and [24].

It is usual to split $SL(2,\mathbb{Z})$ in the following three cases.

Complex case: a characteristic polynomial of such matrices has two complex conjugate roots. There are only three classes of such matrices, they are represented by

$$\begin{pmatrix} 1 & 1 \\ -1 & 0 \end{pmatrix}, \quad \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad \text{and} \quad \begin{pmatrix} 0 & 1 \\ -1 & -1 \end{pmatrix}.$$

Degenerate case: the characteristic polynomial has a double root (that is actually equal to 1). Such matrices are integer conjugate to exactly one of the following family

$$\begin{pmatrix} 1 & n \\ 0 & 1 \end{pmatrix} \quad \text{for } n \geq 0.$$

Totally real case: the case of two real roots is the most complicated. The geometric description is given via continued fractions. So we give necessary definitions at first.

2.1.1. *Ordinary continued fractions.* The expression (finite or infinite)

$$a_0 + 1/(a_1 + 1/(a_2 + \ldots)\ldots))$$

is an *ordinary continued fraction* if $a_0 \in \mathbb{Z}$, $a_k \in \mathbb{Z}_+$ for $k > 0$. Denote it by $[a_0 : a_1; \ldots]$ (or by $[a_0 : a_1; \ldots; a_n]$).

An ordinary continued fraction is *odd* (*even*) if it has an odd (even) number of elements. Any rational number has a unique odd and even ordinary continued fractions:

$$\frac{7}{5} = 1 + \frac{1}{2 + \frac{1}{2}} = 1 + \frac{1}{2 + \frac{1}{1 + 1/1}}.$$

The odd and even continued fractions of the same number coincide except for the very last elements, as in example with 7/5:

$$[a_0 : a_1; \ldots; a_n] = [a_0 : a_1; \ldots; a_n{-}1 : 1].$$

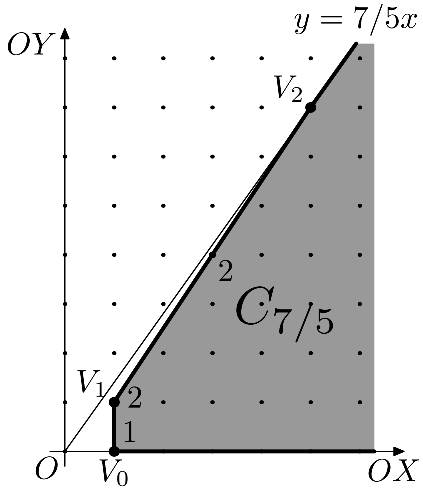
Any irrational number has a unique infinite ordinary continued fraction.

2.1.2. *Integer geometry notation.* Let us briefly recall some notions of integer geometry. A point is *integer* if all its coordinates are integers. A segment or a vector is integer if it has integer endpoints. An angle is *integer* if its vertex is integer and its edges contain integer points distinct to the vertex.

Definition 2.1. The *integer length* of an integer segment AB is the number of inner integer points in the segment plus one, we denote it by $\mathrm{I}\ell(AB)$.
The *integer sine* of an integer angle ABC is the index of the sublattice generated by integer vectors of the edges of the angle in the lattice of integer points, we denote it by $\mathrm{lsin}(ABC)$.

For additional information on lattice trigonometry we refer to [24] and [26].

Definition 2.2. Consider an arbitrary angle C with vertex at the origin. The boundary of the convex hull of all integer points in C except for the origin is called the *sail* for C.



$$\frac{7}{5} = 1 + \cfrac{1}{2 + \cfrac{1}{2}}$$

FIGURE 2. Geometry of the continued fraction for $7/5 = [1 : 2; 2]$.

In general a sail is a broken line that contains a finite or infinite number of vertices.

Definition 2.3. Consider an arbitrary angle with integer vertex. Let the sail for this angle be a broken line with the sequence of vertices (V_i). Denote:

$$\begin{aligned} a_{2k} &= \mathrm{I}\ell\, V_k V_{k+1}, \\ a_{2k-1} &= \mathrm{lsin}\, V_{k-1} V_k V_{k+1} \end{aligned}$$

for all admissible indices. The *lattice length-sine sequence* (*LLS-sequence*, for short) for the sail is the sequence (a_n).

2.1.3. *Geometry of ordinary continued fractions.* An odd or infinite continued fraction of any real number $\alpha \geq 1$ has the following geometric interpretation. Consider the angle in the first orthant defined by two rays $y = \alpha x$ and $y = 0$, we denote it by C_α. Let also the first vertex of the sail for C_α be in the ray $y = 0$ (actually it is the point $(1, 0)$).

Theorem 2.4. *Consider a real number $\alpha \geq 1$. Let $(a_0, \ldots, a_{2n})$ (or $(a_0, a_1, \ldots)$) be the LLS-sequence of C_α. Then*

$$\alpha = [a_0 : a_1; \ldots; a_{2n}] \qquad (\alpha = [a_0 : a_1; \ldots]).$$

□

We refer to [28] for the geometry of continued fractions in a more general situation.

Example 2.5. On Figure 2 we show the example of the sail for $C_{7/5}$. The boundary convex hull consists of two rays and two segments. It contains three vertices V_0, V_1, and V_2. Direct calculations show that

$$\mathrm{I}\ell(V_0V_1) = 1; \quad \mathrm{lsin}(V_0V_1V_2) = 2; \quad \text{and} \quad \mathrm{I}\ell(V_1V_2) = 2.$$

So we have $7/5 = [1 : 2; 2]$.

