On determination of periods of geometric continued fractions for two-dimensional algebraic hyperbolic operators.

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Introduction

A two-dimensional operator A in $SL(2,\mathbb{Z})$ is said to be *hyperbolic*, if its eigenvalues are real and distinct. In the present paper we study the connection between periods of geometric continued fraction in the sense of Klein and reduced operators, described by J. Lewis and D. Zagier in [15]. Actually, a determination of a period of the geometric continued fraction for the operator A is equivalent to a calculation of a period of the ordinary continued fraction for the tangent of an angle between any eigen straight line of A and the horizontal axis.

In present paper for any period of a geometric continued fraction in the sense of Klein we make an explicit construction of a reduced hyperbolic operator in $SL(2,\mathbb{Z})$ with the given period for the geometric continued fraction (Theorem 6). It turns out that reduced operators naturally forms one-parametric families. Further we describe an algorithm to construct a period for an arbitrary operator of $SL(2,\mathbb{Z})$. The base part of the algorithm is to determine a reduced operator that is conjugate to the given (the Gauss Reduction Theory).

In 1993 V. I. Arnold formulated some questions on periods of continued fractions related to the eigenvectors of the $SL(2,\mathbb{Z})$-operators, see for instance in [3] and [5]. The first studies of these problems are made in the article [18] by M. Pavlovskaya, in which the author experimentally investigates statistical questions on geometrical continued fractions (such as verification of the average length of periods of continued fractions). The questions on the distribution of positive integers in minimal periods of quadratic continued fractions were studied by M. O. Avdeeva and B. A. Bykovskii in the works [1] and [2]. In his work [4] V. I. Arnold investigates palindromic properties of continued fraction periods and the connection to the integer forms (the complete proof of the palindromic property is given by F. Aicardi and M. Pavlovskaya). The relation between "$-$"-continued fractions of hyperbolic operators and Fuchian groups, and a few words on the algorithm of integer conjugacy check for the operators in $SL(2,\mathbb{Z})$ is described in the work of S. Katok [10]. Some estimates of the period lengths for ordinary continued fractions for $\sqrt{d}$ are given in [7] by D. R. Hickerson.

In the works [11] and [12] F. Klein generalized geometric continued fraction to the multidimensional case of $SL(n,\mathbb{Z})$-operators. For more information see the works of

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V. I. Arnold [5], E. I. Korkina [13], G. Lachaud [14], J.-O. Moussafir [17], the author [8], etc.

The work is organized as follows. In the first section we give definitions of ordinary continued fractions and geometric continued fractions in the sense of Klein, we show a duality of sails of a geometric continued fraction. Further in the second section we construct families of operators possessing the geometric continued fractions (actually the corresponding LLS-sequences) with the period $(a_1, \ldots, a_{2n-1}, t)$, where t is the parameter of families. Further we give an algorithm of period calculation for geometric continued fractions. In the third section we formulate some questions and show the results of a few experiments related to the algorithm.

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1. Geometric continued fractions in the sense of Klein

Ordinary continued fractions. Consider an arbitrary finite sequence of integers $(a_0, a_1, \ldots, a_n)$, where a_0 is any integer and $a_i > 0$ for $i > 0$. The rational

$$a_0 + 1/(a_1 + 1/(a_2 + \ldots) \ldots))$$

is called the *ordinary continued fraction* and denoted by $[a_0 : a_1; \ldots; a_n]$.

A continued fraction with even (odd) number of elements is called an *even* (*odd*) ordinary continued fraction respectively.

Proposition 1. *For any rational there exist a unique even and a unique odd continued fractions.* □

For instance the even and odd ordinary continued fractions for a rational 47/39 are [1:4; 1; 7] and [1:4; 1; 6; 1] respectively.

Sails of hyperbolic operator. By $[[a, c][b, d]]$ we denote the operator

$$\begin{pmatrix} a & c \\ b & d \end{pmatrix}.$$

An operator in $SL(2, \mathbb{R})$ with two distinct real eigenvalues is called *hyperbolic*.

A vector (segment) is said to be *integer* if it (its endpoints) has integer coordinates.

