ON ASYMPTOTIC REDUCIBILITY IN $SL(3,\mathbb{Z})$

OLEG KARPENKOV

Abstract. Recently we showed that Hessenberg matrices are proper to represent conjugacy classes in $SL(n,\mathbb{Z})$. In this paper we focus on the reducibility properties in the set of Hessenberg matrices of $SL(3,\mathbb{Z})$. We investigate the first interesting open case here: the case of matrices having one real and two complex conjugate eigenvalues.

Contents

Introduction

In current paper we study the $SL(3,\mathbb{Z})$ integer conjugacy classes via *reduced representatives*. Recall that two matrices M_1 and M_2 in $SL(3,\mathbb{Z})$ are *integer conjugate* if there

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exists a matrix X in $GL(3,\mathbb{Z})$ such that

$$M_2 = XM_1X^{-1}.$$

For matrices with a pair of complex conjugate eigenvalues we discovered the following phenomenon: *Hessenberg matrices distinguish corresponding conjugacy classes asymptotically* (Theorem 2.6). We show that a similar statement is not true for the case of operators with three real eigenvalues.

Background. In classical approach to $SL(n,\mathbb{Z})$-conjugacy problem one splits $SL(n,\mathbb{Q})$-conjugacy classes into $SL(n,\mathbb{Z})$ conjugacy classes. After that the problem is reduced to certain problems related to orders of algebraic fields extended by the roots of characteristic polynomial of the corresponding matrices (like computing their class numbers, etc.).

In [18] we introduced multidimensional analog of Gauss Reduction Theorem, it is an alternative approach to the conjugacy problem (for two-dimensional Gauss Reduction Theory we refer to [17], [26], and [27]). In multidimensional Gauss Reduction Theory the key role play Hessenberg matrices, they generalize reduced matrices in Gauss Reduction Theory. Hessenberg matrices are matrices that vanish below the superdiagonal (for more information see in [31]). They appear in the work [13] by K. Hessenberg for the first time, they were later used in QR-algorithms ([12], [32], [30]). In [18] we defined a natural notion of Hessenberg complexity for Hessenberg matrices, which is a nonnegative integer function, and showed that *each integer conjugacy class of irreducible matrices has only finite number of Hessenberg matrices with minimal Hessenberg complexity.*

Description of the paper. We start in Section 1 with general definitions and notation. Further in Section 2 we formulate main results of current paper: Theorem 2.1 on parabolic structure of the sets of Hessenberg matrices with two complex conjugate eigenvalues, and Theorem 2.6 on asymptotic reducibility of matrices in these sets. In Section 3 we prove Theorem 2.1. Further in Section 4 we show some necessary tools that we use in the proof of Theorem 2.6 (Markoff-Davenport characteristic, Klein-Voronoi continued fractions, etc.). Then in Section 5 we give a proof of Theorem 2.6. Finally in Section 6 we formulate several open problems.

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1. Definitions and notation

1.1. **ς-reduced matrices.** A matrix M of the form

$$\begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ 0 & a_{32} & a_{33} \end{pmatrix}$$

is called an (*upper*) *Hessenberg* matrix. We say that the *Hessenberg type* of M is

$$\langle a_{11}, a_{21} | a_{12}, a_{22}, a_{32} \rangle.$$

Definition 1.1. A Hessenberg matrix in $SL(3,\mathbb{Z})$ is said to be *perfect* if we have

$$0 \le a_{11} < a_{21};$$
$$0 \le a_{12} < a_{32};$$
$$0 \le a_{22} < a_{32}.$$

Definition 1.2. The *Hessenberg complexity* of a Hessenberg matrix M of Hessenberg type $\langle a_{11}, a_{21} | a_{12}, a_{22}, a_{32}\rangle$ is the number $a_{12}^2 a_{23}$, we denote it by $\varsigma(M)$.

Notice that $\varsigma(M)$ equals the volume of the parallelepiped spanned by the following vectors $v = (1,0,0)$, $M(v)$, and $M^2(v)$.

Definition 1.3. We say that a perfect Hessenberg matrix M is ς*-reduced* if its Hessenberg complexity is the least possible. Otherwise we say that the matrix is ς*-nonreduced*.

In [18] we proved the following result.

Theorem 1.4. *i). Any conjugacy class of $SL(3,\mathbb{Z})$ contains a ς-reduced matrix.*
ii). The number of ς-reduced matrices is finite in any integer conjugacy class.

The results of current paper give evidences concerning the fact that the majority of Hessenberg matrices with two complex conjugate and one real eigenvalues are ς-reduced.

1.2. **Perfect Hessenberg matrices of a given Hessenberg type.** In this paper we study three-dimensional perfect Hessenberg $SL(3,\mathbb{Z})$-matrices with irreducible characteristic polynomials. There are two main geometrically essentially different cases of $SL(3,\mathbb{Z})$-matrices: the *real spectrum* (or *RS*- for short) *case* when the characteristic polynomials of matrices have only real eigenvalues, and the *nonreal spectrum* (or *NRS*- for short) *case* of matrices with a pair of complex conjugate and one real eigenvalues.

Denote the set of all $SL(3,\mathbb{Z})$-matrices of a fixed a Hessenberg type Ω by $H(\Omega)$. Let $H_v(\Omega)$ be the subset of all NRS-matrices in $NRS(\Omega)$.

Definition 1.5. Let $\Omega = \langle a_{11}, a_{21} | a_{12}, a_{22}, a_{32}\rangle$. Consider $v = (a_{13}, a_{23}, a_{33})$ such that the determinant of the matrix (a_{ij}) equals 1. Denote

$$H_\Omega^v(m,n) = \begin{pmatrix} a_{11} & a_{12} & a_{11}m + a_{12}n + a_{13} \\ a_{21} & a_{22} & a_{21}m + a_{22}n + a_{23} \\ 0 & a_{32} & a_{32}n + a_{33} \end{pmatrix}.$$

It is clear that

$$H(\Omega) = \left\{ H_\Omega^v(m,n) \middle| m \in \mathbb{Z}, n \in \mathbb{Z} \right\}$$

Here to choose v means to choose the origin O in the plane $H(\Omega)$. So the set $H(\Omega)$ has the structure of two-dimensional plane. We denote by OMN the coordinate system corresponding to the parameters (m,n).

Let $\mathcal{D}_\Omega^v(m,n)$ denote the discriminant of the characteristic polynomial of $H_\Omega^v(m,n)$. Then the set $NRS(\Omega)$ is defined by the following inequality in variables n and m:

$$\mathcal{D}_\Omega^v(m,n) < 0.$$

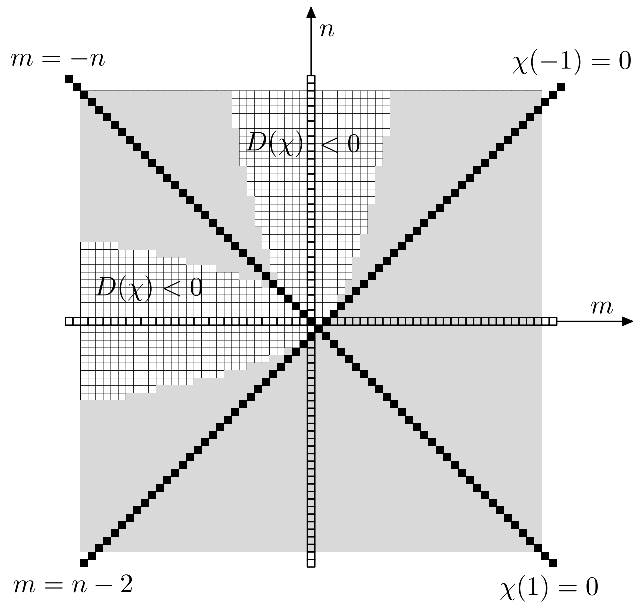

FIGURE 1. The family of matrices of Hessenberg type $\langle 0,1|0,0,1\rangle$.

Example 1.6. In Figure 1 we show the subset of NRS-matrices $NRS(\langle 0,1|0,0,1\rangle)$. For this example we choose $v=(0,0,1)$.

2. FORMULATION OF MAIN RESULTS AND EXAMPLES

We start in Subsection 2.1 with the formulation of a supplementary theorem on parabolic structure of the set of NRS-matrices, we give the proof later in Section 3. Further in Subsection 2.2 we formulate the main result on asymptotic uniqueness of ς-reduced matrices, the proof is shown in Section 5. In Subsection 2.3 with describe examples of families of matrices with fixed Hessenberg type.

2.1. **Parabolic structure of the set of NRS-matrices.**

The set $NRS(\langle 0,1|0,0,1\rangle)$ on Figure 1 "reminds" the set of points with integer coordinates in the union of the convex hulls of two parabolas. Let us formalize this in a general statement.