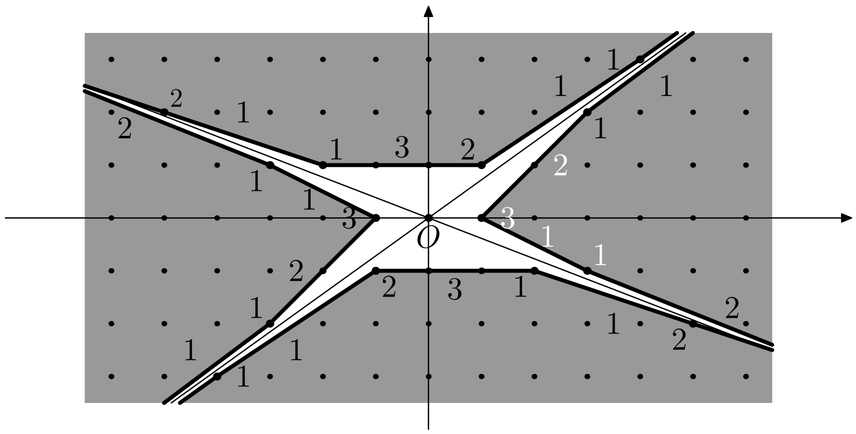


FIGURE 3. The periodic sails of a matrix $\begin{pmatrix} 7 & 18 \\ 5 & 13 \end{pmatrix}$ are periodic. One of its periods is $(1,1,3,2)$.

2.1.4. *Continued fractions for 2-dimensional operators with real eigenvectors.* Let us expand a notion of continued fractions in the following way. Consider an arbitrary $GL(2,\mathbb{R})$-operator with two real distinct eigenvalues. This operator has exactly two eigenlines. The complement to these eigenlines consists of four angles. The boundary of the convex hull of all integer points except the origin inside any of these angles is called a *sail* of the matrix. The set of all four sails is called the *geometric continued fraction*.

Let A be an $SL(2,\mathbb{Z})$-operator with two real distinct eigenvector. Then all four sails of the operators are two-side infinite broken lines. Moreover all four LLS-sequences coincide up to a shift and/or reversing the order. An operator A acts on its sails as a shift, therefore, all the LLS-sequences of A are periodic and the period is defined by the shift.

Example 2.6. On Figure 3 we show an example of a geometric continued fraction for the matrix

$$\begin{pmatrix} 7 & 18 \\ 5 & 13 \end{pmatrix}.$$

The LLS-sequences of all four sails are periodic. Their periods are either $(1,1,3,2)$ or $(2,3,1,1)$ counterclockwise.

2.1.5. *Totally real case.* Note that, there are two subfamilies of totally real $SL(2,\mathbb{Z})$-operators: with positive eigenvectors and with negative eigenvalues. The composition with an operator of symmetry about the origin (i.e. $-Id$) gives a one-to-one correspondence between the sets of all operators with positive and negative eigenvalues. Hence we restrict ourselves to the case of operators with positive eigenvalues.

Definition 2.7. A matrix $\begin{pmatrix} a & c \\ b & d \end{pmatrix}$ in $SL(2,\mathbb{Z})$ is *reduced* if $d > b \geq a \geq 0$.

It is interesting that the LLS-sequence of a reduced operator can be written directly from the coefficients of the matrix.

Theorem 2.8. *Consider a reduced matrix* $M = \left(\begin{smallmatrix} a & c \\ b & d \end{smallmatrix}\right)$. *Suppose* $\frac{b}{a} = [a_1; a_2 : \ldots : a_{2n-1}]$ *and* $\lambda = \lfloor \frac{d-1}{b} \rfloor$ *then*

$$(a_1, a_2, \ldots, a_{2n-1}, \lambda)$$

is one of the periods of the LLS-sequences for the geometric continued fraction of M. □

We say that the reduced matrices are "almost" normal forms, since each matrix could have more than one normal form. Their number is described via LLS-sequence is as follows.

Theorem 2.9. (On almost normal forms.) *The number of reduced matrices in an integer conjugacy class with minimal period* $(a_1, \ldots, a_k)$ *of the corresponding LLS-sequence is* k. □

Remark. Let us say a few words about a relation between reduced and ς-reduced matrices in $SL(2, \mathbb{Z})$. From one hand any reduced totally real matrix is a perfect Hessenberg matrix. From the other hand any ς-reduced matrix is also reduced, although there are certain reduced matrices that are not ς-reduced.

The LLS-sequence itself is the complete invariant of the set of all integer conjugacy classes in $SL(2, \mathbb{Z})$.

Theorem 2.10. (On complete invariant of integer conjugacy classes.) *An even period of the LLS-sequence (up to an even shift and reversing the order) is a complete invariant of an integer conjugacy class of a* $SL(2, \mathbb{Z})$*-matrices with distinct positive real eigenvalues.* □

Notice that all odd periods correspond to matrices with negative determinant.

Example 2.11. For the matrix

$$M = \begin{pmatrix} 1519 & 1164 \\ -1964 & -1505 \end{pmatrix}$$

the period is $(1, 2, 1, 2)$. So there are two reduced matrices: with the period $(1, 2, 1, 2)$ and $(2, 1, 2, 1)$. The coefficients a and b for the matrices are then respectively as follows

$$\frac{b}{a} = [1; 2 : 1] = \frac{4}{3} \quad \text{or} \quad \frac{b}{a} = [2; 1 : 2] = \frac{8}{3}.$$

Now it is not hard to find the elements c and d of the reduced matrices from conditions $\lambda = \lfloor \frac{d-1}{b} \rfloor$ and $ad - bc = 1$. Finally we get all two reduced matrices integer conjugate to M. They are as follows:

$$\begin{pmatrix} 3 & 8 \\ 4 & 11 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} 3 & 4 \\ 8 & 11 \end{pmatrix}.$$

The complexities of these matrices are 4 and 8 respectively. So the first matrix is ς-reduced.

2.1.6. *Matrices with the same Hessenberg type, their geometric continued fractions.* It turns out that matrices of the same Hessenberg type has similar geometric background. We start with the following example.

Example 2.12. Consider the family of matrices of Hessenberg type $\langle 3,4\rangle$:

$$\begin{pmatrix} 3 & 3m+2 \\ 4 & 4m+3 \end{pmatrix}.$$

For $m=-1,-2$ the corresponding operators does not have real eigenvectors. For $m\ge 0$ the periods of the corresponding geometric continued fractions are

$$[2,2], \quad [1,2,1,1], \quad [1,2,1,2], \quad [1,2,1,3], \quad \ldots, \quad [1,2,1,m], \quad \ldots$$

The periods in cases $m=0$ and $m=2$ are twice the minimal. This means that the corresponding matrices are squares of some other integer matrices. For $m=0$ it is a square of some integer matrix with negative determinant, since the minimal period is odd, and for $m=2$ it is a square of some $SL(2,\mathbb{Z})$-matrix with negative determinant. The matrices are ς-reduced for $m\ge 2$.