Consider a hyperbolic operator A with no integer eigenvectors. The operator A has exactly two distinct eigen straight lines. These lines do not contain integer points of the lattice distinct to the origin. The complement to the union of the lines consists of four piece-wise connected components, each of which is an open octant. Consider one of these octants. The boundary of the convex hull of all integer points except the origin in the closure of the octant is called a *sail* of the operator A. The set of all sails is called the *geometric continued fraction in the sense of Klein* for the operator A (see also the works of F. Klein [11], E. I. Korkina [13], and V. I. Arnold [5]). Two sails are said to be *equivalent* if there exists a linear lattice-preserving transformation, taking one of the sails to another. Geometric continued fractions with equivalent sails are called *equivalent*.

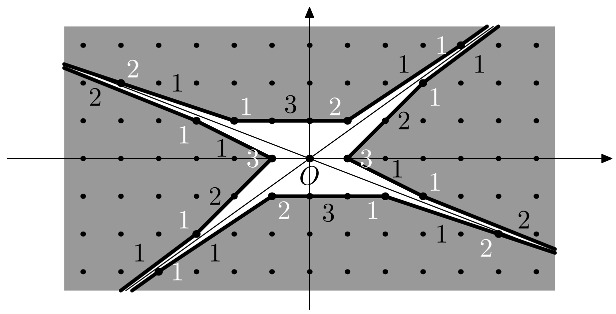


FIGURE 1. Geometric continued fraction of the operator $[[7,18][5,13]]$.

Remark. The majority of constructions of this article can be naturally generalized to the case of operators that have integer eigenvectors. For simplicity of the exposition we do not consider such operators in the article.

LLS-sequence. An *integer length* of an integer segment PQ is the quantity of inner integer points of the segment plus one, it is denoted by $\mathrm{l}\ell(PQ)$. Let integer segments PQ and PR do not lie on the same straight line. An *integer sine* of the angle QPR is an index of the sublattice generated by the integer vectors of the straight lines PQ and PR in the lattice of all integer vectors (we denote it by $\mathrm{lsin}(QPR)$). An integer sine of an angle contained in some straight line is supposed to be equivalent to zero.

Any sail of the hyperbolic operator A with no integer eigen vectors is a two-sides broken line with infinite in both sides number of segments.

Definition 2. Let $\ldots V_{-2}V_{-1}V_0V_1V_2\ldots$ be a sail of some operator. The infinite in both sides sequence of positive integers

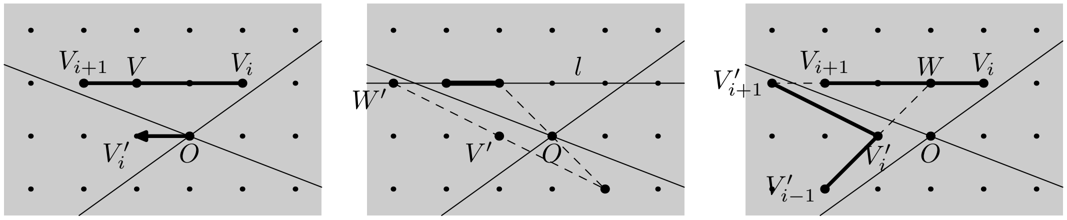
$$(\ldots, \mathrm{l}\ell(V_{-2}V_{-1}), \mathrm{lsin}\,\angle V_{-2}V_{-1}V_0, \mathrm{l}\ell(V_{-1}V_0), \mathrm{lsin}\,\angle V_{-1}V_0V_1, \mathrm{l}\ell(V_0V_1), \mathrm{lsin}\,\angle V_0V_1V_2, \ldots)$$

is called the *LLS-sequence* of the sail (see also in [9]).

Two LLS-sequences are called *equivalent*, if one can be obtained from another by shifting on a finite number of elements and/or reversing the order of all elements. The LLS-sequences of equivalent sails are equivalent, since all integer lengths and sines are invariants under integer-linear transformations.

On Figure 1 we show the geometric continued fraction for the operator $[[7,18][5,13]]$. Integer lengths of edges are denoted by black digits, and integer sines — by white. The LLS-sequences of all four sails are equivalent to $(\ldots,2,1,1,3,2,1,1,\ldots)$.