Consider the matrix $H_\Omega^v(0,0)=(a_{ij})$ and define b_1, b_2, and b_3 as coefficients of characteristic polynomial of this matrix in variable t:

$$-t^3+b_1t^2-b_2t+b_3.$$

In the case of $SL(3,\mathbb{Z})$ we have $b_3=1$, nevertheless we write b_3 for generality reasons. For the family $H_\Omega^v(m,n)$ we define the following two quadratic functions

$$p_{1,\Omega}(m,n)=m-\alpha_1n^2-\beta_1n-\gamma_1;$$
$$p_{2,\Omega}(m,n)=\frac{n}{a_{21}}-\alpha_2\Big(\frac{a_{21}m-a_{11}n}{a_{21}}\Big)^2-\beta_2\Big(\frac{a_{21}m-a_{11}n}{a_{21}}\Big)-\gamma_2,$$

where

$$\begin{cases} \alpha_1 = -\dfrac{a_{32}}{4a_{21}} \\ \beta_1 = \dfrac{a_{11}-a_{22}-a_{33}}{2a_{21}} \\ \gamma_1 = \dfrac{4b_2-b_1^2}{4a_{21}a_{32}} \end{cases}; \qquad \begin{cases} \alpha_2 = \dfrac{a_{32}a_{21}}{4b_3} \\ \beta_2 = -\dfrac{b_2}{2b_3} \\ \gamma_2 = \dfrac{b_2^2-4b_1b_3}{4a_{21}a_{32}b_3} \end{cases}.$$

Denote by $B_R(O)$ the interior of the circle of radius R centered at the origin $(0,0)$ in the real plane OMN of the family $H_\Omega^v(m,n)$. For a real number t we denote

$$\Lambda_t = \{(m,n) \mid (p_{1,\Omega}(m,n)-t)(p_{2,\Omega}(m,n)-t) < 0\}.$$

Theorem 2.1. *For any positive ε there exists $R>0$ such that in the complement to $B_R(O)$ the following inclusions hold*

$$\Lambda_\varepsilon \subset NRS(\Omega) \subset \Lambda_{-\varepsilon}.$$

We give a proof of this theorem in Section 3.

2.2. **Theorem on asymptotic uniqueness of ς-reduced NRS-matrices.** A point is called *integer* if all its coordinates are integers. A ray is said to be *integer* if its vertex is integer and it contains integer points distinct to the vertex.

Definition 2.2. An integer ray in $H(\Omega)$ is said to be *an NRS-ray* if all its integer points correspond to NRS-matrices. A direction is said to be *asymptotic* for the set $NRS(\Omega)$ if there exists an NRS-ray with this direction.

As it is stated in Theorem 2.1, for any Hessenberg type Ω the set $NRS(\Omega)$ almost coincides with the union of the convex hulls of two parabolas. This implies the following statement.

Proposition 2.3. *Let $\Omega = \langle a_{11}, a_{21} | a_{12}, a_{22}, a_{32}\rangle$. There are exactly two asymptotic directions for the set $NRS(\Omega)$, they are defined by the vectors $(-1,0)$ and (a_{11},a_{21}).* □

Let us consider a Hessenberg type $\Omega = \langle a_{11}, a_{21} | a_{12}, a_{22}, a_{32}\rangle$ and an appropriate integer vector v.

Definition 2.4. Consider a family of Hessenberg matrices H_Ω^v. Denote

$$R_{1,\Omega,v}^{m,n} = \left\{H_\Omega^v(m-t,n) \middle| t \in \mathbb{Z}_{\ge 0}\right\};$$
$$R_{2,\Omega,v}^{m,n} = \left\{H_\Omega^v(m+a_{11}t,n+a_{21}t) \middle| t \in \mathbb{Z}_{\ge 0}\right\}.$$

By $R_{1,\Omega,v}^{m,n}(t)$ or respectively by $R_{2,\Omega,v}^{m,n}(t)$ we denote the t-th element in the corresponding family.

Remark 2.5. The families $R_{1,\Omega,v}^{m,n}$ and $R_{2,\Omega,v}^{m,n}$ coincide with the sets of all integer points of some rays with directions $(-1,0)$ and (a_{11},a_{21}) respectively. Conversely, from Proposition 2.3 it follows that the set of integer points of any NRS-ray coincides either with $R_{1,\Omega,v}^{m,n}$ or with $R_{2,\Omega,v}^{m,n}$ for some integers m and n.

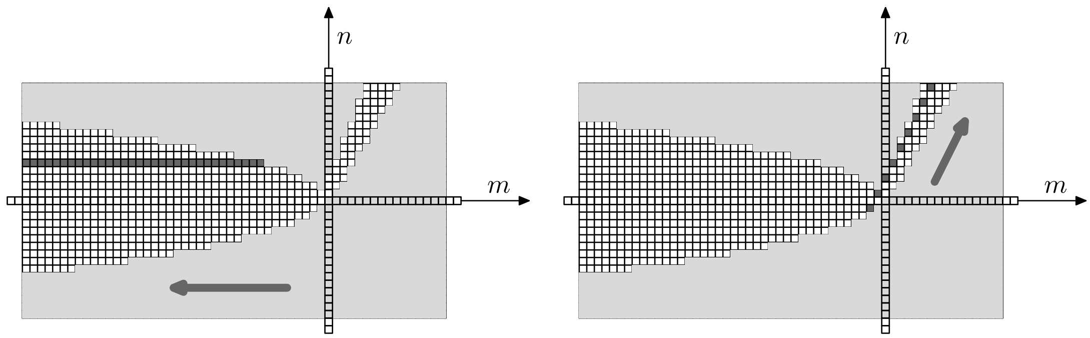

FIGURE 2. Any NRS-ray contains finitely many ς-nonreduced matrices.

On Figure 2 we show in dark gray two NRS-rays: $R^{-9,5}_{1,\langle 1,2|1,1,3\rangle,(0,0,-1)}$ from the left and $R^{-2,-1}_{2,\langle 1,2|1,1,3\rangle,(0,0,-1)}$ from the right.

Now we are ready to formulate the main result on asymptotic behavior of NRS-matrices, we prove it later in Section 5.

Theorem 2.6. (On asymptotic ς-reducibility and uniqueness.) *i). Any NRS-ray (as on Figure 2) contains only finitely many ς-nonreduced matrices.*
ii). Any NRS-ray contains only finitely many matrices that have more than one integer conjugate ς-reduced matrix.

Example 2.7. Any NRS-ray for the Hessenberg type $\langle 0,1|0,0,1\rangle$ contains only ς-reduced perfect matrices. Experiments show that any NRS-ray for $\langle 0,1|1,0,2\rangle$ contains at most one ς-nonreduced matrix (see in Figure 3 on page 7).

2.3. **Examples of NRS-matrices for a given Hessenberg type.** In this subsection we study several examples of families $NRS(\Omega)$ for the Hessenberg types:

$$\langle 0,1|0,0,1\rangle, \quad \langle 0,1|1,0,2\rangle, \quad \langle 0,1|1,1,2\rangle, \quad \text{and} \quad \langle 1,2|1,1,3\rangle.$$

In Figures 3, 4, and 5 the dark gray squares correspond to ς-nonreduced matrices. We also fill with gray the squares corresponding to ς-reduced Hessenberg matrices that are n-th powers ($n \geq 2$) of some integer matrices.

Hessenberg perfect NRS-matrices $H^{(1,0,0)}_{\langle 0,1|0,0,1\rangle}(m,n)$. The Hessenberg complexity of all these matrices is 1, and, therefore, they are all ς-reduced, see the family on Figure 1 on page 4.

Hessenberg perfect NRS-matrices $H^{(1,0,0)}_{\langle 0,1|1,0,2\rangle}(m,n)$. The Hessenberg complexity of these matrices equals 2. Experiments show that 12 of such matrices are ς-nonreduced, see the family in Figure 3. It is conjectured that all others Hessenberg matrices of $NRS\big(\langle 0,1|1,0,2\rangle\big)$ are ς-reduced.