For $m\le -3$ the periods of the corresponding geometric continued fractions are

$$[4,1], \quad [4,2], \quad [4,3], \quad [4,4], \quad \ldots, \quad [4,-2-m], \quad \ldots$$

For $m=-6$ the matrix is a square of some $GL(2,\mathbb{Z})$-matrix. Starting from $m\le -6$ the matrices are ς-reduced.

So almost all matrices of Hessenberg type $\langle 3,4\rangle$ are ς-reduced. This is actually the case for all Hessenberg types in $SL(2,\mathbb{Z})$.

Theorem 2.13. *Almost all matrices of a given Hessenberg type in $SL(2,\mathbb{Z})$ are ς-reduced.* □

This follows from general Theorem 3.6 below and direct calculation of complexities for all vertices of the period, we skip the proof here.

For further information about the two-dimensional case we refer the reader, for instance, to the works [27], [39] and [40].

2.2. Continued fractions in the sense of Klein-Voronoi.

2.2.1. *Background.* In 1839 C. Hermite [19] posed the problem of generalizing ordinary continued fractions to the higher-dimensional case. Since then there were many different definitions generalizing different properties of ordinary continued fractions. A nice geometrical generalization of ordinary continued fraction for operators with all real eigenvalues was made by F. Klein in [31] and [32].

Multidimensional continued fractions in the sense of Klein have many relations with other branches of mathematics. For example, O. N. German [17] and J.-O. Moussafir [42] discussed the connection between the sails of multidimensional continued fractions and Hilbert bases. In [52] H. Tsuchihashi described the relationship between periodic multidimensional continued fractions and multidimensional cusp singularities. M. L. Kontsevich and Yu. M. Suhov studied the statistical properties of random multidimensional continued fractions in [33]. The relations to approximation theory of maximal commutative subgroups is discussed by A. Vershik and the author in [29]. The combinatorial topological generalization of Lagrange theorem was obtained by E. I. Korkina in [34] and its algebraic generalization by G. Lachaud [37]. The book [5] of V. I. Arnold is a good survey of geometric problems and theorems associated with one-dimensional and multidimensional continued fractions in the sense of Klein (see also his articles [2], [3], and [4]).

Approximately at the time of the works by F. Klein G. Voronoi in his dissertation [53] introduced a geometric algorithmic definition for all the cases even for operators with pairs of complex conjugate eigenvalues. In [9] and [10] J. A. Buchmann generalized Voronoi's algorithm making it more convenient for computation of fundamental units in orders. We use ideas of J. A. Buchmann to define the *multidimensional continued fraction in the sense of Klein-Voronoi* for all the cases. Note that if all the eigenvalues of an operator are real numbers then the Klein-Voronoi multidimensional continued fraction is a continued fractions in the sense of Klein.

2.2.2. *General definitions.* Consider an operator A in $GL(n,\mathbb{R})$ with distinct eigenvalues. Suppose that it has k real eigenvalues $r_1,\ldots,r_k$ and $2l$ complex conjugate eigenvalues $c_1,\overline{c}_1,\ldots,c_l,\overline{c}_l$, where $k+2l=n$.

Denote by $T^l(A)$ the set of all real operators commuting with A such that their real eigenvalues are all unit and the absolute values for all complex eigenvalues equal one. Actually, $T^l(A)$ is an abelian group with operation of matrix multiplication.

For a vector v in $\mathbb{R}^n$ we denote by $T_A(v)$ the orbit of v with respect of the action of the group of operators $T^l(A)$. If v is in general position with respect to the operator A (i.e. it does not lie in invariant planes of A), then $T_A(v)$ is homeomorphic to the l-dimensional torus. For a vector of an invariant plane of A the orbit $T_A(v)$ is also homeomorphic to a torus of positive dimension not greater than l, or to a point.

Example 2.14. Suppose that A is a totally real operator. Since all its eigenvectors are real, $T^0(A)$ consists only of the unit operator and $T_A(v)=\{v\}$.

Example 2.15. Now consider an operator A with a pair of complex eigenvalues whose all the other eigenvalues are real. The group $T^1(A)$ correspond to elliptic rotations in the invariant plane of A corresponding to complex eigenvalues. Such rotations are parameterized by an angle of rotation. A general orbit of $T_A(v)$ is an ellipse around the $(n-2)$-dimensional invariant subspace corresponding to real eigenvalues. Any orbit in the invariant subspace of real eigenvalues consists of one point.

Let g_i be a real eigenvector with eigenvalue r_i for $i=1,\ldots,k$; g_{k+2j-1} and g_{k+2j} be vectors corresponding to the real and imaginary parts of some complex eigenvector with eigenvalue c_j for $j=1,\ldots,l$. Consider the coordinate system corresponding to the basis $\{g_i\}$:

$$OX_1X_2\ldots X_kY_1Z_1Y_2Z_2\ldots Y_lZ_l.$$

Denote by π the $(k+l)$-dimensional plane $OX_1X_2\ldots X_kY_1Y_2\ldots Y_l$. Let π_+ be the cone in the plane π defined by the equations $y_i\geq 0$ for $i=1,\ldots,l$. For any v the orbit $T_A(v)$ intersects the cone π_+ in a unique point.

Definition 2.16. A point p in the cone π_+ is said to be *π-integer* if the orbit $T_A(p)$ contains at least one integer point.

Consider all (real) hyperplanes invariant under the action of the operator A. There are exactly k such hyperplanes. In the above coordinates the i-th of them is defined by the equation $x_i = 0$. The complement to the union of all invariant hyperplanes in the cone π_+ consists of 2^k arcwise connected components. Consider one of them.