Duality of sails. Two sails are called *dual* with respect to each other if their LLS-sequences coincide, the sequence of integer lengths of the first sail coincides with the sequence of integer sines of the second, and the sequence of integer sines of the first sail coincides with the sequence of integer lengths of the second. (On the duality for multidimensional continued fractions in the sense of Klein see in the book by G. Lachaud [14].)



a). Construction of the vertex V_i'.   b). Construction of the edge.   c). Calculation of the sine.

FIGURE 2. Duality of adjacent angles.

Proposition 3. *Let A be a hyperbolic operator with no integer eigenvectors. Then the LLS-sequences of all his four octants coincide. Moreover the sails of the opposite octants are equivalent. The sails of the adjacent octants are dual.*

In the proof of Proposition 3 we use the following properties of an integer sine (for further information see in [9]):

$$\operatorname{lsin}(PQR) = \operatorname{lsin}(RQP); \quad \operatorname{lsin}(PQR) = \operatorname{lsin}(\pi - PQR),$$

where by $\pi - PQR$ we denote the angle adjacent to the angle PQR.

The statement of Proposition 3 is known to the author from V. I. Arnold and E. I. Korkina and supposed to be classical, still its formulation and proof seem to be missing in the literature. To avoid further questions we give the proof here.

Proof. The sails of the opposite octants (and so their LLS-sequences) are equivalent, since they are symmetric about the origin.

Consider now the case of adjacent sails. Let now S and S' be adjacent sails of the operator A, where S' is counterclockwise with respect to S while turning around the origin O. Let S be a broken line $\ldots V_{-1}V_0V_1\ldots$ with counterclockwise order of vertices with respect to the origin.

Let us show a natural bijection between the edges of the sail S and vertices of the sail S'. Consider an edge V_iV_{i+1} of the sail S. Denote by V the closest integer point in the segment V_iV_{i+1} to the endpoint V_{i+1} and distinct to V_{i+1}. Denote the integer point $O+\overline{VV_{i+1}}$ by V_i'. Notice that the point V_i' is in the convex hull of S'.

Since the triangle OVV_{i+1} does not contain integer points inside, the triangle $OV_i'V_{i+1}$ does not contain integer points inside. Hence there is no integer points between the parallel lines OV and $V_i'V_{i+1}$. Therefore, V_i' is a vertex of the sail S' (see on Figure 2a).

From the other side V' is a vertex of the sail S'. Denote by l the closest to the line OV' parallel line containing integer points and intersecting W. The line l intersects with the octant containing S' in a ray. Denote by W' the closest to the vertex of the ray integer point of this ray. Then the point $V'+\overline{W'V'}$ is in the octant opposite to the octant with the sail S (otherwise V' is not a vertex). Hence the point $W'+2\overline{V'O}$ symmetric about the origin to the point $V'+\overline{W'V'}$ lies in the octant of the sail S. Therefore, the integer points $W'+\overline{V'O}$ and $W'+2\overline{V'O}$ lie in the octant containing S. Since l is the closest to the line XV' parallel line containing integer points and intersecting S, the segment with the endpoints $W'+\overline{V'O}$ and $W'+2\overline{V'O}$ in contained in the sail (see on Figure 2b).

Therefore the above correspondence between the edges V_iV_{h+2} of the sail S and vertices V_i' of the sail S' is a bijection. Moreover the order of vertices V_i' is clockwise.

Let us prove now that $\mathrm{l}\ell(V_iV_{i+1}) = \mathrm{lsin}(V_{i-1}'V_i'V_{i+1}')$. Denote the point $V_i+\overline{OV_i'}$ by W. By construction, the point W belongs to the segment V_iV_{i+1} (see Figure 2c). Then

$$\mathrm{lsin}(V_{i-1}'V_i'V_{i+1}') = \mathrm{lsin}(\pi - WV_i'V_{i+1}') = \mathrm{lsin}(WV_i'V_{i+1}') = \mathrm{l}\ell(WV_{i+1}') = \mathrm{l}\ell(V_iV_{i+1}).$$

By the analogous argumentation the following equalities also hold:

$$\mathrm{l}\ell(V_i'V_{i+1}') = \mathrm{lsin}(V_{i-1}V_iV_{i+1}).$$

Therefore, the LLS-sequences of the sails S and S' coincide, moreover the sequence of integer lengths (sines) of S coincides with the sequence of integer sines (lengths) of S'. Hence S and S' are dual. Proposition 3 is proved. □

Existence and uniqueness of the equivalence classes of continued fractions with a given LLS-sequences.