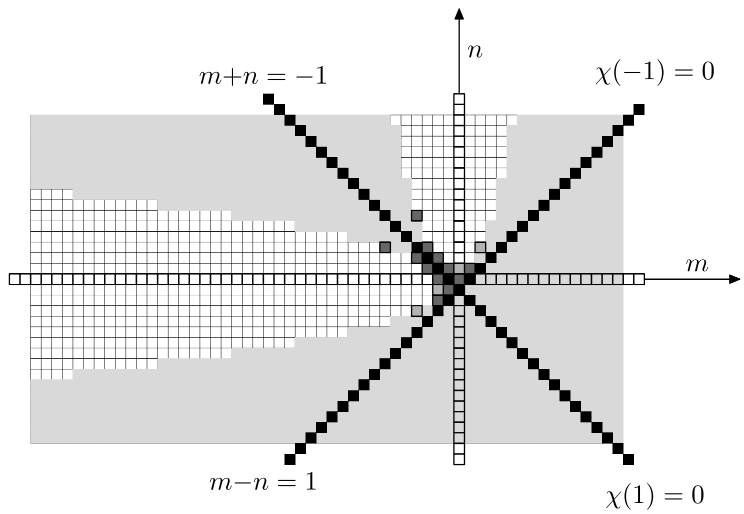

FIGURE 3. The family of Hessenberg matrices $H^{(1,0,0)}_{\langle 0,1|1,0,2\rangle}(m,n)$.

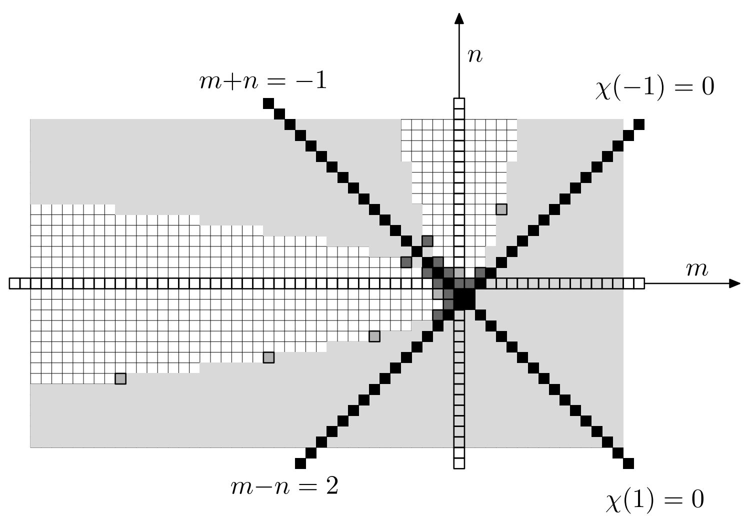

FIGURE 4. The family of Hessenberg matrices $H^{(1,0,1)}_{\langle 0,1|1,1,2\rangle}(m,n)$.

Hessenberg perfect NRS-matrices $H^{(1,0,1)}_{\langle 0,1|1,1,2\rangle}(m,n)$**.** The Hessenberg complexity of these matrices equals 2. We have found 12 ς-nonreduced matrices in the family. It is conjectured that all other Hessenberg matrices of $NRS\big(\langle 0,1|1,1,2\rangle\big)$ are ς-reduced. See in Figure 4.

Hessenberg perfect NRS-matrices $H^{(0,0,-1)}_{\langle 1,2|1,1,3\rangle}(m,n)$**.** This is a more complicated example of a family of Hessenberg perfect NRS-matrices, their complexity equals 12. We have

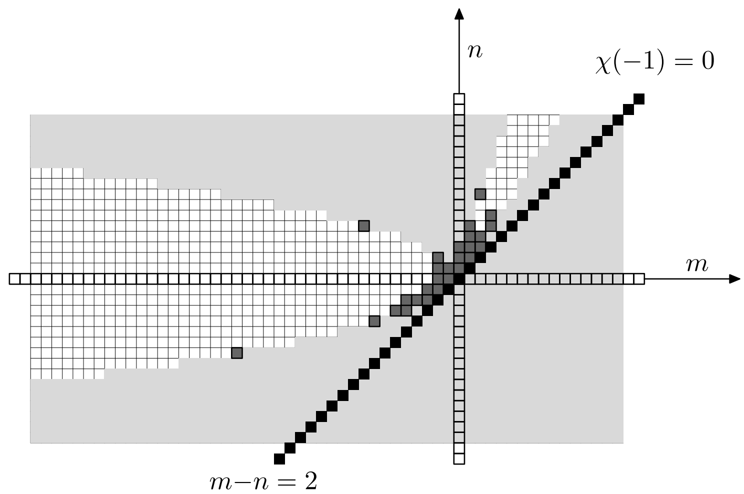

FIGURE 5. The family of Hessenberg matrices $H^{(0,0,-1)}_{\langle 1,2|1,1,3\rangle}$.

found 27 ς-nonreduced matrices in the family. It is conjectured that all other Hessenberg matrices of $NRS\big(\langle 1,2|1,1,3\rangle\big)$ are ς-reduced. See in Figure 5.

3. Proof of Theorem 2.1

We start the proof with several lemmas, but first let us give a small remark.

Remark. The set $NRS(\Omega)$ is defined by the inequality

$$\mathcal{D}^v_\Omega(m,n) < 0.$$

In the left part of the inequality there is a polynomial of degree 4 in variables m and n. Note that the product $16a_{21}^2a_{32}^2b_3\big(p_{1,\Omega}(m,n)p_{2,\Omega}(m,n)\big)$ is a good approximation to $\mathcal{D}^v_\Omega(m,n)$ at infinity: the polynomial

$$\mathcal{D}^v_\Omega(m,n) - 16a_{21}^2a_{32}^2b_3\big(p_{1,\Omega}(m,n)p_{2,\Omega}(m,n)\big)$$

is a polynomial of degree 2 in variables m and n.

Lemma 3.1. *The curve $\mathcal{D}^{(1,0,0)}_{\langle 0,1|0,0,1\rangle}(m,n) = 0$ is contained in the domain defined by the inequalities:*

$$\begin{cases} (m^2-4n+3)(n^2+4m+3) \ge 0 \\ (m^2-4n-3)(n^2+4m-3) - 72 \le 0 \end{cases}$$

Remark. Lemma 3.1 implies that the curve $\mathcal{D}^{(1,0,0)}_{\langle 0,1|0,0,1\rangle}(m,n) = 0$ is contained in some tubular neighborhood of the curve

$$(m^2-4n)(n^2+4m) = 0.$$

Proof. Note that

$$\mathcal{D}^{(1,0,0)}_{\langle 0,1|0,0,1\rangle}(m,n) = (m^2-4n)(n^2+4m) - 2mn - 27.$$

Thus, we have

$$\mathcal{D}^{(1,0,0)}_{\langle 0,1|0,0,1\rangle}(m,n) - (m^2-4n+3)(n^2+4m+3) = -2(n-3)^2 - 2(m+3)^2 - (n+m)^2 \le 0,$$

and

$$\mathcal{D}^{(1,0,0)}_{\langle 0,1|0,0,1\rangle}(m,n) - (m^2-4n-3)(n^2+4m-3) + 72 = 2(n-3)^2 + 2(m+3)^2 + (n-m)^2 \ge 0.$$

Therefore, the curve $\mathcal{D}^{(1,0,0)}_{\langle 0,1|0,0,1\rangle}(m,n) = 0$ is contained in the domain defined in the lemma. □

Lemma 3.2. *For any $\Omega = \langle a_{11}, a_{21}|a_{12}, a_{22}, a_{32}\rangle$ there exists an affine (not necessarily integer) transformation of the plane OMN taking the curve $\mathcal{D}^v_\Omega(m,n) = 0$ to the curve $\mathcal{D}^{(1,0,0)}_{\langle 0,1|0,0,1\rangle}(m,n) = 0$.*

Proof. Let $H^v_\Omega(0,0) = (a_{i,j})$. Note that a matrix $H^v_\Omega(m,n)$ is rational conjugate to the matrix

$$H^{(1,0,0)}_{\langle 0,1|0,0,1\rangle}(a_{23}a_{32} - a_{11}a_{33} + a_{12}a_{21} - a_{22}a_{33} - a_{11}a_{22} + a_{21}a_{32}m - a_{11}a_{32}n, a_{11} + a_{22} + a_{33} + a_{32}n)$$

by the matrix

$$X^v_\Omega = \begin{pmatrix} 1 & a_{11} & a_{11}^2 + a_{12}a_{21} \\ 0 & a_{21} & a_{11}a_{21} + a_{21}a_{22} \\ 0 & 0 & a_{21}a_{32} \end{pmatrix}.$$

Therefore, the curve $\mathcal{D}^v_\Omega(m,n) = 0$ is mapped to the curve $\mathcal{D}^{(1,0,0)}_{\langle 0,1|0,0,1\rangle}(m,n) = 0$ bijectively. In OMN coordinates this map corresponds to the following affine transformation

$$\begin{pmatrix} m \\ n \end{pmatrix} \mapsto \begin{pmatrix} a_{21}a_{32}m - a_{11}a_{32}n \\ a_{32}n \end{pmatrix} + \begin{pmatrix} a_{23}a_{32} - a_{11}a_{33} + a_{12}a_{21} - a_{22}a_{33} - a_{11}a_{22} \\ a_{11} + a_{22} + a_{33} \end{pmatrix}.$$

This completes the proof of the lemma. □

Proof of Theorem 2.1. Consider a family of matrices $H^v_\Omega(-p_{1,\Omega}(0,t) + \varepsilon, t)$ with real parameter t. Direct calculations show that for $\varepsilon \ne 0$ the discriminant of the matrices for this family is a polynomial of the forth degree in variable t, and

$$\mathcal{D}^v_\Omega(-p_{1,\Omega}(0,t) + \varepsilon, t) = \frac{1}{4}a_{21}a_{32}^5\varepsilon t^4 + O(t^3).$$

Therefore, there exists a neighborhood of infinity with respect to the variable t such that the function $\mathcal{D}^v_\Omega(-p_{1,\Omega}(0,t) + \varepsilon, t)$ is positive for positive ε in this neighborhood, and negative for negative ε.