Definition 2.17. The convex hull of all π-integer points except the origin contained in the given arcwise connected component is called a *factor-sail* of the operator A. The set of all factor-sails is said to be the *factor-continued fraction* for the operator A.
The union of all orbits $T_A(*)$ in $\mathbb{R}^n$ represented by the points in the factor-sail is called the *sail* of the operator A. The set of all sails is said to be the *continued fraction* for the operator A in the sense of Klein-Voronoi (see in Figure 4 below).

The intersection of the factor-sail with a hyperplane in π is said to be an m-dimensional face of the factor-sail if it is homeomorphic to the m-dimensional disc.
The union of all orbits in $\mathbb{R}^n$ represented by points in some face of the factor-sail is called the *orbit-face* of the operator A.
Integer points of the sail are said to be *vertices* of this sail.

2.2.3. *Algebraic continued fractions.* Consider now an operator A in the group $GL(n,\mathbb{Z})$ with irreducible characteristic polynomial. Suppose that it has k real roots $r_1, \ldots, r_k$ and $2l$ complex conjugate roots: $c_1, \overline{c}_1, \ldots, c_l, \overline{c}_l$, where $k + 2l = n$. In the simplest possible cases $k{+}l = 1$ any factor-sail of A is a point. If $k{+}l > 1$, than any factor-sail of A is an infinite polyhedral surface homeomorphic to $\mathbb{R}^{k+l-1}$.

Definition 2.18. The group of all $GL(n,\mathbb{Z})$-operators commuting with A is called the *Dirichlet group* and denoted by $\Xi(A)$.
The subgroup of the Dirichlet group $\Xi(A)$ consisting of all matrices whose real eigenvalues are all positive is called the *positive Dirichlet group*. We denote it by $\Xi_+(A)$.

The Dirichlet group $\Xi(A)$ takes the Klein-Voronoi continued fraction to itself but maybe exchange the sails. The positive Dirichlet group $\Xi_+(A)$ consists exactly from operators preserving all the sails. By Dirichlet unit theorem (see, for instance, in [6]) the group $\Xi(A)$ is homomorphic to $\mathbb{Z}^{k+l-1} \oplus G$, where G is some finite commutative group. The group $\Xi_+(A)$ is homeomorphic to $\mathbb{Z}^{k+l-1}$ and its action on any sail is free. The quotient of a sail by the action of $\Xi_+(A)$ is homeomorphic to the $(n{-}1)$-dimensional torus.

Definition 2.19. A *fundamental domain of the Klein-Voronoi continued fraction* is a collection of open orbit-faces such that for any $\Xi(A)$-orbit of orbit-faces of the continued fraction there exists a unique representative in the collection.
A *fundamental domain of a sail* is a collection of open orbit-faces such that for any $\Xi_+(A)$-orbit of orbit-faces of the sail there exists a unique representative in the collection.

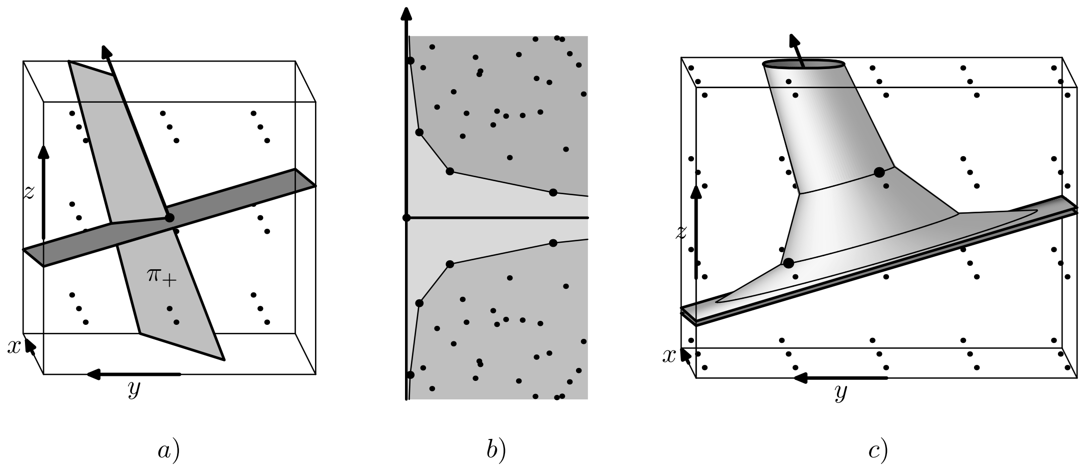


FIGURE 4. A tree-dimensional example: a) the cone π_+ and the eigenplane; b) the continued factor-fraction; c) a sail of the continued fraction.

Example 2.20. Let us study an operator A with a Frobenius matrix

$$\begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 1 \\ 0 & 1 & 3 \end{pmatrix}.$$

This operator has one real and two complex conjugate eigenvalues. Therefore, the cone π_+ for A is a two-dimensional half-plane. In Figure 4a the halfplane π_+ is colored in light gray and the invariant plane corresponding to the pair of complex eigenvectors is in dark gray. The vector shown in Figure 4a with endpoint at the origin is an eigenvector of A.

In Figure 4b we show the cone π_+. The invariant plane separates π_+ onto two parts. The dots on π_+ are the π-integer points. The boundaries of the convex hulls in each part of π_+ are two factor-sails. Actually, one factor-sail is taken to another by the induced action of $-Id$, where Id is an identity operator of $\mathbb{R}^3$.

Finally, in Figure 4c we show one of the sails. Three orbit-vertices shown in the figure correspond to the vectors $(1,0,0)$, $(0,1,0)$, and $(0,0,1)$: the large dark points $(0,1,0)$ and $(0,0,1)$ are visible on the corresponding orbit-vertices.

The positive Dirichlet group $\Xi_+(A)$ in our example is homeomorphic to $\mathbb{Z}$, it is generated by A. The group $\Xi(A)$ is homeomorphic to $\mathbb{Z}\oplus\mathbb{Z}/2\mathbb{Z}$ with generators A and $-Id$. The operator A takes the point $(1,0,0)$ and its orbit-vertex to the point $(0,1,0)$ and the corresponding orbit-vertex. Therefore, a fundamental domain of the continued fraction for the operator A contains one orbit-vertex and one vertex edge. For instance, we can choose the orbit-vertex corresponding to the point $(1,0,0)$ and the orbit-edge corresponding to the "tube" connecting orbit-vectors for the points $(1,0,0)$ and $(0,1,0)$.