Definition 4. The *LLS-sequence* of an operator A is the LLS-sequence for any of its sails.

Proposition 5. *i). For any infinite in two sides sequence of integers there exists a hyperbolic operator whose LLS-sequence coincides with the given.*

ii). Two sails with coinciding LLS-sequences are either equivalent or dual. All sails dual to the given are equivalent to each other.

Proof. Both statements follows directly from Corollary 5.11 of [9] (see also in the work of E. I. Korkina [13]). □

2. Algebraic sails.

Construction of the operator with the given period. Consider now an algebraic case of hyperbolic operators of the group $SL(2,\mathbb{Z})$ with irreducible over rationals characteristic polynomial. Let A be an integer hyperbolic operator of $SL(2,\mathbb{Z})$. Denote by $\Xi(A)$ the group of operators in all $SL(2,\mathbb{Z})$ commuting with A and having positive real eigenvalues. By Dirichlet unity theorem [6] the group $\Xi(A)$ is isomorphic to $\mathbb{Z}$. Any sail of the operator A is invariant under the action of the group $\Xi(A)$, moreover the operators of the group $\Xi(A)$ act on the sails by shifting the edges of the broken line along thy broken line. The sails of the operator A are called *algebraic*.

Therefore LLS-sequences of hyperbolic algebraic operators are periodic. The converse is also true (see Corollary 7 below).

Remark. On Figure 1 we show the sails of hyperbolic algebraic operator $[[7,18][5,13]]$ with a period of the LLS-sequence equals $(2,1,1,3)$.

Theorem 6. *Consider an $ST(2,\mathbb{Z})$-operator $[[a, c+\lambda a][b, d+\lambda b]]$.*
i). Let $a=0$, $b=1$, $d=1$. If $\lambda>2$ then the operator A is hyperbolic and its sails are algebraic. One of the periods of the LLS-sequence of the operator A is

$$(1, \lambda-1).$$

ii). Let $b>a\geq 1$, $0<d\leq b$, $\lambda\geq 1$. Let the odd ordinary continued fraction for b/a equal

$$[a_0{:}a_1;\ldots;a_{2n}].$$

Then the operator A is hyperbolic and its sails are algebraic. One of the periods of the LLS-sequence of the operator A is

$$(a_0, a_1, \ldots, a_{2n}, \lambda).$$

Notice that for any couple of relatively prime integers (a,b) where $b>a\geq 0$ there exists a couple of integers (c,d), satisfying $0<d\leq b$ and $ad-bc=1$.

Remark. For negative values of λ in the case $a=0$ the periods are $(1,|\lambda|-3)$. In the case $a>0$ the periods equal

$$(a'_0, a'_1, \ldots, a'_{2m}, |\lambda|-2),$$

where $[a'_0{:}a'_1;\ldots;a'_{2m}]$ — is the odd ordinary continued fraction for $b/(b-a)$.

Proof. The discriminant of the characteristic polynomial of the operator A equals $((a+\lambda b+d)^2-4$. Since $\lambda\geq 1$, $b\geq 1$, $d\geq 1$, and $a\geq 0$, the discriminant is nonnegative. Besides it equals zero in the exceptional case $a=0$, $b=1$, $\lambda=1$, $d=1$. Therefore the operator A is hyperbolic in all cases. Since t^2-4 for integer $t>2$ is not a square of some integer, the sails of the operator A are algebraic.

Let us now construct a period for the LLS-sequence. Note that both eigenvalues of the operator A:

$$\frac{a+\lambda b+d\pm\sqrt{((a+\lambda b+d)^2-4}}{2}$$

are positive, and thus the operator A takes each sail to itself. Consider the sail S, whose convex hull contains the point $P=(1,0)$.