Hence for a given ε there exists a sufficiently large $N_1 = N_1(\varepsilon)$ such that for any $t > N_1$ there exists a solution of the equation $\mathcal{D}^v_\Omega(m,n) = 0$ at the segment with endpoints

$$\big(-p_{1,\Omega}(0,t) + \varepsilon, t\big) \quad \text{and} \quad \big(-p_{1,\Omega}(0,t) - \varepsilon, t\big)$$

of the plane OMN.

Now we examine the family in variable t for the second parabola:

$$H_\Omega^v\left(t-a_{11}p_{2,\Omega}(t,0)-\frac{a_{11}}{\sqrt{a_{11}^2+a_{21}^2}}\varepsilon, -a_{21}p_{2,\Omega}(t,0)-\frac{a_{21}}{\sqrt{a_{11}^2+a_{21}^2}}\varepsilon\right).$$

By the same reasons, for a given ε there exists a sufficiently large $N_2 = N_2(\varepsilon)$ such that for any $t > N_2$ there exists a solution of the equation $\mathcal{D}_\Omega^v(m,n) = 0$ at the segment with endpoints

$$\begin{array}{ll} \left(t-a_{11}p_{2,\Omega}(t,0)-\frac{a_{11}}{\sqrt{a_{11}^2+a_{21}^2}}\varepsilon, -a_{21}p_{2,\Omega}(t,0)-\frac{a_{21}}{\sqrt{a_{11}^2+a_{21}^2}}\varepsilon\right) & \text{and} \\ \left(t-a_{,1}p_{2,\Omega}(t,0)+\frac{a_{11}}{\sqrt{a_{11}^2+a_{21}^2}}\varepsilon, -a_{21}p_{2,\Omega}(t,0)+\frac{a_{21}}{\sqrt{a_{11}^2+a_{21}^2}}\varepsilon\right) & \end{array}$$

of the plane OMN.

We have shown that for any of the four branches two parabolas defined by $p_{1,\Omega}(m,n) = 0$ and $p_{2,\Omega}(m,n) = 0$ there exists (at least) one branch of $\mathcal{D}_\Omega^v(m,n) = 0$ contained in the ε-tube of the chosen parabolic branch if we are far enough from the origin.

From Lemma 3.1 we know that $\mathcal{D}_{\langle 0,1|0,0,1\rangle}^{(1,0,0)}(m,n) = 0$ is contained in some tubular neighborhood of

$$p_{1,\langle 0,1|0,0,1\rangle}(m,n)p_{2,\langle 0,1|0,0,1\rangle}(m,n) = 0.$$

Then by Lemma 3.2 the curve $\mathcal{D}_\Omega^v(m,n) = 0$ is contained in some tubular neighborhood of the curve

$$p_{1,\Omega}(m,n)p_{2,\Omega}(m,n) = 0$$

outside some ball centered at the origin. Finally, by Viet Theorem, the intersection of the curve $\mathcal{D}_\Omega^v(m,n) = 0$ with each of the parallel lines

$$\ell_t : \quad \frac{a_{11}+a_{21}}{a_{21}}n - m = t$$

contains at most 4 points. Therefore, there exists sufficiently large T such that for any $t \geq T$ the intersection of the curve $\mathcal{D}_\Omega^v(m,n) = 0$ and ℓ_t contains exactly 4 points corresponding to the branches of the parabolas $p_{1,\Omega}(m,n) = 0$ and $p_{2,\Omega}(m,n) = 0$ lying in $\Lambda_{-\varepsilon} \setminus \Lambda_\varepsilon$.

Hence, there exists $R = R(\varepsilon, N_1, N_2, T)$ such that in the complement to the ball $B_R(O)$ we have

$$\Lambda_\varepsilon \subset NRS(\Omega) \subset \Lambda_{-\varepsilon}.$$

The proof of Theorem 2.1 is completed. □

4. Supplementary tools for the proof of Theorem 2.6

In this section we introduce several notions that we use in the proof of Theorem 2.6. In Subsection 4.1 we introduce Markoff-Davenport characteristic that represents the Hessenberg complexity. Further in Subsection 4.2 we show how to construct perfect Hessenberg matrices $(M|v)$ conjugate to a given one. Finally in Subsection 4.3 we give the definition of Klein-Voronoi continued fractions, formulate a theorem on construction of ς-reduced operators via vertices of a fundamental domain of the corresponding Klein-Voronoi continued fraction, and prove one supplementary statement on geometry of continued fractions.

4.1. **MD-characteristics.** The study of the Markoff-Davenport characteristics is closely related to the theory of minima of absolute values of homogeneous forms with integer coefficients in n-variables of degree n. One of the first works in this area was written by A. Markoff [28] for the decomposable forms (into the product of real linear forms) for $n=2$. Further, H. Davenport in series of works [7], [8], [9], [10], and [11] made first steps for the case of decomposable forms for $n=3$.

Consider $A \in SL(n,\mathbb{Z})$. Denote by $P(A,v)$ the parallelepiped spanned by vectors v, $A(v)$, ..., $A^{n-1}(v)$, i.e.,

$$P(A,v)=\left\{O+\sum_{i=0}^{n-1}\lambda_i A^i(v)\,\middle|\,0\le\lambda_i\le 1, i=0,\dots,n-1\right\},$$

where O is the origin.

Definition 4.1. The *Markoff-Davenport characteristic* (or *MD-characteristic*, for short) of an $SL(n,\mathbb{Z})$-operator A is a functional:

$$\Delta_A:\mathbb{R}^n\to\mathbb{R}\qquad\text{defined by}\qquad \Delta_A(v)=V(P(A,v)),$$

where $V(P(A,v))$ is the nonoriented volume of $P(A,v)$.

Remark 4.2. Consider an operator A with Hessenberg matrix M in some integer basis. Then the Hessenberg complexity $\varsigma(M)$ equals the value of MD-characteristic $\Delta_A(1,0,0)$.

We continue with the following general definition.

Definition 4.3. The group of all $GL(3,\mathbb{Z})$-operators commuting with A is called the *Dirichlet group* and denoted by $\Xi(A)$.

For MD-characteristic we have the following invariance property.

Proposition 4.4. *Consider $A\in SL(n,\mathbb{Z})$ and let $B\in\Xi(A)$. Then for an arbitrary v we have*

$$\Delta_A(v)=\Delta_A(B(v)).$$

Basically, this means that the MD-characteristic naturally defines a function over the set of all orbits of the Dirichlet group.

4.2. **Construction of a perfect Hessenberg matrix $(M|v)$ conjugate to a given one.** Let us show how to construct perfect Hessenberg matrices integer conjugate to a given one.

Algorithm to construct perfect Hessenberg matrices.

Input Data. We are given by an $SL(3,\mathbb{Z})$-matrix M of an operator A with irreducible characteristic polynomial over $\mathbb{Q}$ and an integer primitive (i.e., with relatively prime coordinates) vector v.

Step 1. We put $e_1=v$.

Step 2. Choose an integer primitive vector of the plane spanned by v and $A(v)$ on the minimal possible nonzero Euclidean distance from the line spanned by v, denote it by g_2. Find the coordinates q_{11} and a_{21} from the vector decomposition

$$A(e_1) = q_{11}e_1 + a_{21}g_2.$$

Find b_{11} and a_{11} as integer quotients and reminders:

$$q_{11} = |a_{21}|b_{11} + a_{11}.$$

Define

$$e_2 = \operatorname{sign}(a_{21})g_2 + b_{11}e_1.$$

Step 3. Choose an integer primitive vector $g_3 \in \mathbb{R}^3$ on minimal possible nonzero Euclidean distance from the plane spanned by e_1 and e_2. Find the coordinates q_{12}, q_{22}, and a_{32} from the vector decomposition

$$A(e_2) = q_{12}e_1 + q_{22}e_2 + a_{32}g_3.$$

Find b_{12}, b_{22}, a_{12}, and a_{22} as integer quotients and reminders:

$$q_{12} = |a_{32}|b_{12} + a_{12} \quad \text{and} \quad q_{22} = |a_{32}|b_{22} + a_{12}.$$

Then we have

$$e_3 = b_{12}e_1 + b_{22}e_2 + \operatorname{sign}(a_{32})g_3.$$

Output Data. Let C be a transition matrix to the basis $\{e_1, e_2, e_3\}$. In the output we have the perfect Hessenberg matrix CMC^{-1}.