2.3. **Invariants of conjugacy classes.** As we already know, any integer conjugacy class of totally-real $SL(2,\mathbb{Z})$-matrices is defined by a period of the corresponding geometric continued fraction (Theorem 2.10). Similar statement is true in the multidimensional case.

2.3.1. *Theorem on complete invariant.* Let A be a $GL(n,\mathbb{Z})$-operator with distinct eigenvalues. Then any $B \in \Xi(A)$ acts on its Klein-Voronoi continued fraction of A as a periodic shift.

Definition 2.21. Let P be a Klein-Voronoi continued fraction (of some $GL(n,\mathbb{Z})$-opera/-tor). A transformation T of P is said to be a *period* of P if there exists a $GL(n,\mathbb{Z})$-operator A such that

— the Klein-Voronoi continued fraction of A coincides with P;

— the transformation T coincides with the action of A on P.

Let us define now the congruence for continued fractions and their periods.

Definition 2.22. Two Klein-Voronoi continued fractions are said to be *integer* congruent if there exists a $GL(n,\mathbb{Z})$-operator that takes one of them to another.
Two periods of the congruent Klein-Voronoi continued fractions are said to be *integer congruent* if there exists a $GL(n,\mathbb{Z})$-operator that takes the first sail to the second and the period of the first sail to the period of the second.

Periods of Klein continued fractions in a totally-real three dimensional case ($k = 3$) were studied in works [36], [35], [37], [38], [21], [22], etc.

Theorem 2.23. (On complete invariant in general case.) *A Klein-Voronoi continued fraction together with one of its periods is a complete invariant of an integer conjugacy class of $GL(n,\mathbb{Z})$-matrices.*

Remark 2.24. In Theorem 2.25 below we give the description of the set of all periods of a Klein-Voronoi continued fraction.

Proof. If two matrices are integer conjugate then their Klein-Voronoi continued fractions and the corresponding periods are integer congruent.

Suppose now that two matrices A and B have integer congruent continued fractions with congruent periods (let C be the integer congruence of the sails and periods). Denote

$$\tilde{B} = CBC^{-1}.$$

Matrices A and $\tilde{B}$ have the same periods of the same continued fraction. Therefore, their actions coincide for all points of the Klein-Voronoi sail for A. It is clear that points of the Klein-Voronoi continued fractions span $\mathbb{R}^n$, hence by linearity $A = \tilde{B}$. □

2.3.2. *Structure of the set of periods of Klein-Voronoi continued fraction.* Consider a Klein-Voronoi continued fractions S. The composition of two integer linear shifts of S is again an integer shift of P and hence the set of all periods is the group. Denote it by $\Xi(P)$.

Theorem 2.25. *Let A be a matrix with irreducible characteristic polynomial over $\mathbb{Q}$ and P be its Klein-Voronoi continued fraction. Then the group of its periods $\Xi(P)$ coincides with the Dirichlet group $\Xi(A)$.*

Here we should show that distinct Dirichlet groups does not correspond to the same Klein-Voronoi continued fraction. So the proof of the theorem is based on the following proposition.

Proposition 2.26. *Consider operators A and B in $SL(n,\mathbb{Z})$ with irreducible characteristic polynomials. Operators A and B commute if and only if they have the same Klein-Voronoi continued fraction.*

Remark. Recall that Klein-Voronoi continued fractions are defined only for operators with distinct eigenvalues. If the characteristic polynomial of an operator is irreducible over $\mathbb{Q}$, then all its roots are distinct, so any operator of Proposition 2.26 has a Klein-Voronoi continued fraction.

Proof. If operators A and B with irreducible characteristic polynomials commute, then they have the same eigenvectors. Hence an arbitrary operator C commutes with A if and only if C commutes with B. Hence $T_A(v) = T_B(v)$ for any vector v. Therefore, the Klein-Voronoi continued fractions of both operators coincide by construction.

Let us prove the converse statement. Let the Klein-Voronoi continued fractions for A and B coincide as sets.

Suppose A has real eigenvectors $e_1, \ldots, e_k$ and complex conjugate eigenvectors $a_j \pm Ib_j$ for $j = 1, \ldots, l$, where $k + 2l = n$ (here $I = \sqrt{-1}$). Let g_i be a real eigenvector with eigenvalue r_i for $i = 1, \ldots, k$; g_{k+2j-1} and g_{k+2j} be vectors corresponding to the real and imaginary parts of some complex eigenvector with eigenvalue c_j for $j = 1, \ldots, l$. Consider the coordinate system corresponding to the basis $\{g_i\}$:

$$OX_1X_2\ldots X_kY_1Z_1Y_2Z_2\ldots Y_lZ_l.$$

In this coordinates we consider the form Φ_A that in the above coordinates is written as

$$\Phi_A(x_1, \ldots, x_n) = \left(\prod_{i=1}^{k} x_i \prod_{j=1}^{l}(y_j^2 + z_j^2)\right)$$

(we study this form later in Section 3). Similarly we define the form Φ_B for the operator B. From definition it follows that A preserves Φ_A and B preserves Φ_B.

Since asymptotically (in the complement to the balls centered at the origin with the radius increasing to infinity) the Klein-Voronoi continued fraction for A (for B) is tends to the set $\Phi_A = 0$ (and $\Phi_B = 0$ respectively) in a continuous category, the operators A and B have all the same invariant subspaces. In particular, their one-dimensional real eigenspaces corresponding to real eigenvectors and two-dimensional eigenspaces (we denote them by $\pi_1, \ldots, \pi_l$) defined by pairs of complex conjugate roots coincide. This implies that A and B commute if and only if they commute for the vectors of the invariant planes $\pi_1, \ldots, \pi_l$.