Suppose the operator A is of series i). Then the set of the vertices for one of its sails coincides with the set of points $A^n(1,0)$ with an integer parameter n. Simple calculations lead to the result of the theorem.

Let now an operator A be an operator of series ii), i. e. $b>a>0$.

Denote by α the closed convex angle with vertex at the origin and edges passing through the points P and $A(P)$. Consider the boundary of the convex hull of all integer points inside α except the origin. The boundary consists of two rays and a finite broken line. We call the finite broken line in the boundary the *sail of the angle* α and denote it by S_α.

Now we show that the sail of the angle α is completely contained in one of the sails of the operator A. Denote by S_α^∞ the following infinite broken line:

$$\bigcup_{i=-\infty}^{+\infty}\Big(A^i(S_\alpha)\Big).$$

By the construction the convex hull of S_α^∞ coincides with the convex hull of the sail S. It remains to verify if S_α^∞ coincides with the boundary of its convex hull. The broken line S_α^∞ is the boundary of the convex hull if the convex angles generated by adjacent edges of the broken line do not contain the origin. To check this it is sufficient to study all the angles of one of the periods of the broken line S_α^∞, for instance all the angles with vertices at vertices of S_α except the point $A(P)$. All convex angles generated by couples of adjacent edges at inner vertices of the sail S_α do not contain the origin by definition. It remains to check the angle with vertex at $P=(1,0)$.

Since $b/a>1$, the first edge is parallel to the vector $(0,1)$. Consider the second edge PQ. Since the triangle OPQ does not contain integer points distinct to the integer points of the segment PQ and the vertex O, the segment PQ contains the point with coordinates $(x,-1)$. By the convexity of finite broken line $A^{-1}(S_\alpha)$ the value of x is determined by the eigen direction:

$$\left(\frac{-a+\lambda b+d+\sqrt{((a+\lambda b+d)^2-4}}{2b},-1\right),$$

namely,

$$x=\left\lfloor\frac{-a+\lambda b+d+\sqrt{((a+\lambda b+d)^2-4}}{2b}\right\rfloor+1,$$

where $\lfloor t\rfloor$ denotes the maximal integer that does not exceeding t. As one can show, x is contained in the open interval $\big(\lambda+1+(d-1)/b,\lambda+1+d/b\big)$. By condition $b\ge d>0$ we have

$$x=\lambda+1.$$

Hence for $\lambda>1$ the convex angle at vertex P does not contain the origin. Therefore, the broken line S_α^∞ coincides with the sail.

In the paper [9] it is shown that the sail S_α consists of $n+1$ segments. The integer lengths of the consecutive segments equal $a_0,a_2,\ldots,a_{2n}$, and the integer sines of the corresponding angles equal $a_1,a_3,\ldots,a_{2n-1}$ respectively. Now note, that from the explicit value of x it follows that the integer sine for the angle at point P equals λ. Hence the LLS-sequence of the sail S has a period

$$(a_0,a_1,\ldots,a_{2n},\lambda).$$

Therefore the LLS-sequence of the operator A has the prescribed period. □

Corollary 7. *A sail with the periodic LLS-sequence is algebraic* (*i. e. a sail of some algebraic hyperbolic operator*).

Proof. In Theorem 6 we constructed the algebraic operators for all finite sequences as periods. Then in Proposition 5 we showed that the sails with equivalent LLS-sequences are either equivalent or dual. Therefore any sail with periodic LLS-sequence is algebraic. □

Remark. Consider some sail with periodic LLS-sequence. Let a minimal period of LLS-sequence is even and consists of $2n$ elements. Then there exists an $SL(2,\mathbb{Z})$-operator A with positive eigenvalues, that makes an n-edge shift of the sail along the sail. Precisely this operator generates the group $\Xi(A)$ of the sail shifts (see above). Let a minimal period of LLS-sequence is odd and consists of $2n+1$ elements (in particular this implies that the sail is equivalent to any dual sail). Then there exists a $GL(2,\mathbb{Z})$-operator B with negative discriminant, whose square makes an $(2n+1)$-edge shift of the sail along the sail. Moreover, the operator B^2 generates the group $\Xi(T)$.