Definition 4.5. Consider an $SL(3,\mathbb{Z})$-matrix M with irreducible characteristic polynomial over $\mathbb{Q}$ and an integer primitive vector v. Starting from M and v the above algorithm generates a perfect Hessenberg matrix, we denote it by $(M|v)$

Remark 4.6. In [18] we showed that any perfect Hessenberg matrix integer conjugate to M is represented as $(M|v)$ for a certain integer primitive vector v.

4.3. **Klein-Voronoi continued fractions.** In the proof of Theorem 2.6 we essentially use the geometric construction of Klein-Voronoi continued fractions. In [19] and [20] F. Klein proposed a multidimensional generalization of continued fractions to totally real case. First attempts to find analogous construction in other cases were made by G. Voronoi in his dissertation [33]. In 1985 J. A. Buchmann in his papers [5] and [6] proposed to use Voronoi's extension to compute of fundamental units in orders. We use a slightly modified definition of Klein-Voronoi continued fraction from the paper [18].

4.3.1. *RS-case.* Let us first briefly recall Klein's definition of two-dimensional continued fraction in totally real case. Consider an operator A in $GL(3,\mathbb{Z})$ with three real distinct eigenvalues. This operator has three distinct invariant planes passing through the origin. The complement to the union of these planes consists of 8 open orthants. Let us choose an arbitrary orthant.

Definition 4.7. The boundary of the convex hull of all integer points except the origin in the closure of the orthant is called the *sail*. The set of all 8 sails of the space $\mathbb{R}^3$ is called the 2-*dimensional continued fraction in the sense of Klein*.

For further information on Klein continued fractions we refer to the following literature: [24], [3], [1], [2], [21] [23], [22], [25], [14], [15], [16], [29] etc.

4.3.2. *NRS-case.* Consider an operator A in $GL(3,\mathbb{R})$ with distinct eigenvalues. Suppose that it has a real eigenvalue r and complex conjugate eigenvalues c and $\bar{c}$.

Denote by $T^1(A)$ the set of all real operators commuting with A such that they have a real eigenvalue equals 1 are with absolute value of both complex eigenvalues equal one. Actually, $T^l(A)$ is an abelian group with operation of matrix multiplication isomorphic to S^1.

For $v \in \mathbb{R}^3$ we denote

$$T_A(v) = \{B(v) \mid B \in T^1(A)\}.$$

If v is a real eigenvector then $T_A(v)$ consists of one point. Otherwise (in general case) $T_A(v)$ is homeomorphic to S^1.

Let g_1 be a real eigenvector with eigenvalue r; g_2 and g_3 be vectors corresponding to the real and imaginary parts of some complex eigenvector with eigenvalue c. Consider the coordinate system $OXYZ$ corresponding to the basis $\{g_i\}$. Denote by π the $(k+l)$-dimensional plane OXY, and by π_+ — the half-plane of π defined by $y \geq 0$.

Proposition 4.8. *For any v the orbit $T_A(v)$ intersects the half-plane π_+ in a unique point.* □

Definition 4.9. A point $p \in \pi_+$ is said to be π-*integer* if the orbit $T_A(p)$ contains at least one integer point.

The invariant hyperplane $x = 0$ of operator A divides π_+ into two arcwise connected components.

Definition 4.10. The convex hull of all π-integer points except the origin contained in a given arcwise connected component is called a *factor-sail* of the operator A. The set of both factor-sails is said to be the *factor-continued fraction* for the operator A.
The union of all orbits $T_A(*)$ in $\mathbb{R}^n$ represented by the points in the factor-sail is called the *sail* of the operator A. The set of all sails is said to be the *continued fraction* for the operator A in the sense of Klein-Voronoi (see in Figure 6 below).

It is clear that the factor-sail is a broken line. The corresponding sail is the surface of elliptic rotation of the factor-sail around the eigenline of A. The cones corresponding to rotation of edges (vertices) are called *factor-edges* (*factor-vertices*).

4.3.3. *Algebraic continued fractions.* Consider an operator A in $GL(3,\mathbb{Z})$ with a real eigenvalue r and two complex conjugate distinct eigenvalues c and $\bar{c}$. Suppose also that the characteristic polynomial of A is irreducible over $\mathbb{Q}$.

The Dirichlet group $\Xi(A)$ (of $GL(3,\mathbb{Z})$-operators commuting with A) takes the Klein-Voronoi continued fraction to itself but maybe exchange the sails. By Dirichlet unit theorem (see in [4]) the Dirichlet group $\Xi(A)$ is always homomorphic to $\mathbb{Z}\oplus\mathbb{Z}/2\mathbb{Z}$.

Definition 4.11. A *fundamental domain of the Klein-Voronoi continued fraction* for A is a collection of open orbit-vertices and orbit-edges such that for any orbit-face F of the continued fraction there exists a unique orbit-face F' in this collection and an operator $T\in\Xi(A)$ such that $F=T(F')$.

Example 4.12. Consider an operator

$$A=\begin{pmatrix} 0 & 0 & 1\\ 1 & 0 & 1\\ 0 & 1 & 3\end{pmatrix}.$$

It has one real and two complex conjugate eigenvalues. In Figure 6a we show in light gray the halfplane π_+, the invariant plane for A corresponding to complex conjugate eigenvalues is colored in dark gray. The boundary line of the halfplane π_+ is an invariant line of A, it contains real eigenvectors of A.

The halfplane π_+ is shown in Figure 6b. The invariant plane intersects π_+ in a ray separating π_+ into two connected components. A point of π_+ is colored in black if and only if it is a π-integer point. The boundaries of the convex hulls in each part of π_+ are two factor-sails. Notice that, one factor-sail is taken to another by the induced action of the operator $-Id$, where Id is the identity operator.

In Figure 6c we show one of the sails of Klein-Voronoi continued fraction for A. There are three visible orbit-vertices, they correspond to integer vectors $(1,0,0)$, $(0,1,0)$, and $(0,0,1)$: the large dark points $(0,1,0)$ and $(0,0,1)$ are visible on the corresponding orbit-vertices, the point $(1,0,0)$ is on the backside of the continued fraction.

A fundamental domain of the operator consists of one orbit-vertex and one orbit edge. For instance, one can take the orbit-vertex corresponding to the point $(1,0,0)$ and the orbit-edge corresponding to the "tube" connecting orbit-vertices for the points $(1,0,0)$ and $(0,1,0)$.

In the proof of Theorem 2.6 we use the following result on construction of ς-reduced operators via vertices of fundamental domains of Klein-Voronoi continued fractions.

Theorem 4.13. ([18]) *Consider an $SL(n,\mathbb{Z})$=operator A with matrix M having distinct eigenvalues. Let U be a fundamental domain of the Klein-Voronoi continued fractions for A. Then we have:*

(i) For any ς-reduced matrix $\hat{M}$ integer conjugate to M there exists $v\in U$ such that $\hat{M}=(M|v)$.

(ii) Let $v\in U$. The matrix $(M|v)$ is ς-reduced if and only if the MD-characteristic $\Delta_A(v)$ attains its minimal value. □

4.3.4. *One general fact on fundamental domain of Klein-Voronoi continued fractions for NRS-matrices of $SL(3,\mathbb{Z})$.* Further we use the following statement.

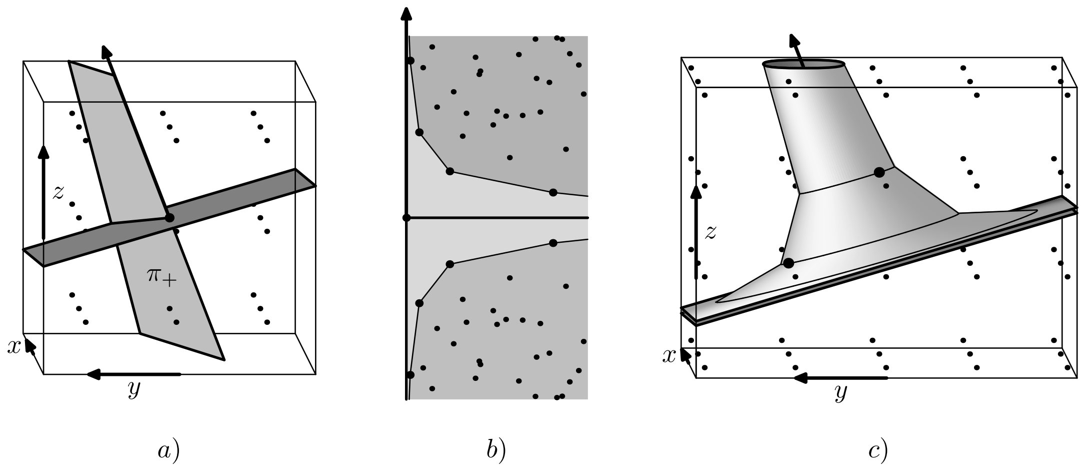

FIGURE 6. A Klein-Voronoi continued fraction: a) the cone π_+ and the eigenplane; b) the continued factor-fraction; c) one of the sails.