Let us show that A and B restricted to the plane π_i ($i = 1, \ldots, l$) commute. Consider the section of the Klein-Voronoi continued fraction by the plane passing through v and parallel to the invariant subspace spanned by all complex eigenvectors of A, we denote it by T_v. From construction $T_v = T_A(v)$. By the above the invariant subspace spanned by all complex eigenvectors of A coincide with invariant subspace spanned by all complex eigenvectors of B, and hence $T(v) = T_B(v)$. Therefore, $T(v) = T_A(v) = T_B(v)$.

It is clear the forms Φ_A and Φ_B are constant at the orbit $T(v)$ for any v. This follows from the fact that any operator P of $T^l(A)$ (respectively $T^l(B)$) preserves Φ_A (respectively, Φ_B), since all eigenvalues of P are of unit absolute values and P is diagonalizable in the eigenbasis of A (respectively B). Therefore, from linearity reasoning $\Phi_A = c\cdot\Phi_B$ for some nonzero constant c. Therefore, the operator A preserves the form Φ_B.

Consider now the plane π_j for some $1 \le j \le l$ and take coordinates OXY such that the restriction of the form $\Phi_B(v)$ to this plane is

$$x^2 + y^2 = \lambda.$$

Direct calculations show that there are two types of operators that preserve this form. They are written in OXY coordinates as follows

$$\begin{pmatrix} \alpha & \beta \\ -\beta & \alpha \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} -\alpha & \beta \\ \beta & \alpha \end{pmatrix}$$

with parameters α and β. The operators of the second family have two real eigenvalues in the plane π_j, which is by the above not the case for the operator B. Therefore, both A and B are from the first family. All operators of the first family commute. Hence A commutes with B in planes π_j for $1 \le j \le l$.

Therefore, A and B are both diagonalisable in the same complex basis. Hence A and B commute. □

Remark. It is interesting to notice that for certain A there exist some operators corresponding to the second family in the proof of Proposition 2.26. These operators preserve the Klein-Voronoi continued fraction of A, although they have different Klein-Voronoi continued fractions or even multiple roots.

Proof of Theorem 2.25. From one hand, by Proposition 2.26 all $SL(n,\mathbb{Z})$-matrices defining the shift commute with each other and also with A. Therefore, they are in $\Xi(A)$. From the other hand, any matrix of $\Xi(A)$ defines a period of Klein-Voronoi continued fraction P. □

3. Algorithmic aspects of reduction to ς-reduced matrices

In Section 1 above we show the existence of and finiteness of ς-reduced matrices in each integer conjugacy class of $SL(n,\mathbb{Z})$-matrices. The aim of this section is to show the techniques to construct ς-reduced matrices integer conjugate to a given one. In Subsection 3.1 we give a geometric interpretation of the Hessenberg complexity as a volume of a certain simplex, which is called the MD-characteristics. Further in Subsection 3.2 we use the MD-characteristics to show that that all ς-reduced matrices are obtained from integer vertices of Klein-Voronoi continued fraction by applying the algorithm of Subsection 1.2 to them. The corresponding techniques is discussed in Subsection 3.3.

3.1. **Markoff-Davenport characteristics.** In this Subsection we characterize the Hessenberg complexity in terms of Markoff-Davenport characteristics.

3.1.1. *Definition of the MD-characteristics and its invariance under the action of the Dirichlet group.* The study of the Markoff-Davenport characteristics is closely related to the theory of minima of absolute values of homogeneous forms with integer coefficients in n-variables of degree n. One of the first works in this area was written by A. Markoff [41] for the decomposable forms (into the product of real linear forms) for $n = 2$. Further, H. Davenport in series of works [11], [12], [13], [14], and [15] made first steps for the case of decomposable forms for $n = 3$.

Consider $A \in SL(n, \mathbb{Z})$. Denote by $P(A, v)$ the parallelepiped spanned by vectors v, $A(v)$, ..., $A^{n-1}(v)$, i.e.,

$$P(A, v) = \Big\{ O + \sum_{i=0}^{n-1} \lambda_i A^i(v) \Big| 0 \le \lambda_i \le 1, i = 0, \dots, n-1 \Big\},$$

where O is the origin.

Definition 3.1. The *Markoff-Davenport characteristics* (or *MD-characteristic*, for short) of an $SL(n, \mathbb{Z})$-operator A is a functional:

$$\Delta_A : \mathbb{R}^n \to \mathbb{R} \qquad \text{defined by} \qquad \Delta_A(v) = V(P(A, v)),$$

where $V(P(A, v))$ is the nonoriented volume of $P(A, v)$.

Proposition 3.2. *Consider $A \in SL(n, \mathbb{Z})$ and let $B \in \Xi(A)$. Then for an arbitrary v we have*

$$\Delta_A(v) = \Delta_A(B(v)).$$

Remark. This means that the MD-characteristics naturally defines a function over the set of all orbits of the Dirichlet group.

Proof. Since $B \in \Xi(A)$ we have $A^n B(v) = BA^n(v)$. Hence the parallelepiped $P(A, B(v))$ coincides with $B(P(A, v))$. Since $B \in SL(n, \mathbb{Z})$, the volume of the parallelepiped is preserved. Therefore,

$$\Delta_A(v) = \Delta_A(B(v)).$$

□

3.1.2. *Homogeneous forms associated to $SL(n, \mathbb{Z})$-operators.* Let $\{e_i\}$ be an integer basis of $\mathbb{R}^n$. Consider any $SL(n, \mathbb{Z})$-operator A with irreducible characteristic polynomial. Suppose that it has k real eigenvalues $r_1, \dots, r_k$ and $2l$ complex conjugate eigenvalues $c_1, \overline{c}_1, \dots, c_l, \overline{c}_l$, where $k + 2l = n$. Let us now define a new basis of vectors $g_1, \dots, g_{k+2l}$ in the following way. For $i = 1, \dots, k$ we choose g_i to be an eigenvector corresponding to the eigenvalue r_i. For $j = 1, \dots, l$ we choose g_{k+2j-1} and g_{k+2j} to be the real and the imaginary parts of some complex eigenvector corresponding to the eigenvalue c_j. Consider the system of coordinates

$$OX_1X_2 \dots X_kY_1Z_1Y_2Z_2 \dots Y_lZ_l$$

corresponding to the basis $\{g_i\}$.