Remark. Let us say a few words about non-hyperbolic operators in $SL(2,\mathbb{Z})$. It turns out that each of such operators is equivalent to exactly one of the operators of the following list:
— $[[1,1][-1,0]]$;
— $[[0,1][-1,0]]$;
— $[[0,1][-1,-1]]$;
— $[[1,n][0,1]]$, where n is any integer.

Algorithm of finding a period of the LLS-sequence for a hyperbolic algebraic operator.

Definition 8. An operator $[[a,c][b,d]]$ in $SL(2,\mathbb{Z})$ is said to be *reduced*, if the following holds: $d > b \geq a \geq 0$.

Remark. The definition of a reduced operator is slightly different to one given in the works [15] and [16]: *an operator in $SL(2,\mathbb{Z})$ is reduced iff it has non-negative entries which are non-decreasing downwards and to the right.*

The main idea of the calculation of the period is to find a reduced operator with the sails equivalent to the sails of the given one. Then it remains to calculate the period of the reduced operator by Theorem 6.

Data. Suppose we know the integer entries of an operator $[[a,c][b,d]]$ with unit determinant and positive discriminant. Let also the characteristic polynomial does not has roots ± 1. From the listed conditions it follows that the operator $[[a,c][b,d]]$ is hyperbolic operator in $SL(2,\mathbb{Z})$ with irreducible characteristic polynomial.

It is requested to construct one of the periods of the LLS-sequence of the hyperbolic algebraic operator $[[a,c][b,d]]$.

Description of the algorithm.

Step 1. If $b < 0$, then we multiply the operator $[[a,c][b,d]]$ by $[[-1,0][0,-1]]$. The LLS-sequence does not change at that.

Step 2. So, now the element b is positive. Conjugate the operator $[[a,c][b,d]]$ by the operator $[[1,-\lfloor a/b \rfloor][0,1]]$. We obtain the operator $[[a',b'][c',d']]$, where $0 \leq a' \leq b'$.

Step 3.1. Suppose $b' = 1$, then $a' = 0$, $c' = -1$. Moreover we have $|d| > 2$, since otherwise the original operator is not algebraic. Therefore a period of the LLS-sequence equals $(3, |d| - 2)$.

Step 3.2.1. Suppose, $b' > 1$. If $d' > b'$, then we have found a reduced operator, now we go to Step 4.

Step 3.2.2. Suppose, $b' > 1$. If $d' < -b'$, then we conjugate by the operator $[[-1, 1][0, 1]]$. Finally we have the operator $[[a'', c''][b'', d'']]$ with $b'' = b'$, $a'' = b' - a'$, and $d'' = -b' - d' > 0$, further we should go to Step 3.2.1, or to Step 3.2.3.

Step 3.2.3. Suppose, $b' > 1$. The case $|d'| \leq |b'|$. Note that the absolute values of b' and d' do not coincide since the determinant of the operator does not have divisors distinct to the unity. Therefore it remains the case $|d| < |b|$. In this case we have:

$$|c'| = \left|\frac{a'd' - 1}{b'}\right| \leq \frac{(b' - 1)^2 + 1}{b'} \leq b' - 1.$$

We conjugate the operator $[[a', c'][b', d']]$ with the operator $[[0, -1][-1, 0]]$ and obtain $[[d', b'][c', a']]$, where $|c'| < |b'|$. Now we return back to Step 1 with the obtained operator $[[d', b'][c', a']]$.

Step 4. We obtained a reduced operator $[[\tilde{a}, \tilde{c}][\tilde{b}, \tilde{d}]]$, $\tilde{b} > 1$ with the LLS-sequence equivalent to the LLS-sequence of the original operator. By Theorem 6 to construct one of the periods of the LLS-sequences of the reduced operator we should construct the odd ordinary continued fraction for $\tilde{b}/\tilde{a}$, and find the integer $\lfloor (\tilde{d} - 1)/\tilde{b} \rfloor$.