Consider an NRS-operator A in $SL(3,\mathbb{Z})$ and any integer point x distinct from the origin. Denote by $\Gamma_A^0(p)$ the convex hull of the union of two orbits corresponding to the points p and $A(p)$. For any integer k we denote by $\Gamma_A^k(p)$ the set $A^k(\Gamma_A^0(x))$.

Proposition 4.14. *Let A be an NRS-operator in $SL(3,\mathbb{Z})$ and p be an integer point distinct from the origin. Then there exists a fundamental domain of the Klein-Voronoi continued fraction for A with all (integer) orbit-vertices contained in the set $\Gamma_A^0(p)$.*

The proof is based on the following lemma. Let

$$\Gamma_A(p) = \bigcup_{k\in\mathbb{Z}} \Gamma_A^k(p).$$

Lemma 4.15. *Consider $A \in SL(3,\mathbb{Z})$ with NRS-matrix and let p be any integer point distinct from the origin. Then one of the Klein-Voronoi sails for A is contained in the set $\Gamma_A(p)$.*

Proof. Notice that the set $\Gamma_A(p)$ is a union of orbits. Let us project $\Gamma_A(p)$ to the halfplane π_+. The set $\Gamma_A(p)$ projects to the closure of the complement of the convex hull for the points $\pi(A^k(p))$ for all integer number k in the angle defined by eigenspaces. Since all the points $A^k(p)$ are integer, their convex hull is contained in the convex hull of all points corresponding to integer orbits in the angle. Hence $\pi(\Gamma_A(p))$ contains the projection of the sail. Therefore, the set $\Gamma_A(p)$ contains one of the sails. □

Proof of Proposition 4.14. Since $-Id$ exchange the sails, one can choose a fundamental domain entirely contained in one sail. Let this sail contains a point p. By Lemma 4.15 $\Gamma_A(p)$ contains this sail. Therefore all the orbit-vertices of a fundamental domain for Klein-Voronoi continued fraction can be chosen from the factor-set $\Gamma_A^0(p)$. □

5. Proof of Theorem 2.6

5.1. **Geometry of Klein-Voronoi continued fractions for matrices of $R^{m,n}_{1,\Omega,v}$.** Let us show the following statement.

Proposition 5.1. *Consider an NRS-ray $R^{m,n}_{1,\Omega,v}$. Then there exists $C>0$ such that for any $t>C$ there exists a fundamental domain for the Klein-Voronoi continued fraction of the matrix $R^{m,n}_{1,\Omega,v}(t)$ such that all integer points in this domain are contained in the triangle with vertices $(1,0,0)$, $(a_{11},a_{21},0)$, and $(-a_{11},-a_{21},0)$.*

We begin with the case of matrices of Hessenberg type $\Omega_0=\langle 0,1|0,0,1\rangle$. Such matrices form a family $H(\Omega_0)$ with real parameters m and n as before:

$$H^{(1,0,0)}_{\langle 0,1|0,0,1\rangle}(m,n)=\begin{pmatrix} 0 & 0 & 1\\ 1 & 0 & m\\ 0 & 1 & n\end{pmatrix}.$$

Here $v_0=(1,0,0)$.

Lemma 5.2. *Let $R^{m,n}_{1,\Omega_0,v_0}$ be an NRS-ray. Then for any $\varepsilon>0$ there exists $C>0$ such that for any $t>C$ the convex hull of the union of two orbit-vertices*

$$T_{R^{m,n}_{1,\Omega_0,v_0}(t)}(1,0,0) \quad \textit{and} \quad T_{R^{m,n}_{1,\Omega_0,v_0}(t)}(0,1,0)$$

is contained in the ε-tubular neighborhood of the convex hull of three points $(1,0,0)$, $(0,1,0)$, $(0,-1,0)$.

Remark. Actually Lemma 5.2 means that the corresponding domain tends to be flat while the parameter t tends to infinity.

Proof. Let us find the asymptotics of eigenvectors and eigenplanes for operators $R^{m,n}_{1,\Omega_0,v_0}(t)$ while t tends to $+\infty$. Denote the real eigenvector of $R^{m,n}_{1,\Omega_0,v_0}(t)$ by $e(t)$. We have

$$e(t)=\mu\big((1,0,0)+O(t^{-1})\big)$$

for some nonzero real μ.

Consider the unique invariant real plane of the operator $R^{m,n}_{1,\Omega_0,v_0}(t)$ (it corresponds to the pair of complex conjugate eigenvalues). Notice that this plane is a union of all orbits $T_{R^{m,n}_{1,\Omega_0,v_0}(t)}(w)$ for arbitrary vectors w of this plane. Any such orbit is an ellipse with axes $\lambda g_{\max}(t)$ and $\lambda g_{\min}(t)$ for some positive real number λ, where

$$g_{\max}(t)=(0,t,0)+O(1),$$
$$g_{\min}(t)=(0,0,t^{1/2})+O(t^{-1/2}).$$

Actually, the vectors $g_{\max}(t)\pm I g_{\min}(t)$ are two complex eigenvectors of $R^{m,n}_{1,\Omega_0,v_0}(t)$. For the ratio of the lengths of maximal and minimal axes of any orbit we have the following asymptotic estimate:

$$\frac{\lambda|g_{\max}(t)|}{\lambda|g_{\min}(t)|}=|t|^{1/2}+O(|t|^{-1/2}).$$

Since
$$(1,0,0)-\frac{1}{\mu}e(t)=O(|t|^{-1}),$$
the minimal axis of the orbit-vertex $T_{R^{m,n}_{1,\Omega_0,v_0}(t)}(1,0,0)$ is asymptotically not greater than $O(t^{-1})$. Therefore, the length of the maximal axis is asymptotically not greater than some function of type $O(|t|^{-1/2})$. Hence, the orbit of the point $(1,0,0)$ is contained in the $(C_1|t|^{-1/2})$-ball of the point $(1,0,0)$, where C_1 is a constant that does not depend on t.

We have
$$(0,1,0)-\frac{1}{t}g_{\max}(t)=O(|t|^{-1}).$$
Therefore, the length of the maximal axis of the orbit-vertex $T_{R^{m,n}_{1,\Omega_0,v_0}(t)}(1,0,0)$ is asymptotically not greater than some function $1+O(t^{-1/2})$. Hence, the length of the minimal axis is asymptotically not greater than some function $O(|t|^{-1/2})$. This implies that the orbit of the point $(0,1,0)$ is contained in the $(C_2|t|^{-1/2})$-tubular neighborhood of the segment with vertices $(0,1,0)$ and $(0,-1,0)$, where C_2 is a constant that does not depend on t.

Therefore, the convex hull of the union of two orbit-vertices
$$T_{R^{m,n}_{1,\Omega_0,v_0}(t)}(1,0,0) \quad \text{and} \quad T_{R^{m,n}_{1,\Omega_0,v_0}(t)}(0,1,0)$$
is contained in the C-tubular neighborhood of the triangle with vertices $(1,0,0)$, $(0,1,0)$, $(0,-1,0)$, where $C=\max(C_1,C_2)|t|^{-1/2}$. This concludes the proof of the lemma. □

Let us now formulate a similar statement for the general case of Hessenberg matrices.

Corollary 5.3. *Let $\Omega=\langle a_{11},a_{21}|a_{12},a_{22},a_{32}\rangle$ and $R^{m,n}_{1,\Omega,v}$ an NRS-ray. Then for any $\varepsilon>0$ there exists $C>0$ such that for any $t>C$ the convex hull of the union of two orbit-vertices*
$$T_{R^{m,n}_{1,\Omega,v}(t)}(1,0,0) \quad \textit{and} \quad T_{R^{m,n}_{1,\Omega,v}(t)}(a_{11},a_{21},0)$$
is contained in the ε-tubular neighborhood of the convex hull of three points $(1,0,0)$, $(a_{11},a_{21},0)$, $(-a_{11},-a_{21},0)$.