A form

$$\alpha\left(\prod_{i=1}^{k}x_i\prod_{j=1}^{l}(y_j^2+z_j^2)\right)$$

with nonzero α is said to be *associated* to the operator A.

Proposition 3.3. *Let A be an $SL(n,\mathbb{Z})$-operator with irreducible characteristic polynomial. Then the MD-characteristics of A is an absolute value of a form associated to A for a certain nonzero α.*

Proof. Let us consider the formulas of MD-characteristics of A in the eigen-basis of vectors

$$g_1,\ldots,g_k,g_{k+1}+Ig_{k+2},g_{k+1}-Ig_{k+2},\ldots,g_{k+2l-1}+Ig_{k+2l},g_{k+2l-1}-Ig_{k+2l}$$

in $\mathbb{C}^n$, where $I=\sqrt{-1}$. Let the coordinates in this eigen-basis be $\{t_i\}$.

Then for any vector $v=(t_1,\ldots,t_n)$ we have

$$A^j(x)=(r_1^jt_1,\ldots,r_k^jt_k,c_1^jt_{k+1},\overline{c}_1^jt_{k+2},\ldots,c_l^jt_{k+2l-1},\overline{c}_l^jt_{k+2l}).$$

Therefore,

$$\Delta_A(t_1,\ldots,t_n)=\alpha\left|\prod_{i=1}^{k}t_i\prod_{j=1}^{l}(t_{k+2j-1}t_{t+2j})\right|=\frac{\alpha}{4^l}\left|\prod_{i=1}^{k}x_i\prod_{j=1}^{l}(y_j^2+z_j^2)\right|$$

Simple calculations show that $\alpha\neq 0$. $\square$

3.1.3. *Hessenberg complexity in terms of MD-characteristics.*

Proposition 3.4. *Consider an operator A with Hessenberg matrix M in some integer basis $\{e_i\}$. The Hessenberg complexity $\varsigma(M)$ equals the value of MD-characteristics $\Delta_A(e_1)$.*

Proof. Suppose that the Hessenberg type of the matrix M is

$$\big\langle a_{1,1},a_{1,2}\big|a_{2,1},a_{2,2},a_{2,3}\big|\cdots\big|a_{n-1,1},\ldots,a_{n-1,2},a_{n-1,n}\big\rangle.$$

Denote by V_k the plane $\operatorname{Span}\big(v,A(v),A^2(v),\ldots,A^{k-1}(v)\big)$.

Let us inductively show that

$$A^k(e_1)=\left(\prod_{i=1}^{k}a_{i,i+1}\right)e_{k+1}+v_k\quad\text{where } v_k\in V_k.$$

Base of induction. We have $A(e_1)=a_{1,2}e_2+a_{1,1}e_1$.

Step of induction. Suppose that the statement holds for $k=m$, i.e.,

$$A^m(e_1)=\left(\prod_{i=1}^{m}a_{i,i+1}\right)e_{m+1}+v_m,\quad\text{and } v_m\in V_m.$$

Let us show the statement for $m+1$. Since M is Hessenberg, $A(v_m)$ is in V_{m+1}. Therefore, we have

$$\begin{aligned} A^{m+1}(e_1) &= A\left(\left(\prod_{i=1}^{m} a_{i,i+1}\right)e_{m+1}\right)+A(v_m)= \\ &\left(\prod_{i=1}^{m+1} a_{i,i+1}\right)e_{m+1}+\left(A(v_m)+\left(\prod_{i=1}^{m} a_{i,i+1}\right)\big(A(e_{m+1})-a_{m+1,m+2}e_{m+2}\big)\right). \end{aligned}$$

The second summand in the last expression is in V_{m+1}. We have shown the step of induction.

Therefore,

$$\Delta_A(e_1)=\prod_{i=1}^{n-1}|a_{i+1,i}|^{n-i}=\varsigma(M).$$

This concludes the proof of the proposition. □

Further we use the following corollary.

Corollary 3.5. *Consider an operator A with Hessenberg matrix M in some integer basis $\{e_i\}$. Let v be any primitive integer vector. Then we have*

$$\varsigma(M|v)=\Delta_A(v),$$

where $(M|v)$ is the matrix constructed by the algorithm of Subsection 1.2. □

3.2. **Klein-Voronoi continued fractions and minima of MD-characteristics.** In the following theorem we use Klein-Voronoi continued fractions to find minima of MD-characteristics.

Theorem 3.6. *Consider a matrix $M\in SL(n,\mathbb{Z})$ with distinct eigenvalues. Let U be a fundamental domain of the Klein-Voronoi continued fractions for M (see Definition 2.19). Then we have:*

(i) For any ς-reduced matrix $\hat{M}$ integer conjugate to M there exists $v\in U$ such that $\hat{M}=(M|v)$.

(ii) Let $v\in U$. The matrix $(M|v)$ is ς-reduced if and only if the MD-characteristic $\Delta_A(v)$ attains its minimal value.

Proof. Notice that Theorem 3.6(ii) is a direct corollary of Corollary 3.5.

Let us prove Theorem 3.6(i). Let A be an $SL(n,\mathbb{Z})$-operator with irreducible characteristic polynomial defined by the matrix M. By Proposition 3.3 there exists a nonzero constant α such that MD-characteristics Φ_A at any point in the system of coordinates $OX_1X_2\ldots X_kY_1Z_1Y_2Z_2\ldots Y_lZ_l$ is

$$\alpha\left|\prod_{i=1}^{k}x_i\prod_{i=1}^{l}(y_i^2+z_i^2)\right|.$$

Suppose that the minimal absolute value of F on the set of integer points except the origin equals m_0.