3. Some questions and examples

A question on complexity of the minimal period. Note that for any operator there exist finitely many reduced operators with the same trace and LLS-sequence. If we study the reduced operators that make shifts of sails on a minimal possible period, then the number of such operators coincides with the length of the minimal period (see also in [15]). Let $[[a, c][b, d]]$ be a reduced operator. We call the integer b — its *complexity*.

Problem 1. Study the minimal complexity for reduced operators with LLS-sequence having a length n period $(a_1, \ldots, a_n)$.

Remark. The minimal complexity coincides with the minimal positive value of the integer sine of the angles POQ, where O is the origin, $P = (x, y)$ is an arbitrary integer point distinct to O, and $Q = A(P)$. Therefore the minimal complexity, considered as the minimal possible integer sine, is well defined for all operators and it is invariant under conjugations.

If n is even, then the problem is equivalent to finding the minimal numerator among the numerators of the rationals:

$$[a_1 : \ldots ; a_{n-1}], \quad [a_2 : \ldots ; a_n], \quad [a_3 : \ldots ; a_n ; a_1], \quad \ldots \quad , [a_n : a_1 ; \ldots ; a_{n-2}].$$

Example 1. Let the period contains two elements: (a, b), where $a < b$, then the minimal complexity equals a.

Example 2. Let the period contains four elements: (a, b, c, d). Let d is not smaller than the other elements of the period, let also $d > a$ except for the case $a = b = c = d$. Then the rational with the minimal numerator can be found from the following table.

(a, b, c, d) $d \geq a, b, c$	Rational with the minimal numerator
$d > a, b, c$	$[a{:}b; c]$
$d = c; b < a < d$	$[d{:}a; b]$
$d = c; a < b < d$	$[a{:}b; c]$
$d = c; a = b < d$	$[a{:}b; c]$ and $[d{:}a; b]$
$d = b; d > a, c$	$[a{:}b; c]$ and $[c{:}d; a]$
$d = c = b; d > a$	$[a{:}b; c]$ and $[c{:}d; a]$
$d = c = b = a$	all

If n is odd, then the problem is equivalent to finding the minimal numerator among the numerators of the rationals (we define $a_{n+k} = a_k$):

$$[a_1{:} \ldots ; a_{2n-1}], \quad [a_2{:} \ldots ; a_{2n}], \quad [a_3{:} \ldots ; a_{2n}; a_1], \quad \ldots \quad , [a_{2n}{:}a_1; \ldots ; a_{2n-2}].$$

Example 3. Let the period consists of three elements: (a, b, c), where $c \geq a, b$. Then the fraction $[a{:}b; c; a; b]$ has the minimal numerator (or of one of some equivalent minimal numerators in the case of $a = c$ or $b = c$).

One can suppose that we should skip one of the maximal elements of the period, but that is not true for the six element sequence: $(1, 4, 5, 4, 1, 4)$. The minimum of the numerators is attained at the fraction $[1 : 4; 5; 4; 1]$, and not at the fraction $[4 : 1; 4; 1; 4]$.

On frequencies of occurrences of the reduced operators. First we describe a proper probabilistic space. Let $P = (a_1, a_2, \ldots , a_{2n-1}, a_{2n})$ be some period. Denote by $\#_N(P)$ the quantity of all operators satisfying the following conditions:
i). The absolute value of any entry of the operator does not exceed N.
ii). The sequence P is one of the periods of SL-sequence for the operator.
iii). Starting from the operator, the algorithm of the previous section constructs the reduced operator $[[a, b][c, d]]$, where $(a, b) = (0, 1)$ for the case $P = (1, a_2)$; and

$$b/a = [a_1 : a_2; \ldots ; a_{2n-1}]$$

in the remaining cases.

Then one studies the relative statistics of $\#_N(P)$ while N tends to infinity. The following questions are of interest.

Problem 2. a). Which one of the reduced operators with a given trace (or with a fixed LLS-sequence) is the most frequent as a result of the algorithm of the previous section?
b). What is the probability of that?
c). Is it true that the maximal possible probability is attained at reduced operators with minimal complexity?