Proof. Denote $\Omega=\langle a_{11},a_{21}|a_{12},a_{22},a_{32}\rangle$ and choose
$$X=\begin{pmatrix} a_{21}a_{32} & -a_{32}a_{11} & a_{11}a_{22}-a_{21}a_{12} \\ 0 & a_{32} & -a_{11}-a_{22} \\ 0 & 0 & 1 \end{pmatrix}.$$

A direct calculation shows that
$$H^{(1,0,0)}_{\langle 0,1|0,0,1\rangle}\left(l_1(m,n)-\frac{t}{a_{21}a_{32}},l_2(n_0)\right)=XH_\Omega(m-t,n)X^{-1},$$
where l_1 and l_2 are linear functions with coefficients depending only on a_{11}, a_{21}, a_{12}, a_{22}, and a_{32}. Therefore, the ray $R^{m,n}_{1,\Omega,v}$ after the described change of coordinates and a homothety is taken to the ray $R^{\tilde m,\tilde n}_{1,\Omega_0,v_0}$ of matrices with Hessenberg type $\Omega_0=\langle 0,1|0,0,1\rangle$ for certain $\tilde m$ and $\tilde n$.

Lemma 5.2 implies the following. For any $\varepsilon > 0$ there exists a positive constant such that for any t greater than this constant the convex hull of the union of two orbit-vertices

$$T_{R^{\tilde m,\tilde n}_{1,\Omega_0,v_0}(t)}(1,0,0) \quad \text{and} \quad T_{R^{\tilde m,\tilde n}_{1,\Omega_0,v_0}(t)}(0,1,0)$$

is contained in the ε-tubular neighborhood of the triangle with vertices $(1,0,0)$, $(0,1,0)$, $(0,-1,0)$.

Now if we reformulate the last statement for the family of operators in old coordinates, then we get the statement of the corollary. □

Proof of Proposition 5.1. We note that the operator $R^{m,n}_{1,\Omega,v}(t)$ takes the point $(1,0,0)$ to the point $(a_{11}, a_{21}, 0)$. Therefore, the convex hull of the union of two orbit-vertices

$$T_{R^{m,n}_{1,\Omega,v}(t)}(1,0,0) \quad \text{and} \quad T_{R^{m,n}_{1,\Omega,v}(t)}(a_{11},a_{21},0)$$

(we denote it by $W(t)$) coincides with the set $\Gamma^0_{R^{m,n}_{1,\Omega,v}(t)}(1,0,0)$.

From Proposition 4.14 it follows that there exists a fundamental domain for the continued fraction with all its orbit-vertices contained in $W(t)$. Choose a sufficiently small ε_0 such that the ε_0-tubular neighborhood of the triangle with vertices

$$(1,0,0), \qquad (a_{1,1}, a_{2,1}, 0), \quad \text{and} \quad (-a_{11}, -a_{21}, 0)$$

does not contain integer points distinct from the points of the triangle. From Corollary 5.3 it follows that for a sufficiently large t the set $W(t)$ is contained in the ε_0-tubular neighborhood of the triangle. This implies the statement of Proposition 5.1. □

5.2. Geometry of Klein-Voronoi continued fractions for matrices of $R^{m,n}_{2,\Omega,v}$.

Now let us study the remaining case of the rays of matrices with asymptotic direction (a_{11}, a_{21}).

Proposition 5.4. *Consider an NRS-ray $R^{m,n}_{2,\Omega,v}$. Then there exists $C > 0$ such that for any $t > C$ there exists a fundamental domain for the Klein-Voronoi continued fraction of the matrix $R^{m,n}_{2,\Omega,v}(t)$ such that all integer points in this domain are contained in the triangle with vertices $(1,0,0)$, $(-1,0,0)$, and $(a_{11}, a_{21}, 0)$.*

The proof of this proposition is based on the corollary of the following lemma. We remind that $\Omega_0 = \langle 0,1|0,0,1\rangle$.

Lemma 5.5. *Let $R^{m,n}_{2,\Omega_0,v_0}$ be an NRS-ray. Then for any $\varepsilon > 0$ there exists $C > 0$ such that for any $t > C$ the convex hull of the union of two orbit-vertices*

$$T_{R^{m,n}_{2,\Omega_0,v_0}(t)}(1,0,0) \quad \text{and} \quad T_{R^{m,n}_{2,\Omega_0,v_0}(t)}(0,1,0)$$

is contained in the ε-tubular neighborhood of the convex hull of three points $(1,0,0)$, $(-1,0,0)$, $(0,1,0)$.

Proof. First, we note that the continued fractions for the operators A and A^{-1} coincide.

Secondly, the following holds:

$$H^{(1,0,0)}_{\langle 0,1|0,0,1\rangle}(m, n+t) = X H^{(1,0,0)}_{\langle 0,1|0,0,1\rangle}(-n-t, -m) X^{-1},$$

where

$$X = \begin{pmatrix} 0 & -1 & -n-t \\ -1 & 0 & -m \\ 0 & 0 & -1 \end{pmatrix}.$$

Thus, in the new coordinates we obtain the equivalent statement for the ray $R^{-n,-m}_{1,\Omega_0,v_0}(t)$. Now Lemma 5.5 follows directly from Lemma 5.2. □

Corollary 5.6. *Let* $\Omega = \langle a_{11}, a_{21} | a_{12}, a_{22}, a_{32}\rangle$ *and* $R^{m,n}_{2,\Omega,v}$ *be an NRS-ray. Then for any* $\varepsilon > 0$ *there exists* $C > 0$ *such that for any* $t > C$ *the convex hull of the union of two orbit-vertices*

$$T_{H^v_\Omega(m+a_{11}t,n+a_{21}t)}(1,0,0) \quad \textit{and} \quad T_{H^v_\Omega(m+a_{11}t,n+a_{21}t)}(a_{11},a_{21},0)$$

is contained in the ε*-tubular neighborhood of the triangle with vertices* $(1,0,0)$*,* $(-1,0,0)$*, and* $(a_{11},a_{21},0)$*.* □

Remark. We omit the proofs of Corollary 5.6 and Proposition 5.4, since they repeat the proofs of Corollary 5.3 and Proposition 5.1.

5.3. **Conclusion of the proof.** Let us finally conclude the proof of Theorem 2.6. Let A be an operator with Hessenberg matrix M in $SL(3,\mathbb{Z})$. By Theorem 4.13 any ς-reduced matrix congruent to M is constructed as the matrix $(M|v)$, where v is an integer vector in an arbitrary chosen fundamental domain of the Klein-Voronoi continued fractions for A, in addition v should be the minimum of the absolute value of MD-characteristic on the integer lattice except the origin. To calculate ς-reduce matrices we find all such minima of MD-characteristics in appropriate fundamental domains.

The case of NRS-rays with asymptotic direction $(-1,0)$. Consider an NRS-ray $R^{m,n}_{1,\Omega,v}$. By Proposition 5.1 there exists $C > 0$ such that for any integer $t > C$ we can choose a fundamental domain for the Klein-Voronoi continued fraction of $R^{m,n}_{1,\Omega,v}(t)$ such that all its integer points are in the triangle with vertices $(1,0,0)$, $(a_{11},a_{21},0)$, and $(-a_{11},-a_{21},0)$.

This triangle contains only finitely many integer points, all of them have the last coordinate equal to zero. The value of the MD-characteristic for a point $(x,y,0)$ equals:

$$(a_{21}x - a_{11}y)a_{32}^2y^2t + \tilde{C},$$

where the constant $\tilde{C}$ does not depend on t, it depends only on x, y, and Ω. Therefore, for any point $(x,y,0)$ the MD-characteristic is linear with respect to the parameter t, and it increases with growth of t. The only exceptions are the points of type $\lambda(1,0,0)$ and $\mu(a_{11},a_{21},0)$ (for integers λ and μ). The values of MD-characteristic are constant in these points with respect to the parameter t.

Since there are finitely many integer points in the triangle $(1,0,0)$, $(a_{11},a_{21},0)$, and $(-a_{11},-a_{21},0)$, for sufficiently large t the MD-characteristic at points of the triangle attains the minima only at $(1,0,0)$ and at $(a_{11},a_{21},0)$. Since $R^{m,n}_{1,\Omega,v}(t)$ takes the point $(1,0,0)$ to the point $(a_{11},a_{21},0)$, a fundamental domain may contain only one of these two points, let it be $(1,0,0)$.