Choose the coordinates $OX_1 \ldots X_k Y_1 Y_2, \ldots, Y_l$ in the cone π_+. Consider a projection of $\mathbb{R}^n$ to the cone along the $T_A(v)$ orbits. Since we project along the $T_A(v)$ orbits on which the MD-characteristics is constant, the projection of the MD-characteristics is well-defined, denote it by $\tilde{\Phi}_A$. In the chosen coordinates of π_+, the function $\tilde{\Phi}_A$ is written as follows:

$$\alpha \Big| \prod_{j=1}^{k} x_j \prod_{j=1}^{l} y_j^2 \Big|.$$

The obtained function is convex in any orthant of the cone π_+. Since any factor sails are boundary of a convex hull in each orthant, all the minima of the convex function $\tilde{\Phi}_A$ restricted to the convex hulls are attained at the boundary, i.e., at π-integer points of factor-sails. Therefore, all integer minima of Φ_A are at vertices of the Klein-Voronoi continued fraction.

By Corollary 3.5, the Hessenberg complexity $\varsigma(M|v)$ coincides with MD-characteristics $\Delta_A(v)$. Since any matrix integer conjugate to M has a presentation in the form $(M|v)$ and all the integer minima of MD-characteristics attained at vertices of the Klein-Voronoi continued fraction, any ς-reduced operator $\widetilde{M}$ is represented as $(M|v_0)$ for some vertex v_0 of Klein-Voronoi continued fraction. By Corollary 1.7, for any $B \in \Xi(A)$ we have:

$$(M|B(v_0)) = (M|v_0),$$

since all such B commutes with A. Hence a vector v_0 can be chosen from the fundamental domain D. This concludes the proof. □

Let us give an example of two ς-reduced perfect Hessenberg matrices integer conjugate to each other.

Example 3.7. The ς-reduced Hessenberg matrices (with Hessenberg complexity equal to 3)

$$M_1 = \begin{pmatrix} 0 & 1 & 2 \\ 1 & 0 & 0 \\ 0 & 3 & 5 \end{pmatrix} \qquad \text{and} \quad M_2 = \begin{pmatrix} 0 & 2 & 3 \\ 1 & 1 & 1 \\ 0 & 3 & 4 \end{pmatrix}$$

are integer conjugate.

The reason for this is as follows. Consider the Klein-Voronoi continued fraction of A with matrix M_1. It contains integer vertices $p_1 = (1,0,0)$ and $p_2 = (0,1,-1)$. It turns out that p_1 and p_2 are not in the same orbit of the Dirichlet group but have the same MD-characteristics equals 3. Hence we get distinct two integer conjugate ς-reduced Hessenberg matrices: $M_1 = (M_1|(1,0,0))$ and $M_2 = (M_1|(0,1,-1))$.

3.3. **Construction of ς-reduced matrices by Klein-Voronoi continued fraction.** Any ς-reduced Hessenberg matrix for the operator A is constructed starting from some vertex in a fundamental domain of the Klein-Voronoi multidimensional continued fraction as follows.

Techniques to find ς-reduced matrices in an integer conjugacy class.

Step 1. Find a fundamental domain of the Klein-Voronoi continued fraction for the operator A (see Remark 3.9).

Step 2. Take all vertices of the constructed fundamental domain and find among them all vertices with minimal value of the MD-characteristics (say $v_1, \ldots, v_k$).

Step 3. By Theorem 3.6(i) and (ii) all the ς-reduced matrices integer conjugate to M are $(M|v_1), \ldots, (M|v_k)$. They are all constructed by the algorithm described in Subsection 1.2.

Remark 3.8. For the case of $SL(2,\mathbb{Z})$ we have geometric Gauss Reduction Theory: each vertex of geometric continued fraction corresponds to a starting point of the period. The corresponding matrix is written according to Theorem 2.8.

Remark 3.9. Currently Step 1 is the most complicated. The totally real case of matrices with all eigenvalues being real is studied quite good. For the algorithms of constructing multidimensional continued fractions in this case, we refer to the papers by R. Okazaki [44], J.-O. Moussafir [43] and the author [25]. E. Korkina in [36] and [35], G. Lachaud in [37], [38], A. D. Bruno and V. I. Parusnikov in [8], [46], [47] and [48] the author in [21] and [22] produced a large number of fundamental domains for periodic algebraic two-dimensional continued fractions (see also the site [7] by K. Briggs). Some fundamental domains in three dimensional case are found for instance in [23]. The case with complex conjugate eigenvalues is relatively new, we are planning to study it in our forthcoming paper.

Example 3.10. Let us consider an example of an operator A defined by the matrix

$$\begin{pmatrix} -2 & -4 & -3 \\ 1 & 2 & 2 \\ -1 & -1 & 3 \end{pmatrix}.$$

The characteristic polynomial of this operator has three distinct real roots. Therefore, the Klein-Voronoi continued fraction consists of 8 sails. The compositions of operators $-Id$, A, and $2Id + A^{-1}$ define equivalence between all these sails (here Id is the identity operator) and hence all ς-reduced operators can be written from vertices of one sail. Consider a sail containing the point $[1,0,0]$. There are exactly three distinct orbits of the Dirichlet group containing the vertices in this sail. They are defined by the following points

$$[0,0,1], \quad [1,0,0], \quad \text{and} \quad [3,-1,1].$$

(We skip all the calculations of convex hulls corresponding to the sail, see the algorithms in [43], [25], [49], [44]). The MD-characteristics of these vectors are respectively: 1, 2, and 4. So the minimum of the MD-characteristics (which is 1 in this case) is attained on the vertices of the orbit of the Dirichlet group containing $[0,0,1]$ (and on the corresponding orbits for the rest seven sails). Therefore, there exists a unique ς-reduced Hessenberg matrix, which is

$$\begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 1 \\ 0 & 1 & -3 \end{pmatrix}.$$

The perfect Hessenberg matrices for the vertices $[1,0,0]$ and $[3,-1,1]$ are respectively

$$\begin{pmatrix} 0 & 1 & -1 \\ 1 & 0 & 0 \\ 0 & 2 & -3 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} 1 & 0 & -1 \\ 2 & 0 & 3 \\ 0 & 1 & -4 \end{pmatrix},$$

their ς-complexities are 2 and 4.

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