Absolute value of the trace	Notation for classes of equivalent operators	Period P	Operator $[[a,b][c,d]]$	Value of $\#_{25000}(P)$
3	L_3	$(1,1)$	$[[0,1][-1,3]]$	663160
4	L_4	$(1,2)$	$[[0,1][-1,4]]$	834328
		$(2,1)$	$[[1,2][1,3]]$	304776
5	L_5	$(1,3)$	$[[0,1][-1,5]]$	818200
		$(3,1)$	$[[1,3][1,4]]$	194528
6	$L_{6,1}$	$(1,4)$	$[[0,1][-1,6]]$	777128
		$(4,1)$	$[[1,4][1,5]]$	141784
	$L_{6,2}$	$(2,2)$	$[[1,2][2,5]]$	446432
7	$L_{7,1}$	$(1,5)$	$[[0,1][-1,7]]$	734904
		$(5,1)$	$[[1,5][1,6]]$	110848
	$L_{7,2}$	$(1,1,1,1)$	$[[2,3][3,5]]$	201744
8	$L_{8,1}$	$(1,6)$	$[[0,1][-1,8]]$	695560
		$(6,1)$	$[[1,6][1,7]]$	90688
	$L_{8,2}$	$(2,3)$	$[[1,2][3,7]]$	435472
		$(3,2)$	$[[1,3][2,7]]$	310872
9	L_9	$(1,7)$	$[[0,1][-1,9]]$	660984
		$(7,1)$	$[[1,7][1,8]]$	76552
10	$L_{10,1}$	$(1,8)$	$[[0,1][-1,10]]$	630592
		$(8,1)$	$[[1,8][1,9]]$	66064
	$L_{10,2}$	$(2,4)$	$[[1,2][4,9]]$	408216
		$(4,2)$	$[[1,4][2,9]]$	239712
	$L_{10,3}$	$(1,1,1,2)$	$[[2,3][5,8]]$	260872
		$(2,1,1,1)$	$[[2,5][3,8]]$	114084
		$(1,2,1,1)$	$[[3,4][5,7]]$	149832
		$(1,1,2,1)$	$[[3,5][4,7]]$	114084

TABLE 1. Values of $\#_{25000}(P)$ for the operators with small absolute values of the traces.

In Table 1 we give some results of calculation of $\#_{25000}(P)$ for the operators with small absolute value of the trace. We remind that the minimal absolute value of the trace of hyperbolic $SL(2,\mathbb{Z})$-operator equals 3.

It is interesting to note that $SL(2,\mathbb{Z})$-operators corresponding to $P=(1,2)$ are more frequent than the $SL(2,\mathbb{Z})$-operators corresponding to $P=(1,1)$. This occurs since the sails whose LLS-sequences has the period $(1,1)$ are equivalent to their duals. If we enumerate the operators with multiplicities equivalent to the number of equivalent sails for the operators then we get:

$$4\#_{25000}(1,1) > 2\#_{25000}(1,2) + 2\#_{25000}(2,1).$$

In conclusion we formulate the following question. Denote by $GK(P)$ the probability of the sequence $P=(a_1,a_2,\ldots,a_{2n-1})$ in the sense of Gauss-Kuzmin:

$$GK(P)=\frac{1}{\ln(2)}\ln\left(\frac{(\alpha_1+1)\alpha_2}{\alpha_1(\alpha_2+1)}\right),$$

where $\alpha_1=[a_1{:}a_2;\ldots;a_{2n-2};a_{2n-1}]$, $\alpha_2=[a_1{:}a_2;\ldots;a_{2n-2};a_{2n-1}+1]$.

Problem 3. Let

$$\begin{array}{ll} P_1=(a_1,a_2,\ldots a_{2n-1},a_{2n}), & P_1'=(a_1,a_2,\ldots a_{2n-1}),\\ P_2=(a_2,a_3,\ldots a_{2n},a_1), & P_2'=(a_2,a_3,\ldots a_{2n}). \end{array}$$

Is the following true:

$$\lim_{n\to\infty}\frac{\#_n(P_1)}{\#_n(P_2)}=\frac{GK(P_1')}{GK(P_2')}\ ?$$

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