Therefore, for sufficiently large t the minimum of MD-characteristic at the integer points of the chosen fundamental domain is unique and it is attained at point $(1,0,0)$. Hence by Theorem 4.13 for sufficiently large t the matrix

$$\Big(H^{(1,0,0)}_{\langle a_{11},a_{21}|a_{12},a_{22},a_{32}\rangle}(m-t,n)\Big|(1,0,0)\Big)=H^{(1,0,0)}_{\langle a_{11},a_{21}|a_{12},a_{22},a_{32}\rangle}(m-t,n)$$

is the only ς-reduced matrix in the conjugacy class. This implies both statements of Theorem 2.6 for the ray $R^{m,n}_{1,\Omega,v}$.

Therefore, Theorem 2.6 holds for any NRS-ray with asymptotic direction $(-1,0)$.

The case of NRS-rays with asymptotic direction (a_{11},a_{21}). This case is similar to the case of NRS-rays with asymptotic direction $(-1,0)$, so we omit the proof here.

Proof of Theorem 2.6 is completed. □

6. Open problems

In this section we formulate open questions on the structure of the sets of NRS-matrices and briefly describe the situation for RS-matrices.

NRS-matrices. As we have shown in Theorem 2.6 the number of ς-nonreduced matrices in NRS-rays is always finite. Here we conjecture a stronger statement.

Conjecture 1. Let Ω be an arbitrary Hessenberg type. All but a finite number of NRS-matrices of type Ω are ς-reduced.

If the answer to this conjecture is positive we immediately have the following general question.

Problem 2. Study the asymptotics of the number of ς-nonreduced NRS-matrices with respect to the growth of Hessenberg complexity.

Denote the conjectured number of ς-nonreduced NRS-matrices of Hessenberg type Ω by $\#(\Omega)$. Numerous calculations give rise to the following table for all types with Hessenberg complexity less than 5.

Ω	$\langle 0,1\|0,0,1\rangle$	$\langle 0,1\|1,0,2\rangle$	$\langle 0,1\|1,1,2\rangle$	$\langle 0,1\|1,0,3\rangle$	$\langle 0,1\|1,1,3\rangle$	$\langle 0,1\|1,2,3\rangle$
$\varsigma(\Omega)$	1	2	2	3	3	3
$\#(\Omega)$	0	12	12	6	10	10

Ω	$\langle 0,1\|2,0,3\rangle$	$\langle 0,1\|2,1,3\rangle$	$\langle 0,1\|2,2,3\rangle$	$\langle 1,2\|0,0,1\rangle$	$\langle 0,1\|1,0,4\rangle$	$\langle 0,1\|1,1,4\rangle$
$\varsigma(\Omega)$	3	3	3	4	4	4
$\#(\Omega)$	14	10	10	94	6	8

Ω	$\langle 0,1\|1,2,4\rangle$	$\langle 0,1\|1,3,4\rangle$	$\langle 0,1\|3,0,4\rangle$	$\langle 0,1\|3,1,4\rangle$	$\langle 0,1\|3,2,4\rangle$	$\langle 0,1\|3,3,4\rangle$
$\varsigma(\Omega)$	4	4	4	4	4	4
$\#(\Omega)$	10	8	10	12	8	8

RS-matrices. We conclude this paper with a few words about real spectra matrices (i.e., about $SL(3,\mathbb{Z})$-matrices with three distinct real roots). Mostly we consider the family $H(\langle 0,1|1,0,2\rangle)$, the situation with the other Hessenberg types is similar.

Recall that

$$H^{(1,0,1)}_{\langle 0,1|1,0,2\rangle}(m,n)=\begin{pmatrix} 0 & 1 & n+1\\ 1 & 0 & m\\ 0 & 2 & 2n+1\end{pmatrix}.$$

This matrix is of Hessenberg type $\langle 0,1|1,0,2\rangle$, its Hessenberg complexity equals 2. Hence $H^{(1,0,1)}_{\langle 0,1|1,0,2\rangle}(m,n)$ is ς-reduced if and only if it is not integer conjugate to some matrix of unit Hessenberg complexity, all such matrices are of Hessenberg type $\langle 0,1|0,0,1\rangle$.

In Figure 7 we show all matrices $H^{(1,0,1)}_{\langle 0,1|1,0,2\rangle}(m,n)$ with

$$-20\le m,n\le 20.$$

The square in the intersection of the m-th column with the n-th row corresponds to the matrix $H^{(1,0,1)}_{\langle 0,1|1,0,2\rangle}(m,n)$. It is colored in black if the characteristic polynomial has rational roots. The square is colored in gray if the characteristic polynomial is irreducible and there exists an integer vector (x,y,z) with the coordinates satisfying

$$-1000\le x,y,z\le 1000,$$

such that the MD-characteristic of $H_{\langle 0,1|1,0,2\rangle}(1,0,1)(m,n)$ equals 1 at (x,y,z). All the rest squares are white.

If a square is gray, then the corresponding matrix is ς-nonreduced, see Remark 4.2. If a square is white, then we cannot conclude whether the matrix is ς-reduced or not (since the integer vector with unit MD-characteristic may have coordinates with absolute values greater than 1000).

It is most probable that white squares in Figure 7 represent ς-reduced matrices. We have checked explicitly all the squares with

$$-10\le m,n\le 10.$$

These matrices are contained inside the big black square shown on the figure. All white squares inside it correspond to ς-reduced matrices.

We show a boundary broken line between the NRS- and RS-squares in gray.

Remark. In Figure 7 the *NRS-domain* is easily visualized, it almost completely consists of white squares. While *RS-domain* contains relatively large number of black squares. This indicates a significant difference between RS- and NRS-cases.

Direct calculations of the corresponding MD-characteristic show that the following proposition holds.

Proposition 6.1. *If an integer $m+n$ is odd, then $H^{(1,0,1)}_{\langle 0,1|1,0,2\rangle}(m,n)$ is ς-reduced.* □

Remark. From one hand Proposition 6.1 implies the existence of rays entirely consisting of ς-reduced matrices. From the other hand in contrast to the NRS-matrices this is not

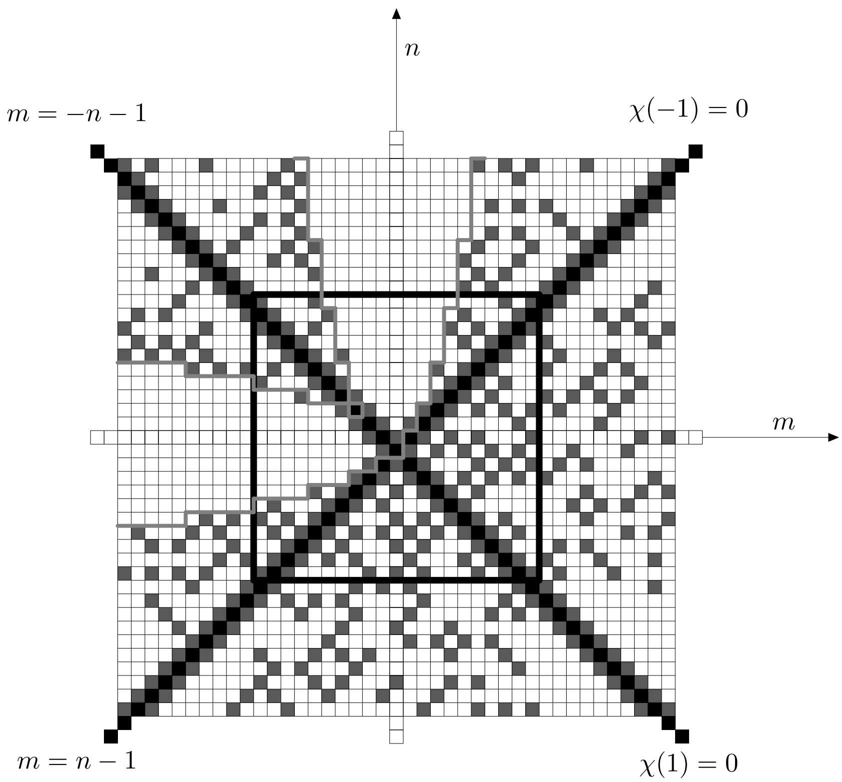

FIGURE 7. The family of matrices of Hessenberg type $\langle 0,1|1,0,2\rangle$.

always the case for RS-matrices. For instance, all matrices corresponding to integer points of the lines

$$1)m=n;\quad 2)m=n+2;\quad 3)m=-n;\quad 4)m=-n-2;\quad 5)n=3m-4;\quad 6)m=3n+6$$

are ς-reduced (we do note state that the list of such lines is complete).

So Theorem 2.6 does not have a direct generalization to the RS-case and we end up with the following problem.

Problem 3. What is the percentage of ς-reduced matrices among matrices of a given Hessenberg type Ω?

It is more likely that almost all Hessenberg matrices are ς-reduced (except for some measure zero subset).

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E-mail address, Oleg Karpenkov: `karpenk@mccme.ru`

TU Graz /Kopernikusgasse 24, A 8010 Graz, Austria/